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Characterization of the Density Matrix in Terms of Statistical Tensor Parameters

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Abstract

It was shown in literatures that every density matrix for a spin-j system can be defined in terms of the Fano statistical tensor parameters. On the other hand, this spherical tensor parameters can be uniquely determined by units vectors. In this study, we construct the density matrix for the most general spin-1 system in terms of unit vectors. We also give some cases to show symmetric properties of spherical tensor parameters.

Keywords: The Density Matrix, Spherical Tensor Parameter, Fano Representation.

Introduction

The concept of density matrix, ρ, first introduced independently by Von Neumann[1], Landau[2] and Dirac[3] in eminently suited to describe polarized spin assemblies. With having$N$ two level atoms, each atom can be represented as a spin-1/2 system and theoretical analysis can be donein terms of collective spin operator $\vec{J} = \frac{1}{2}\sum_{\alpha=1}^{N} \vec{\sigma}_\alpha$. Here $\vec{\sigma}_\alpha$ denote the Pauli spin operator of the αth qubit. For a system characterizing by a state $|\psi\rangle$ in the Hilbert space, $\rho = |\psi\rangle\langle\psi|$ is the density matrix or density operator associated with the quantum state $|\psi\rangle$. Since $\rho = \rho^\dagger$ and $Tr\rho = 1$ number of real parameters needed to specify a density matrix is $N^2 - 1$ where $N = 2j + 1$ is the dimension of the Hilbert space.

It is very well known that ρ for a system of spin-1/2 particles has the form

$$\rho = \frac{Tr\rho}{2}\left[1 + \vec{\sigma}.\vec{P}\right] \quad (1)$$

In terms of the Pauli spin matrices $\vec{\sigma}$and the polarization vector $\vec{P}$. Clearly, ρ assumes a diagonal from if the z −axis is selected parallel to $\vec{P}$and $\frac{1+|\vec{P}|}{2}$, $\frac{1-|\vec{P}|}{2}$ correspond to the number of spin-up and spin-down particles in the assembly. This concept extended to a spin-j system and probabilities $p(m)$ are assigned to the $(2j + 1)$ states $|jm\rangle; m = +j \dots -j$, there by leading to the notion of oriented systems[4].

A standard form for the density matrix ρ for any arbitrary spin-j has been obtained by Fano[5,6], wherein the state of polarization is completely described in terms of what are known as Fano statistical tensors t_q^k where $k = 0,1,2 \dots 2j$ and $q = -k \dots +k$.

Fano Representation of Density Matrix

The systematic use of tensor operators was first suggested by Fano[6]. In this representation it is well − known[5,6,7,8] that the density matrix ρ of a spin-j assembly can be stated in the form

$$\rho = \frac{Tr\rho}{N}\sum_{k=0}^{2j}\sum_{q=-k}^{+k} t_q^k\, \tau_q^{k\dagger}(\vec{J}) \quad (2)$$

Where $N = 2j + 1$ is the dimension of the Hilbert space. The complex spherical tensor parameters t_q^k denote the average expectation values

$$t_q^k = \langle \tau_q^k(\vec{J}) \rangle = \frac{Tr(\rho)\, \tau_q^k(\vec{J})}{Tr\rho} \quad (3)$$

Of the irreducible (spherical) tensor operators $\tau_q^k(\vec{J})$[9].
The $\tau_q^{k'}s$ satisfy the orthogonally relations

$$Tr\left(\tau_q^{k\dagger}\, \tau_{q'}^{k'}\right) = (2j + 1)\delta_{kk'}\delta_{qq'} \quad (4)$$

Here the normalization are in agreement with Madison convention[10]. Also, we have

$$\tau_q^{k\dagger}(\vec{J}) = (-1)^q\, \tau_{-q}^k(\vec{J}) \quad (5)$$

And hermit city of the density matrix together with equation (5) demands

$$t_q^{k^*} = (-1)^q t_{-q}^k. \quad (6)$$

The matrix element of the operator $\tau_q^k(\vec{J})$ take the form

$$\langle jm'|\, \tau_q^k(\vec{J})|jm\rangle = [k]C(jkj; mqm') \quad (7)$$

Where $[k] = \sqrt{2k+1}$ and $C(jkj; mqm')$ are the well known Clebsch-Gordan Coefficients. Using equation (7), the matrix elements of ρ can be written as,

$$\rho_{m'm} = \frac{Tr\rho}{N}\sum_{k,q}[k]C(jkj; m'qm)\, t_q^k \quad (8)$$

And conversely t_q^k can be expressed in terms of $\rho_{m'm}$ using equation (3) as,

