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Paremeter Estimation and Prediction of the Exponential Life Time Model via Bayesian Approach

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Abstract

In this paper the Bayesian analysis for Exponential model assuming uninformative conjugate (Jeffrey's and Uniform) and informative conjugate (inverted gamma and two component inverted gamma mixture) priors is presented. The comparison between the two approaches is made on the basis of Bayes estimates, posterior risks, credible intervals and highest posterior density regions for different sample sizes and parameter values. The Bayesian predictive intervals assuming informative priors are calculated for different combinations of the hyperparameters to discover the range of hyper-parameters that lead towards more precise estimates of the parameters of interest.

Keywords: Exponential distribution, informative prior, conjugate prior, Bayesian estimation, hyper parameters.

Introduction

The Exponential distribution is often used to analyze the lifetime data of objects, specially, for electric components and it was the first lifetime model for which statistical methods were extensively developed. In this paper, we compare Bayes estimators based on uninformative and informative priors using the Posterior Variances, Credible Intervals and Highest Posterior Density Regions. The credible Intervals and HPDs for the parameter of Rayleigh distribution, exponential distribution and normal distribution have been discussed by researchers[1]. Bayesian analysis of the Rayleigh life time model which incorporates square root inverted gamma prior and mixture of two component square root inverted gamma prior[2]. Various other researchers have considered Bayesian analysis for Rayleigh model[3,4]. Some researchers have considered a mixture prior that combined with likelihood to give mixture posterior distribution and some considered neighborhood classes of mixture priors[5]. Robust Bayesian inference by using the two components of mixture priors have been discussed by researchers[6]. The trend of hyper-parameters is determined by calculating the 95% Bayesian predictive intervals. A 95% Predictive Intervals for various sets of values of the hyper-parameters using the sample of size 100 from mixture model was constructed and analyzed[7]. More details about Bayesian prediction interval for a Rayleigh distribution, evaluation of the Bayesian Predictive Intervals of the Rayleigh mixture assuming the Inverted Chi, the Inverted Rayleigh and the Square Root Inverted Gamma priors can be found in the literature[8,9]. A Bayesian analysis of power function mixture distribution has been done and some properties of Trivariate Pseudo Rayleigh distribution have also been defined[10].

Methodology

An informative prior expresses specific and definite information about the unknown parameter of interest. The inverted gamma (IG) and two component IG mixture are used as informative conjugate priors while uniform and Jeffreys are used as uninformative priors.

Bayes Estimates of Exponential Model using IG Prior: Consider exponential distribution with scale parameter θ and location parameter (assuming known)

$$f(x/\theta)=\frac{1}{\theta}e^{-\frac{x}{\theta}}, \qquad 0<x \ \ and \ \ \theta>0 \tag{1a}$$

It is assumed that θ follows IG distribution given below

$$p(\theta) = \frac{b^{a}}{\Gamma(a)}\theta^{-(a+1)}e^{-\frac{b}{\theta}}, \qquad \theta > 0 \tag{1}$$

The posterior distribution is obtained by incorporating prior with the likelihood as under

$$p(\theta\,|\,\mathbf{x})=\frac{(b+\Sigma x)^{a+n}}{\Gamma(a+n)}\theta^{-((a+n)+1)}e^{-\frac{b+\Sigma x}{\theta}}, \ \theta>0 \tag{2}$$

This is the density of the IG distribution where a and b are the hyper-parameters to be elicited.

$$\hat{\theta}=\frac{b+\Sigma x}{a+n-1} \tag{3}$$

Expression for its variance is as under.

$$Var_{(\theta|x)}(\theta)=\frac{(b+\Sigma x)^{2}}{(a+n-1)^{2}(a+n-2)} \tag{4}$$

The predictive intervals using the IG prior
The predictive distribution of the future observation y is

$$p(y|\mathbf{x}) = \int_0^\infty p(\theta|\mathbf{x})p(y|\theta)d\theta \tag{5}$$

where $p(y|\theta)$ is Exponential model under consideration and $p(y|\mathbf{x})$ is the posterior distribution. A $(1-\alpha)100\%$ Bayesian Predictive Interval (L,U) is obtained by solving the following two equations

$$\int_0^L p(y|\mathbf{x})dy = \frac{\alpha}{2}, \quad \int_U^\infty p(y|\mathbf{x})dy = \frac{\alpha}{2}$$

