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Short Communication

Construction of Association Schemes and Coherent Configuration from Williamson's Hadamard Matrices and their Properties

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Abstract

Association Schemes and Coherent Configurations have been constructed from Williamson's H-matrices. We have also described their properties.

Keywords: Hadamard matrices, Symmetric Hadamard matrix, Paley type II Hadamard matrices, Association Scheme, Amorphic 3-AS, Williamson matrices, Williamson's Hadamard matrix, Williamson's.

Introduction

We begin with the following definitions:

Hadamard Matrices (Or H-Matrices): A (1, -1) matrix H of order m such that $HH^T = mI_m$ is called a Hadamard matrix (or an H-matrix). H-matrix of order m = 4b exists for every b ≥ 1 (vide Hall[1] , Hedayat and Wallis[2]). For recent constructions vide Horadam[3].

Hadamard Matrices have application in Discrete Mathematical Modeling especially in Error Correcting Codes, Signal Processing and Cryptography.

Symmetric H-matrix: An H-matrix H is said to be symmetric if $H = H^T$.

Paley type II H-matrix: There exists an H-matrix $P_{2(a+1)}$ of order 2(a + 1) where a is a prime power of the form 4b + 1 such that $P_{2(a+1)} = \begin{bmatrix} S - I_{a+1} & S + I_{a+1} \\ S + I_{a+1} & -S + I_{a+1} \end{bmatrix}$ where $S = \begin{bmatrix} 1 & r \\ r^T & Q \end{bmatrix}$ and $r = [1 \quad 1 \quad - \quad - \quad - \quad 1]$ is a 1 X a array.

Association Scheme (AS) (vide Hanaki[4], Godsil and Song[5]): Let $R_0, R_1, \ldots, R_m$ be binary relations on a set V ={1,2,…., v}. Let $A_i = [a_{ij}]$ be the (0,1) matrix defined as $a_{jk} = \begin{cases} 1, \text{if } (j,k)\in R_i \\ 0, \text{otherwise} \end{cases}$. The matrix A_i is called adjacency matrix of the relation R_i. The set P = $(R_0, R_1, \ldots, R_m)$ is called an m –class association scheme if the adjacency matrices A_i of R_i (i = 0, 1, 2,…., m) satisfying: i. $A_0 = I$ (Identity Matrix) and $A_i \neq 0, \forall\, i$, ii. $\sum_{i=0}^{m} A_i = J$, where J is all-1 matrix, iii. $A_i^T = A_i \;\forall$ i ε {0,1,2,…,m}, iv. There are numbers p_{ij}^k such that $A_i A_j = \sum_{k=0}^{m} p_{ij}^k A_k$

Amorphic 3-AS: Let a 3-AS be defined by the association matrices I, A_1, A_2, A_3. Then 3-AS is called amorphic if each of A_1, A_2, A_3 is an adjacency matrix of a strongly regular graph.

Williamson Matrices (vide Craigen and Kharghani[6], Turyn[7]): Four m x m symmetric and circulant (1, -1) matrices W, X, Y, Z satisfying the condition $A^2 + B^2 + C^2 + D^2 = 4nI_n$ are called Williamson matrices.

Hadamard Matrix of Williamson form: If W, X, Y, Z are Williamson matrices then H = $\begin{bmatrix} W & X & Y & -Z \\ -X & W & Z & Y \\ Y & Z & -W & X \\ -Z & Y & -X & -W \end{bmatrix}$ is called an H-matrix of Williamson form.

Williamson's AS: Let I_m be the unit matrix and α be a circulant matrix of the form Circ (0,1,0,............,0) of order m. Let $w_k = \alpha^k + \alpha^{m-k}$,1 ≤ k ≤ (m-1)/2. Then $w_k = w_{m-k} = w_{-k}$, $w_k^2 = w_{2k} + 2I_m$ and, $w_k w_j = w_{k+j} + w_{|k-j|}$ Where lower suffices 2k, k+ j are to be reduced mod m, whenever they are greater than (m-1)/2. Clearly, W= { $I_m, w_1, w_2, \ldots\ldots,$ $w_{(m-1)/2}$ } is a set of symmetric matrices and defines an p- AS which will be called Williamson's AS, where $p = (m-1)/2$. Williamson matrices W, X, Y, Z are suitable (1,-1) linear combinations of $I_m, w_1, \ldots\ldots\ldots, w_{(m-1)/2}$, which form blocks of a Williamson Hadamard matrix (vide Hall[1]).