$$t_q^k = \sum_{m=-j}^{+j} [k] C(jkj; m'qm)\, \rho_{m'm} \quad (9)$$

It is well-known that corresponding to a given set of unit vectors $\hat{Q}_i$; $i = 1,2 \dots k$, a spherical tensor $s_q^k(\hat{Q}_1 \dots \hat{Q}_k)$ of rank k can be associated through

$$s_q^k(\hat{Q}_1, \hat{Q}_2 \dots \hat{Q}_k) = (\dots((\hat{Q}_1 \otimes \hat{Q}_2) \otimes \hat{Q}_3) \otimes \hat{Q}_k)_q^k \quad (10)$$

Which is unique and completely symmetric in the k indices. Therefore, it was shown[11] that

$$t_q^k = P_k \left(\dots \left(\left(\hat{Q}(\theta_1, \varphi_1) \otimes \hat{Q}(\theta_2, \varphi_2) \right)^2 \otimes \hat{Q}(\theta_3, \varphi_3) \right)^3 \otimes \dots \right)^{k-1} \otimes \hat{Q}(\theta_k, \varphi_k)_q^k \quad (11)$$

Where:

$$\left(\hat{Q}(\theta_1, \varphi_1) \otimes \hat{Q}(\theta_2, \varphi_2)\right)_q^2 = \sum_{q_1} C(11k; q_1 q_2 q) \left(\hat{Q}(\theta_1, \varphi_1)\right)_{q_1}^1 \left(\hat{Q}(\theta_2, \varphi_2)\right)_{q_2}^2 \quad (12)$$

and P_k is real in order to ensure about equation (6). Also, the spherical components of $\hat{Q}$ are given by,

$$\left(\hat{Q}(\theta, \varphi)\right)_q^1 = \sqrt{\frac{4\pi}{3}}\, Y_q^1(\theta, \varphi). \quad (13)$$

Here $Y_q^1(\theta, \varphi)$ are the well-known spherical harmonics.

Note that t_q^k; $q = -k \dots +k$, satisfying equation (6) contains $(2k+1)$ real parameters and the new parameters $\hat{Q}(\theta_i, \varphi_i)$; $i = 1,2 \dots k$, together with P_k constitute exactly the same number of parameters.

Standard Expression of Density Matrix for Spin-1 System

Let us now consider spin-1 state. In this case $j = 1$ and hence according to equation (2) we can construct the density matrix in terms of spherical tensor parameters. Since in the case of spin-1, $k = 0,1,2$ and $q = -2, -1 \dots 1,2$ we get

$$\rho = \frac{1}{3}\left[t_0^0\, \tau_0^{0\dagger} + t_0^1\, \tau_0^{1\dagger} + t_1^1\, \tau_1^{1\dagger} + t_{-1}^1\, \tau_{-1}^{1\dagger} + t_0^2\, \tau_0^{2\dagger} + t_1^2\, \tau_1^{2\dagger} + t_{-1}^2\, \tau_{-1}^{2\dagger} + t_2^2\, \tau_2^{2\dagger} + t_{-2}^2\, \tau_{-2}^{2\dagger} \right] \quad (14)$$

For $j = 1$, matrix representation of irreducible tensor operators τ_q^k's, according to equation (7), are,

$$\tau_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_0^1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\tau_1^1 = -\sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tau_{-1}^1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_0^2 = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau_1^2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tau_{-1}^2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \tau_2^2 = \begin{pmatrix} 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tau_{-2}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{3}0 & 0 & 0 \end{pmatrix}.$$

Therefore, Matrix form of the density matrix can be establish with knowing about matrix elements of $\tau_q^{k'}s$,

$$\rho = \frac{1}{3} \begin{pmatrix} 1 + \sqrt{\frac{3}{2}}\, t_0^1 + \frac{t_0^2}{\sqrt{2}} & \sqrt{\frac{3}{2}}\, (t_{-1}^1 + t_{-1}^2) & \sqrt{3}\, t_{-2}^2 \\ -\sqrt{\frac{3}{2}}\, (t_1^1 + t_1^2) & 1 - \sqrt{2}\, t_0^2 & \sqrt{\frac{3}{2}}\, (t_{-1}^1 - t_{-1}^2) \\ \sqrt{3}\, t_2^2 & -\sqrt{\frac{3}{2}}\, (t_1^1 - t_1^2) & 1 - \sqrt{\frac{3}{2}}\, t_0^1 + \frac{t_0^2}{\sqrt{2}} \end{pmatrix} \quad (15)$$