On simplification these equations become

$$\frac{\alpha}{2} = 1 - \left(\frac{b+\Sigma X}{b+\Sigma X+L}\right)^{a+n} \tag{6}$$

$$\frac{\alpha}{2} = \left(\frac{b+\Sigma X}{b+\Sigma X+U}\right)^{a+n} \tag{7}$$

The Credible Interval and the HPD Region assuming the IG Prior: From equation-2 it follows that

$\frac{2\left(b+\sum_{i=1}^n X_i\right)}{\theta} \sim \chi^2\left(2(a+n)\right)$. Therefore, the following is stated

$$1-\alpha = P\left[\chi^2\left(1-\frac{\alpha}{2},2(a+n)\right) < \frac{2\left(b+\sum_{i=1}^n X_i\right)}{\theta} < \left(\frac{\alpha}{2},2(a+n)\right)\right]$$

$$\left[C_L^{(\theta)}, C_U^{(\theta)}\right] = \left[\frac{2\left(b+\sum_{i=1}^n X_i\right)}{\chi^2\left(\frac{\alpha}{2},2(a+n)\right)}, \frac{2\left(b+\sum_{i=1}^n X_i\right)}{\chi^2\left(1-\frac{\alpha}{2},2(a+n)\right)}\right] \tag{8}$$

is the $(1-\alpha)100\%$ credible interval for θ.

The Highest Posterior Density (HPD) Region is obtained by solving the following two equations simultaneously.

$$1-\alpha = P(\theta_1 < \theta < \theta_2) = \int_{\theta_1}^{\theta_2} \frac{(b+\Sigma x)^{a+n}}{\Gamma(a+n)} \theta^{-\{(a+n)+1\}} e^{-\frac{(b+\Sigma x)}{\theta}} d\theta \tag{9}$$

$$\frac{(b+\sum x)^{a+n}}{\Gamma(a+n)} \theta_1^{-\{(a+n)+1\}} e^{-\frac{(b+\Sigma x)}{\theta_1}} = \frac{(b+\sum x)^{a+n}}{\Gamma(a+n)} \theta_2^{-\{(a+n)+1\}} e^{-\frac{(b+\Sigma x)}{\theta_2}} \tag{10}$$

Bayesian Estimation using Two Component IG mixture Prior: It is assumed that θ follows mixture of two components IG distribution

$$p(\theta) = P\frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{-(a_1+1)}e^{-\frac{b_1}{\theta}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\theta^{-(a_2+1)}e^{-\frac{b_2}{\theta}} \quad \theta > 0 \tag{11}$$

The posterior distribution is obtained as under.

$$P(\theta|\mathbf{x}) = \frac{1}{C}\left\{P\frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{-(a_1+n+1)}e^{-\frac{(b_1+\Sigma x_i)}{\theta}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\theta^{-(a_2+n+1)}e^{-\frac{(b_2+\Sigma x_i)}{\theta}}\right\} \quad \theta > 0$$

(12)

Where

$$c = P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n)}{(b_1+\sum X)^{a_1+n}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n)}{(b_2+\sum X)^{a_2+n}}$$

a_1, a_2 and b_1, b_2 are the hyper-parameters to be elicited. The expression for the Bayes estimator of θ by using the squared error loss function is as below.

$$E_{\theta|x}(\theta) = \frac{1}{C}\left\{P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n-1)}{(b_1+\Sigma X)^{a_1+n-1}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n-1)}{(b_2+\Sigma X)^{a_2+n-1}}\right\} \tag{13}$$

While expression for its variance is

$$Var_{\theta|x}(\theta) = \frac{1}{C}\left\{P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n-2)}{(b_1+\Sigma X)^{a_1+n-2}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n-2)}{(b_2+\Sigma X)^{a_2+n-2}}\right\} - \left[\frac{1}{C}\left\{P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n-1)}{(b_1+\sum X)^{a_1+n-1}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n-1)}{(b_2+\sum X)^{a_2+n-1}}\right\}\right]^2 \tag{14}$$