Method of construction of Association schemes from Williamson's Hadamard Matrices: 3-class Association Schemes from Symmetric Hadamard matrix of Paley type II: Theorem 1: Consider Hadamard matrix of Paley type II

$$H = \begin{bmatrix} S-I & S+I \\ S+I & -S+I \end{bmatrix} \text{ where } S = \begin{bmatrix} 0 & r \\ r^T & Q \end{bmatrix}, r = (1,1,1,\ldots,1)$$

Let $Q = \gamma_1 - \gamma_2$ where I, γ_1 , γ_2 are association matrices then

$A_1 = \gamma_1 X \gamma_2 + \gamma_2 X \gamma_1$, $A_2 = \gamma_1 X \gamma_1 + \gamma_2 X \gamma_2$ and $A_3 = I X L + L X I$ where $K = \gamma_1 + \gamma_2$ define a 3-AS.

Proof: when 4m-1 is equal to $p^r = a$, where p being a prime and let x be a primitive element of Galois field GF (a). Let $\{1, x^2, (x^2)^2, \ldots, (x^2)^{(a-3)/2}\}$ mod (4m-1) is a difference set. We denote this difference set as $\{1, d_1, d_2, \ldots, d_k\}$ mod (4m-1) where $k = (a - 3)/2$.

Let α = circ (0100……0), $\gamma_1 = \alpha + \alpha^{d_1} + \alpha^{d_2} + \ldots + \alpha^{d_k}$ and $\gamma_2 = \alpha^{-1} + \alpha^{-d_1} + \alpha^{-d_2} + \ldots + \alpha^{-d_k}$.

Then $\gamma_1\gamma_2 = \{(a-1)/4\}L$.
Since $\gamma_1 + \gamma_2 = L$
$\Rightarrow \gamma_1\gamma_2 = \gamma_1 (L - \gamma_1) = \{(a-1)/4\} L$
$\Rightarrow \gamma_1 L - \gamma_1^2 = \{(a-1)/4\}L$ (1)

As γ_1 and γ_2 are regular (0, 1) matrices

So $\gamma_1 J = J \gamma_1 = \{(a-1)/2\} J$

Also $L = J - I$

So $\gamma_1 L = \gamma_1 (J - I) = \gamma_1 J - \gamma_1 = \{(a-1)/2\} J - \gamma_1 = \{(a-1)/2\} I + \{(a-1)/2\} L - \gamma_1$
(1) $\Rightarrow \{(a-1)/2\} I + \{(a-1)/2\} L - \gamma_1 - \gamma_1^2 = \{(a-1)/4\} L$
$\Rightarrow \gamma_1^2 = \{(a-1)/2\} I + \{(a-1)/2\} L - \gamma_1 - \{(a-1)/4\} L$
$= \{(a-1)/2\} I + \{(a-1)/4\} L - \gamma_1$
$= \{(a-1)/2\} I + \{(a-5)/4\} \gamma_1 + \{(a-1)/4\} \gamma_2$

Similarly, $\gamma_2^2 = \{(a-1)/2\} I + \{(a-5)/4\} \gamma_2 + \{(a-1)/4\} \gamma_1$
If $b = (a - 3)/4$ then $(a - 1)/4 = (2b + 1)/2$, $(a - 5)/4 = (2b - 1)/2$ and $(a - 1)/2 = (2b + 1)$
So $\gamma_1^2 = (2b+1) I + \{(2b - 1)/2\} \gamma_2 + \{(2b + 1)/2\} \gamma_1$ (2)
And $\gamma_2^2 = (2b+1) I + \{(2b - 1)/2\} \gamma_1 + \{(2b + 1)/2\} \gamma_2$ (3)

And $\gamma_1 \gamma_2 = \{(2b + 1)/2\} L$ (4)

Let $A_1 = \gamma_1 X \gamma_2 + \gamma_2 X \gamma_1$, $A_2 = \gamma_1 X \gamma_1 + \gamma_2 X \gamma_2$ and $A_3 = I X L + L X I$

Then $A_1^2 = \gamma_1^2 X \gamma_2^2 + \gamma_2^2 X \gamma_1^2 + 2(\gamma_1 \gamma_2 X \gamma_2 \gamma_1)$
$= (2b + 1)^2 I + (2b + 1) \{(2b-1)/2\} I X \gamma_1 + (2b + 1)\{(2b+1)/2\} I X \gamma_2 + (2b + 1)\{(2b+1)/2\} \gamma_1 X I$
$+ \{(2b+1)/2\} \{(2b-1)/2\} \gamma_1 X \gamma_1 + \{(2b + 1)/2\}^2 \gamma_1 X \gamma_2 + (2b + 1) \{(2b-1)/2\} \gamma_2 X I + \{(2b - 1)/2\}^2$
$\gamma_2 X \gamma_1 + \{(2b+1)/2\} \{(2b-1)/2\} \gamma_2 X \gamma_2 + (2b + 1)^2 I + (2b + 1) \{(2b + 1)/2\} I X \gamma_1 + (2b + 1)\{(2b$
$- 1)/2\} I X \gamma_2 + (2b + 1)\{(2b-1)/2\} \gamma_1 X I + \{(2b+1)/2\} \{(2b-1)/2\} \gamma_1 X \gamma_1 + \{(2b - 1)/2\}^2 \gamma_1 X \gamma_2 +$
$(2b + 1) \{(2b+1)/2\} \gamma_2 X I + \{(2b + 1)/2\}^2 \gamma_2 X \gamma_1 + \{(2b+1)/2\} \{(2b-1)/2\} \gamma_2 X \gamma_2 + 2 \{(2b+$
$1)/2\}^2 L X L$
$= (2b + 1)^2 I + (4 b^2 + 2b + 1) A_1 + (4 b^2 + 2b) A_2 + 2b (2b + 1) A_3$