According to equation (11), we can formulate $t_q^{k'}s$ in the case of spin-1 state as follows:

$$t_0^1 = P_1 Q_0^1(\theta_1, \varphi_1) = P_1 \mathrm{Cos}\theta_1 \quad (16)$$

$$t_{\pm 1}^1 = P_1 Q_{\pm 1}^1(\theta_1, \varphi_1) = \frac{\mp P_1}{\sqrt{2}} \mathrm{Sin}\theta_1 e^{\pm i\varphi_1} \quad (17)$$

$$t_0^2 = P_2\left[Q(\theta_2, \varphi_2) \otimes Q(\theta_3, \varphi_3)\right]_0^2 = P_2\left[C(112; 000) Q_0^1(\theta_2, \varphi_2) Q_0^1(\theta_3, \varphi_3) + C(112; 1-10) Q_1^1(\theta_2, \varphi_2) Q_{-1}^1(\theta_3, \varphi_3) + C(112; -110) Q_{-1}^1(\theta_2, \varphi_2) Q_1^1(\theta_3, \varphi_3)\right] = P_2\left[\sqrt{\frac{2}{3}} \mathrm{Cos}\theta_2 \mathrm{Cos}\theta_3 - \frac{1}{2\sqrt{6}}\left(\mathrm{Sin}\theta_2 \mathrm{Sin}\theta_3 e^{i\varphi_2} e^{-i\varphi_3} + \mathrm{Sin}\theta_2 \mathrm{Sin}\theta_3 e^{-i\varphi_2} e^{i\varphi_3}\right)\right] \quad (18)$$

$$t_1^2 = P_2\left[Q(\theta_2, \varphi_2) \otimes Q(\theta_3, \varphi_3)\right]_1^2 = P_2\left[C(112; 101) Q_1^1(\theta_2, \varphi_2) Q_0^1(\theta_3, \varphi_3) + C(112; 011) Q_0^1(\theta_2, \varphi_2) Q_1^1(\theta_3, \varphi_3)\right] = \frac{-P_2}{2}\left[\mathrm{Sin}\theta_2 \mathrm{Cos}\theta_3 e^{i\varphi_2} + \mathrm{Sin}\theta_3 \mathrm{Cos}\theta_2 e^{i\varphi_3}\right] \quad (19)$$

$$t_{-1}^2 = P_2\left[Q(\theta_2, \varphi_2) \otimes Q(\theta_3, \varphi_3)\right]_{-1}^2 = P_2\left[C(112; -101) Q_{-1}^1(\theta_2, \varphi_2) Q_0^1(\theta_3, \varphi_3) + C(112; 011) Q_0^1(\theta_2, \varphi_2) Q_{-1}^1(\theta_3, \varphi_3)\right] = \frac{P_2}{2}\left[\mathrm{Sin}\theta_2 \mathrm{Cos}\theta_3 e^{-i\varphi_2} + \mathrm{Sin}\theta_3 \mathrm{Cos}\theta_2 e^{-i\varphi_3}\right] \quad (20)$$

$$t_2^2 = P_2\left[Q(\theta_2, \varphi_2) \otimes Q(\theta_3, \varphi_3)\right]_2^2 = P_2\left[C(112; 112) Q_1^1(\theta_2, \varphi_2) Q_1^1(\theta_3, \varphi_3)\right] = \frac{P_2}{2}\left[\mathrm{Sin}\theta_2 \mathrm{Sin}\theta_3 e^{i\varphi_2} e^{i\varphi_3}\right] \quad (21)$$

$t^2_{-2} = P_2[Q(\theta_2,\varphi_2)\otimes Q(\theta_3,\varphi_3)]^2_{-2} = P_2[C(112;-1-1-2)Q^1_{-1}(\theta_2,\varphi_2)Q^1_{-1}(\theta_3,\varphi_3)] = \frac{P_2}{2}[\mathrm{Sin}\theta_2\mathrm{Sin}\theta_3 e^{-i\varphi_2}e^{-i\varphi_3}]$ (22)