Predictive Intervals using the Two Component IG Mixture Prior: The predictive distribution of the future observation y is

$$p(y|\mathbf{x}) = \int_0^\infty p(\theta|\mathbf{x})\ p(y|\theta)d\theta \tag{15}$$

where $p(y|\theta) = \frac{1}{\theta}e^{-\frac{y}{\theta}}, \quad 0 < y < \infty, \theta > 0$

is the Exponential model.

$$p(y|\mathbf{x}) = \frac{1}{C}\left\{P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n+1)}{(b_1+\sum x_i+y)^{a_1+n+1}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n+1)}{(b_2+\sum x_i+y)^{a_2+n+1}}\right\}$$

$$y > 0 \tag{16}$$

is the posterior distribution where the constant of proportionality is as under.

$$C = P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n)}{(b_1+\sum X)^{a_1+n}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n)}{(b_2+\sum X)^{a_2+n}}.$$

A $(1-\alpha)100\%$ Bayesian Predictive Interval (L,U) is obtained by solving the following two equations, $\int_0^L p(y|\mathbf{x})dy = \frac{\alpha}{2}$ and $\int_U^\infty p(y|\mathbf{x})dy = \frac{\alpha}{2}$. On simplification these equations yield

$$\frac{\alpha}{2} = -\frac{1}{C}\left[\begin{array}{l} P\frac{b_1^{a_1}}{\Gamma(a_1)}\Gamma(a_1+n)\left\{\frac{1}{(b_1+\sum x_i+L)^{a_1+n}} - \frac{1}{(b_1+\sum x_i)^{a_1+n}}\right\} \\ +(1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\Gamma(a_2+n)\left\{\frac{1}{(b_2+\sum x_i+L)^{a_2+n}} - \frac{1}{(b_2+\sum x_i)^{a_2+n}}\right\} \end{array}\right] \quad (17)$$

$$\frac{\alpha}{2} = \frac{1}{C}\left[P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n)}{(b_1+\sum x_i+U)^{a_1+n}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n)}{(b_2+\sum x_i+U)^{a_2+n}}\right] \quad (18)$$

Credible Interval using the Two Components IG mixture Prior: From equation (12) it follows that

$$\frac{2\left(b_1+\sum_{i=1}^n X_i\right)}{\theta} \sim \chi^2\left(2(a_1+n)\right)$$ and

$$\frac{2\left(b_2+\sum_{i=1}^n X_i\right)}{\theta} \sim \chi^2\left(2(a_2+n)\right).$$ Hence

$$\text{P}\,\chi^2\left(2(a_1+n)\right) + (1-P)\chi^2\left(2(a_2+n)\right) \sim \overset{*}{\chi^2}$$ and

$$1-\alpha = P\left[\overset{*}{\chi_1^2} < \frac{1}{\theta} < \overset{*}{\chi_2^2}\right] = P\left[\frac{1}{\overset{*}{\chi_2^2}} < \theta < \frac{1}{\overset{*}{\chi_1^2}}\right],$$

Thus

$$\left[C_L^{(\theta)}, C_U^{(\theta)}\right] = P\left[\frac{1}{\overset{*}{\chi^2}_{\alpha/2}} < \theta < \frac{1}{\overset{*}{\chi^2}_{1-\alpha/2}}\right] \quad (19)$$

is $(1-\alpha)100\%-$credible interval for θ where

$$\overset{*}{\chi^2}_{\alpha/2} = \frac{1}{2C}\left[\begin{array}{l} P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n)}{(b_1+\sum x)^{a_1+n+1}}\chi^2_{2(a_1+n),\alpha/2} + (1-P) \\ \frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n)}{(b_2+\sum x)^{a_2+n+1}}\chi^2_{2(a_2+n),\alpha/2} \end{array}\right]$$

$$\overset{*}{\chi^2}_{1-\alpha/2} = \frac{1}{2C}\left[\begin{array}{l} P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n)}{(b_1+\sum x)^{a_1+n+1}}\chi^2_{2(a_1+n),1-\alpha/2} + \\ (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n)}{(b_2+\sum x)^{a_2+n+1}}\chi^2_{2(a_2+n),1-\alpha/2} \end{array}\right]$$

and

$$c = P\frac{b_1^{a_1}}{\Gamma(a_1)}\frac{\Gamma(a_1+n)}{(b_1+\sum X)^{a_1+n}} + (1-P)\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{\Gamma(a_2+n)}{(b_2+\sum X)^{a_2+n}}$$

Bayesian Estimation of Exponential Model using uninformative priors: The uninformative priors are recommended when no formal prior information is available. The most commonly used uninformative priors are uniform and Jeffrey's prior.