And $A_2^2 = \gamma_1^2 X \gamma_1^2 + \gamma_2^2 X \gamma_2^2 + 2(\gamma_1 \gamma_2 X \gamma_2 \gamma_1)$
$= (2b + 1)^2 I + (2b + 1) \{(2b-1)/2\} I X \gamma_1 + (2b + 1) \{(2b+1)/2\} I X \gamma_2 + (2b + 1) \{(2b+1)/2\} \gamma_1 X I$
$+ \{(2b+1)/2\} \{(2b-1)/2\} \gamma_1 X \gamma_1 + \{(2b - 1)/2\}^2 \gamma_1 X \gamma_1 + (2b + 1) \{(2b+1)/2\} \gamma_2 X I + \{(2b + 1)/2\}^2 \gamma_2 X \gamma_2 + \{(2b+1)/2\} \{(2b-1)/2\} \gamma_2 X \gamma_1 + (2b + 1)^2 I + (2b + 1) \{(2b + 1)/2\} I X \gamma_1 + (2b + 1)\{(2b - 1)/2\} I X \gamma_2 + (2b + 1) \{(2b+1)/2\} \gamma_1 X I + \{(2b+1)/2\} \{(2b-1)/2\} \gamma_1 X \gamma_2 + \{(2b + 1)/2\}^2 \gamma_1 X \gamma_1 + (2b + 1) \{(2b-1)/2\} \gamma_2 X I + \{(2b + 1)/2\}^2 \gamma_1 X \gamma_1 + \{(2b+1)/2\} \{(2b-1)/2\} \gamma_1 X \gamma_2 + 2 \{(2b + 1)/2\}^2 L X L$
$= (2b + 1)^2 I + (4 b^2 + 2b) A_1 + (4 b^2 + 2b + 1) A_2 + 2b (2b + 1) A_3$

And $A_3^2 = I X L^2 + L^2 X I + 2(L X L)$
$= 2 (2b + 1) I + 2b I X \gamma_1 + 2b I X \gamma_2 + \{(2b + 1)/2\} I X L + 2 (2b + 1) I + 2b \gamma_1 X I + 2b \gamma_2 X I +$
$\{(2b + 1)/2\} L X I + 2 (A_1 + A_2)$
$= 4(2b + 1) I + \{(6b + 1)/2\} A_3 + 2(J - A_3 - I)$

And $A_1A_2 = \gamma_1^2 X \gamma_2 \gamma_1 + \gamma_1 \gamma_2 X \gamma_2^2 + \gamma_2^2 X \gamma_1 \gamma_2 + \gamma_2 \gamma_1 X \gamma_1^2$
$= (2b + 1) \{(2b+1)/2\} I X L + \{(4b^2 -1)/4\} \gamma_1 X L + \{(2b + 1)^2/4\} \gamma_2 X L + (2b + 1) \{(2b+1)/2\} L X$
$I + \{(4b^2 -1)/4\} L X \gamma_2 + \{(2b + 1)^2/4\} L X \gamma_1 + (2b + 1) \{(2b+1)/2\} L X$
$I + \{(4b^2 -1)/4\} L X \gamma_1 + \{(2b + 1)^2/4\} L X \gamma_2 + (2b + 1) \{(2b+1)/2\} I X L + \{(4b^2 -1)/4\} \gamma_2 X L +$
$\{(2b + 1)^2/4\} \gamma_1 X L$
$= 2(2b + 1)\{(2b+1)/2\} A_3 + 2(2 b^2 + b)(A_1 + A_2)$