Thus, the matrix elements of the density matrix $\rho_{m'm}$ (equation (8)) in terms of P_1, P_2 and (θ_i,φ_i) in the $|1m>$basis will be,

$$\rho_{11} = \frac{1}{3}\left[1+\sqrt{\frac{3}{2}}P_1\mathrm{Cos}\theta_1+\frac{P_2}{\sqrt{3}}\mathrm{Cos}\theta_2\mathrm{Cos}\theta_3-\frac{P_2}{4\sqrt{3}}\mathrm{Sin}\theta_2\mathrm{Sin}\theta_3\left(e^{i\varphi_2}e^{-i\varphi_3}+e^{-i\varphi_2}e^{i\varphi_3}\right)\right] \quad (23)$$

$$\rho_{10} = \frac{1}{3}\left[\sqrt{\frac{3}{2}}\left(\frac{P_1}{\sqrt{2}}\mathrm{Sin}\theta_1 e^{-i\varphi_1}+\frac{P_2}{2}\left[\mathrm{Sin}\theta_2\mathrm{Cos}\theta_3 e^{-i\varphi_2}+\mathrm{Sin}\theta_3\mathrm{Cos}\theta_2 e^{-i\varphi_3}\right]\right)\right] \quad (24)$$

$$\rho_{1-1} = \frac{1}{3}\left[\frac{\sqrt{3}\,P_2}{2}\left(\mathrm{Sin}\theta_2\mathrm{Sin}\theta_3 e^{-i\varphi_2}e^{-i\varphi_3}\right)\right] \quad (25)$$

$$\rho_{01} = \frac{1}{3}\left[\sqrt{\frac{3}{2}}\left(\frac{P_1}{\sqrt{2}}\mathrm{Sin}\theta_1 e^{i\varphi_1}+\frac{P_2}{2}\left[\mathrm{Sin}\theta_2\mathrm{Cos}\theta_3 e^{i\varphi_2}+\mathrm{Sin}\theta_3\mathrm{Cos}\theta_2 e^{i\varphi_3}\right]\right)\right] \quad (26)$$

$$\rho_{00} = \frac{1}{3}\left[1-P_2\left[\frac{2}{\sqrt{3}}\mathrm{Cos}\theta_2\mathrm{Cos}\theta_3-\frac{1}{2\sqrt{3}}\left(\mathrm{Sin}\theta_2\mathrm{Sin}\theta_3 e^{i\varphi_2}e^{-i\varphi_3}+\mathrm{Sin}\theta_2\mathrm{Sin}\theta_3 e^{-i\varphi_2}e^{i\varphi_3}\right)\right]\right] \quad (27)$$

$$\rho_{0-1} = \frac{1}{3}\left[\sqrt{\frac{3}{2}}\left(\frac{P_1}{\sqrt{2}}\mathrm{Sin}\theta_1 e^{-i\varphi_1}-\frac{P_2}{2}\left[\mathrm{Sin}\theta_2\mathrm{Cos}\theta_3 e^{-i\varphi_2}+\mathrm{Sin}\theta_3\mathrm{Cos}\theta_2 e^{-i\varphi_3}\right]\right)\right] \quad (28)$$

$$\rho_{-11} = \frac{1}{3}\left[\frac{\sqrt{3}P_2}{2}\left(\mathrm{Sin}\theta_2\mathrm{Sin}\theta_3 e^{i\varphi_2}e^{i\varphi_3}\right)\right] \quad (29)$$

$$\rho_{-10} = \frac{1}{3}\left[\left(\frac{P_1}{\sqrt{2}}\mathrm{Sin}\theta_1 e^{i\varphi_1}-\frac{P_2}{2}\left[\mathrm{Sin}\theta_2\mathrm{Cos}\theta_3 e^{i\varphi_2}+\mathrm{Sin}\theta_3\mathrm{Cos}\theta_2 e^{i\varphi_3}\right]\right)\right] \quad (30)$$

$$\rho_{-1-1} = \frac{1}{3}\left[1-+\sqrt{\frac{3}{2}}P_1\mathrm{Cos}\theta_1+\frac{P_2}{\sqrt{3}}\mathrm{Cos}\theta_2\mathrm{Cos}\theta_3-\frac{P_2}{4\sqrt{3}}\mathrm{Sin}\theta_2\mathrm{Sin}\theta_3\left(e^{i\varphi_2}e^{-i\varphi_3}+e^{-i\varphi_2}e^{i\varphi_3}\right)\right] \quad (31)$$

In the figures (1) and (2) we plot spherical tensor parameters$Re(t^1_1)$ and$Re(t^1_{-1})$ in terms of angles θ and φ, respectively. Here we choose P_1 to be $\frac{1}{\sqrt{2}}$.

Symmetric Properties of $t^{k'}_q s$: As we mentioned in previous sections, spherical tensor parameters are symmetrized product of unit vectors $\hat{Q}(\theta_i,\varphi_i); i = 1,2 \ldots k$. Let us now consider some examples to show symmetric properties of $t^{k'}_q s$.