Posterior distribution, Bayes estimator and variance assuming uniform prior: The uniform prior for the unknown parameter of Exponential model is $p(\theta) \propto 1,\ \theta > 0$. Although the said prior is improper but when incorporated with the likelihood, the proper pdf of posterior distribution is obtained. The expressions for the posterior distribution, Bayes estimator and its variance assuming uniform prior can be obtained from equations-2, 3 and 4 respectively setting hyper-parameters as $a=-1,\ b=0$.

The predictive intervals assuming the uniform prior: The predictive interval assuming uniform prior can be obtained solving equations (6) and (7) setting $a=-1,\ b=0$.

The credible intervals and HPD region assuming the uniform prior: The credible interval is given by equation-8 and the equations to find HPD region are equations-9 and 10 provided that hyper-parameters are set as $a=-1,\ b=0$.

Posterior distribution, Bayes estimator and variance assuming Jeffrey's prior: The Jeffrey's prior for the exponential model is $p(\theta) \propto \theta^{-1},\ \theta > 0$. The expressions for the posterior distribution, Bayes estimator and its variance assuming Jeffrey's prior can be obtained from equations-2, 3 and 4 respectively setting hyper-parameters as $a=0=b$.

The predictive intervals assuming the Jeffrey's prior: The predictive intervals can be had from equations-6 and 7 with $a=0=b$.

The credible intervals HPD region assuming the Jeffrey's prior: The credible interval is given by equation-8 and the equations to find HPD region are equations-9 and 10 provided that the hyper-parameters are set as $a=0=b$.

Results and Discussion

In order to compare estimates based on the two approaches, an empirical study is conducted. As one data set is usually unable to clarify performance of the method, random samples of size $n=50$, $n=100$ and $n=150$ are taken from the Exponential distribution with parameter values, $\theta=0.6$, $\theta=1.8$ and $\theta=3.2$.

Bayesian Predictive Intervals using the Inverted Gamma Prior: Bayesian Predictive Interval using the IG Prior are evaluated using equation-6 and 7 for different combinations of the hyper-parameters, a and b, and the results are arranged in table-1. The lower and upper limits of the predictive intervals are observed as a function of various combinations of a and b. It is interesting to note that higher values of a and lower values of b lead towards narrowest Bayesian predictive intervals. The maximum precision is obtained when a is largest and b is the least.

Bayesian Predictive Intervals using the Two Components IG Mixture Prior: Bayesian Predictive Interval using the two components IG mixture prior are evaluated using equation-17 and 18 for different combinations of the hyper-parameters are arranged in table-2. It is observed that the lower values of b_1 and b_2 and higher values of a_1 and a_2 produce narrow predictive intervals.

Comparison of Uninformative and Informative Priors: Bayes estimates, posterior variances, credible intervals and highest posterior density (HPD) region are obtained for uninformative and informative priors and are presented in Table (3- 5). Based on the results and findings from table-1 and table-2, we choose the hyper-parameters as $a=25$ and $b=1$ in case of Inverted gamma prior, and $a_1=10, a_2=25, b_1=1, b_2=10$ in case of two components IG prior.

Comparison of Bayes Estimates: Bayes estimates are evaluated for uninformative (Jeffrey's and Uniform) and informative (IG and two components IG mixture) priors, for different sample sizes and different parameter values. These estimates are summarized in table-3. These estimates tend towards the true value of parameter as the sample size increases. Jeffrey's prior estimates are observed to be slightly more accurate than the uniform prior estimates.

Comparison of Variances: Posterior variances for different sample and parameter sizes are summarized in table-4. As the sample size increases, the variances tend to decrease. Variances of informative Bayes estimates are smaller than that of uninformative counterparts. It is observed that the variance of the mixture prior Bayes estimates is larger as compared to that of IG (or Informative prior) prior counterpart.