And $A_1A_3 = \gamma_1 X L \gamma_2 + L\gamma_1 X \gamma_2 + \gamma_2 X L \gamma_1 + L\gamma_2 X \gamma_1$
$= (2b + 1) \gamma_1 X I + (2b + 1) \gamma_1 X L - \gamma_1 X \gamma_2 + (2b + 1) I X \gamma_1 + (2b + 1) L X \gamma_1 - \gamma_1 X \gamma_2 +(2b + 1) \gamma_2$
$X I + (2b + 1) \gamma_2 X L - \gamma_2 X \gamma_1 + (2b + 1)I X \gamma_2 + (2b + 1) L X \gamma_2 - \gamma_2 X \gamma_1$
$= 4b A_1 + (4b +2) A_2 + (2b + 1) A_3$

And $A_2A_3 = \gamma_1 X L \gamma_1 + L\gamma_1 X \gamma_1 + \gamma_2 X L \gamma_2 + L\gamma_2 X \gamma_2$
$= (2b + 1) \gamma_1 X I + (2b + 1) \gamma_1 X L - \gamma_1 X \gamma_1 + (2b + 1) I X \gamma_1 + (2b + 1) L X \gamma_1 - \gamma_1 X \gamma_1 + (2b + 1) \gamma_2$
$X I + (2b + 1) \gamma_2 X L - \gamma_2 X \gamma_2 + (2b + 1) I X \gamma_2 + (2b + 1) L X \gamma_2 - \gamma_2 X \gamma_2$
$= (4b + 2) A_1 + 4b A_2 + (2b + 1) A_3$

So A_1, A_2, A_3 define an amorphic 3- AS.

Properties of Association Matrices: We have find Eigen values of Adjacency matrices.

Eigen values of the Adjacency matrix A of a Strongly Regular Graph: If $A^2 = LI + \zeta A + \eta(J - A - I)$ where $L = p_{11}^0$, $\zeta = p_{11}^1$, $\eta = p_{11}^2$

The eigen values of A are L, l, m where l and m are roots of $x^2 + (\eta - \zeta) x + (\eta - l) = 0$, $l \geq 0$, $m \leq -1$

L, l, m have multiplicities 1, c, d

where $1 + c + d = v$ and $L + cl + dm = 0$

Eigen value of A_3

$A_3^2 = 4(2b + 1) I + (4b + 1) A_3 + 2(J - A_3 - I)$

So $L = 4(2b + 1)$, $\zeta = (4b + 1)$, $\eta = 2$

Then $x^2 + (\eta - \zeta) x + (\eta - l) = 0$
$\Rightarrow x^2 + (1 - 4b) x - 2 (1 + 4b) = 0$

So $x = \dfrac{-(1-4b) \pm \sqrt{16b^2 + 24b + 9}}{2}$
$= 4b + 1, -2$
$\Rightarrow l = 4b + 1$ and $m = -2$
Now, $1 + c + d = v \Rightarrow 1 + c + d = 4b + 3$
$L + cl + dm = 0 \Rightarrow (8b + 4) + (4b + 1) c - 2 d = 0$

Solving these we get, $c = 0$ and $d = 4b + 2$

Eigen Values	Multiplicities
L= 8b+ 4	1
l = 4b + 1	0
m = -2	4b + 2

Results and Discussion

Association schemes were discovered by R. C. Bose and his followers in 1920. We have tried to obtain all Association schemes and Coherent Configurations defined by minimum number of relations leading to the construction of an H-matrix of given form. We can obtain Association Schemes and Coherent Configurations from Williamson matrices. In this paper we have find the Eigen values of Association matrices with their multiplicities. Our work is motivated by the fact that the construction of H-matrices of Williamson form uses a family of Association schemes.

Conclusion

There are several methods of finding Association Schemes. Our method is different. Association schemes have application in Mathematics, in Information Technology and in Computer Science. The Association schemes thus obtained are used in the construction of Pairwise Balanced Designs which have applications in Design Theory.

References

1. Hall M. (1986). Combinatorial Theory. Wiley, New York 2nd edition.
2. Hedayat A. and Wallis W.D. (1978). Hadamard matrices and their applications. *Ann.Statist.,* **6,** 1184-1238
3. Horadam K.J. (2007). Hadamard Matrices and Their Applications. Princeton University Press.
4. Hanaki A. (2000). Skew-Symmetric Hadamard Matrices and Association Scheme. *SUT Journal of Mathematics*, 36(2), 251-258
5. Godsil D. and Song S.Y. (2007). Association Scheme in Handbook of Combinatorial Designs. ed. C.J.Colbourn and J.H. Dinitz, 2nd ed. Chapman and Hall, CRC, Boca Raton
6. Craigen R. and Kharghani H. (2007). Hadamard matrices and Hadamard designs in Handbook of Combinatorial designs. eds C.J. Colbourn and J.H. Dinitz, 2nd edition, Chapman and Hall/CRC, Boca Raton.
7. Turyn R.J. (1972). An infinite class of Williamson Matrices. *J.Combinatorial Theory Ser.A*, 12, 319-321.