A: $K = 2$ and $q = 1$

$$t^2_1 = \frac{-P_2}{2}[Sin\theta_2 e^{i\varphi_2}Cos\theta_3 + Sin\theta_3 e^{i\varphi_3}Cos\theta_2]$$
$$= \frac{-P_2}{2}[z_3(x_2+iy_2)+z_2(x_3+iy_3)] \quad (32)$$

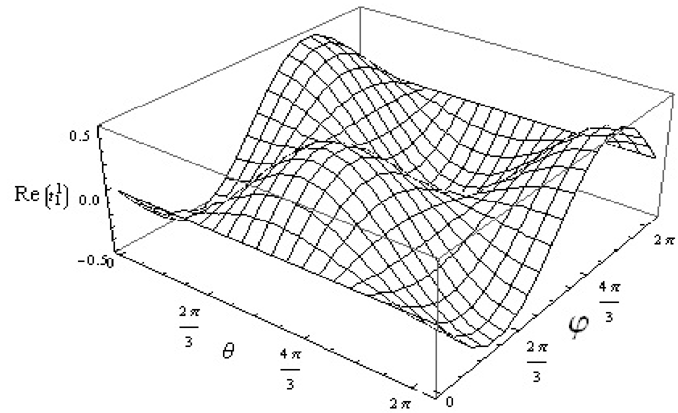

Figure-1

$Re(t^1_1) = \frac{-1}{2}$ Sinθ Cosφin terms of $0 \le \theta \le 2\pi$and$0 \le \varphi \le 2\pi$.

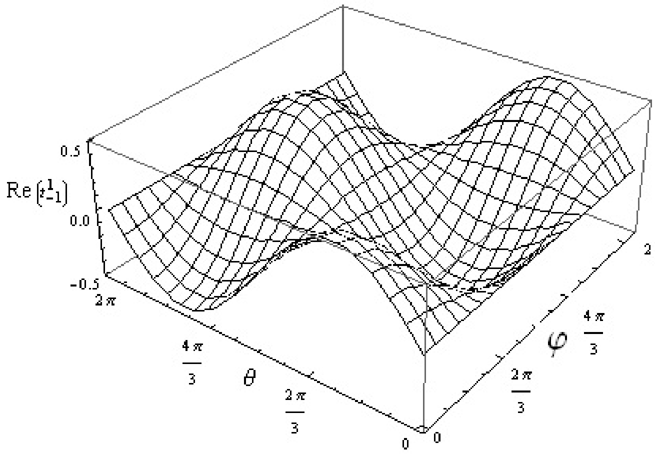

Figure-2

$Re(t^1_{-1}) = \frac{1}{2}$ Sinθ Cosφin terms of $0 \le \theta \le 2\pi$and$0 \le \varphi \le 2\pi$.

B: $K = 2$ and $q = 0$

$$t^2_0 = P_2\left[\sqrt{\frac{2}{3}}\mathrm{Cos}\theta_2\mathrm{Cos}\theta_3 - \frac{1}{2\sqrt{6}}\left(\mathrm{Sin}\theta_2\mathrm{Sin}\theta_3 e^{i\varphi_2}e^{-i\varphi_3} + \mathrm{Sin}\theta_2\mathrm{Sin}\theta_3 e^{-i\varphi_2}e^{i\varphi_3}\right)\right]$$

$$= \mathrm{P}_2\left[\sqrt{\frac{2}{3}}\mathrm{z}_2\mathrm{z}_3 - \frac{1}{2\sqrt{6}}((\mathrm{x}_2 + \mathrm{iy}_2)(\mathrm{x}_3 - \mathrm{iy}_3) + (\mathrm{x}_2 - \mathrm{iy}_2)(\mathrm{x}_3 + \mathrm{iy}_3))\right]. \quad (33)$$

With $\widehat{\mathrm{Q}}(\theta,\varphi) \equiv (\mathrm{Sin}\theta\ \mathrm{Cos}\varphi, \mathrm{Sin}\theta\ \mathrm{Sin}\varphi, \mathrm{Cos}\theta)$ and $\hat{Q}.\hat{Q} = 1$.

Conclusion

The paper is devoted to study the density matrix in terms of spherical tensor parameters. Since these parameters can be uniquely determined by unit vectors, therefore, we constructed the standard form of the density matrix for spin-1 system in terms of unit vectors. The symmetric properties of spherical tensor parameters was shown in some cases.

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