Comparison of Credible Intervals and HPD Regions: The 95% credible intervals for θ assuming uninformative and informative priors for different sample sizes and parameter values are presented in table-5. It is evident that with increasing sample size, the interval length becomes narrower. The results show that the informative prior gives narrower intervals than the uninformative priors. A 95% HPD for θ assuming the uninformative and informative priors for different sample sizes and parameter values are given in table-6. The HPD regions become narrower as increases the sample size. In case of informative prior, HPD regions have narrower spread than the uninformative prior case.

Conclusion

In this paper we presented Bayesian analysis for Exponential model assuming uninformative conjugate (Jeffrey's and Uniform) and informative conjugate (inverted gamma and two component inverted gamma mixture). Bayes estimates, posterior risks, credible intervals and highest posterior density regions for different sample sizes and parameter values, based on two approaches, are compared.

We observed that Bayesian Predictive Intervals using the IG Prior are narrower for higher values of shape hyper-parameter while lower values of scale hyper-parameter. Bayesian Predictive Intervals using the two components IG mixture prior have minimum spread for the lower values of scale hyper-parameters and higher values of shape hyper-parameters. Shape hyper-parameters bring about rapid changes in size of Bayesian Predictive Intervals especially the upper limits. Informative Bayes estimates are found to be more precise than uninformative counterparts. The informative prior give narrower credible intervals and HPD regions than the uninformative prior.

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Table-1
Bayesian Predictive Intervals for different values of the hyper parameters (n=250)

Hyper parameters	a=1	a=5	a=10	a=15	a=20	**a = 25**
b=1	L=0.07631 U=11.2001 δ=11.1238	L=0.07511 U=11.0232 δ=10.9480	L=0.0736 U=10.8097 δ=10.7360	L=0.07227 U=10.6043 δ=10.5320	L=0.07094 U=10.4066 δ=10.3356	L=0.06965 U=10.2161 δ=10.1464
b=5	L=0.07671 U=11.2593 δ=11.1825	L=0.07551 U=11.0815 δ=11.0059	L=0.074058 U=10.86687 δ=10.79281	L=0.072660 U=10.66041 δ=10.58774	L=0.07131 U=10.4616 δ=10.3903	L=0.07001 U=10.2701 δ=10.2001
b=10	L=0.0772 U=11.333 δ=11.2561	L=0.07600 U=11.1543 δ=11.0782	L=0.07454 U=10.9383 δ=10.8617	L=0.07313 U=10.7305 δ=10.6573	L=0.0717 U=10.530 δ=10.4586	L=0.0704 U=10.337 δ=10.2671
b=20	L=0.0782 U=11.481 δ=11.402	L=0.0769 U=11.300 δ=11.2230	L=0.07551 U=11.0812 δ=11.0056	L=0.07409 U=10.8706 δ=10.7965	L=0.07272 U=10.6679 δ=10.5952	L=0.07139 U=10.4727 δ=10.4013
b=25	L=0.07873 U=11.5555 δ=11.4767	L=0.07749 U=11.3729 δ=11.2954	L=0.07600 U=11.1526 δ=11.0765	L=0.07457 U=10.9407 δ=10.8661	L=0.07319 U=10.7367 δ=10.6635	L=0.07185 U=10.5402 δ=10.4683

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Table-2
Bayesian Predictive Intervals for different values of the hyper parameters (n=250)

Hyper paramter	$a_1=5$ $a_2=5$	$a_1=10$ $a_2=10$	$a_1=15$ $a_2=15$	$a_1=20$ $a_2=20$	$a_1=25$ $a_2=25$
$b_1=1$ $b_2=1$	L=0.075113 U=11.02321 δ=10.94809	L=0.073668 U=10.80972 δ=10.736052	L=0.07227 U=10.6043 δ=10.53203	L=0.07094 U=10.4066 δ=10.33566	L=0.06965 U=10.2161 δ=10.14645
$b_1=5$ $b_2=5$	L=0.07551 U=11.0815 δ=11.0059	L=0.07405 U=10.8669 δ=10.79285	L=0.072663 U=10.6609 δ=10.58823	L=0.071315 U=10.4616 δ=10.39028	L=0.070018 U=10.2701 δ=10.20008
$b_1=10$ $b_2=10$	L=0.076006 U=11.1543 δ=11.078294	L=0.0745450 U=10.9383 δ=10.863755	L=0.073138 U=10.730503 δ=10.657365	L=0.071786 U=10.5307 δ=10.458914	L=0.070478 U=10.3377 δ=10.26722
$b_1=15$ $b_2=15$	L=0.0765032 U=11.2272 δ=11.1506968	L=0.075032 U=11.0097 δ=10.934668	L=0.073616 U=10.8006 δ=10.726984	L=0.072252 U=10.5992 δ=10.526948	L=0.07093 U=10.4063 δ=10.3353
$b_1=20$ $b_2=20$	L=0.076999 U=11.30008 δ=11.22308	L=0.075518 U=11.0812 δ=11.00568	L=0.074093 U=10.8707 δ=10.79660	L=0.072721 U=10.66799 δ=10.59527	L=0.071399 U=10.4728 δ=10.40140

Table-3
Comparison of Bayes Estimates

N	**P**	**θ**	**Bayes Estimates**			
			Jeffreys Prior	**Uniform Prior**	**Informative Prior**	**Mixture Prior**
50	0.05	1.8	1.83 (0.07202)	1.8704 (0.076086)	1.23 (0.0211)	1.3541 (0.023881)
50	0.2	0.8	0.8207 (0.01425)	0.8352 (0.015063)	0.56 (0.0043)	0.6777095 (0.006307)
100	0.05	1.8	1.8128 (0.033980)	1.848 (0.034967)	1.46 (0.01748)	1.55318 (0.0197)
100	0.1	0.8	0.808 (0.00670)	0.8134 (0.006988)	0.65 (0.00348)	0.68381 (0.003802)
250	0.1	1.8	1.8082 (0.01328018)	1.81458 (0.013358)	1.65 (0.00996)	0.707162 (0.001832)
250	0.3	0.8	0.801 (0.002619)	0.8077 (0.0026222)	0.73 (0.001991)	2.91076 (0.033894)

Table-4
Comparison of 95% Credible Intervals for Uninformative and Informative Priors

No.	n	p	Θ	Credible Intervals			
				Jeffreys	Uniform	Informative Prior	Mixture Prior
1	50	0.1	1.8	(1.35, 2.36)	(1.38, 2.41)	(0.96, 1.52)	(1.05, 1.65)
2	50	0.2	0.8	(0.62, 1.08)	(0.63, 1.11)	(0.45, 0.71)	(0.54, 0.85)
3	100	0.05	1.8	(1.51, 2.24)	(1.53, 2.24)	(1.25, 1.78)	(1.30, 1.85)
4	100	0.1	0.6	(0.62, 0.919)	(0.63, 0.93)	(0.52, 0.73)	(0.57, 0.81)
5	150	0.1	0.8	(0.71, 0.98)	(0.71, 0.98)	(0.62, 0.84)	(0.66, 0.89)
6	150	0.3	3.2	(2.92, 4.02)	(2.93, 4.05)	(2.55, 3.43)	(2.75, 3.74)
7	250	0.1	0.8	(0.652, 0.84)	(0.654, 0.84)	(0.60, 0.76)	(0.63, 0.796)
8	250	0.3	3	(2.68, 3.43)	(2.69, 3.45)	(2.46, 3.12)	(2.58, 3.29)

Table-5
Comparison of 95% HPD with Uninformative and Informative Priors

N	Θ	Highest Posterior Density (HPD) Regions		
		Jeffreys Prior	Uniform Prior	Informative Prior
50	1.8	(1.317, 2.292)	(1.336, 2.3559)	(0.9436, 1.4985)
50	0.8	(0.604, 1.056)	(0.6116, 1.0804)	(0.4402, 0.6981)
100	1.8	(1.492, 2.219)	(1.5074, 2.2395)	(1.2365, 1.7619)
100	0.6	(0.613, 0.906)	(0.617, 0.9186)	(0.5105, 0.7277)
150	0.8	(0.699, 0.974)	(0.707, 0.976)	(0.6177, 0.8321)
150	3.2	(2.90, 3.97)	(2.9072, 4.0147)	(2.5258, 3.4039)
250	0.8	(0.632, 0.790)	(0.6475, 0.8391)	(0.5996, 0.7562)
250	3	(2.669, 3.293)	(2.675, 3.432)	(2.448, 3.106)