Research Journal of Mathematical and Statistical Sciences ISSN 2320-6047
Vol. **9(2),** 1-5, May **(2021)** *Res. J. Mathematical and Statistical Sci.*

Parameter estimation of length biased Nakagami distribution

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Available online at: www.iscamaths.com, www.isca.in, www.isca.me
Received 24th June 2020, revised 2nd April 2021, accepted 10th May 2021

Abstract

In this paper, the length biased Nakagami distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter have been derived under squared error, precautionary and weighted loss functions by using quasi and inverted gamma priors.

Keywords: Length biased Nakagami distribution, Bayesian method, quasi and inverted gamma priors, squared error, precautionary and weighted loss functions.

Introduction

Nakagami distribution is proposed by Nakagami[1]. The length biased Nakagami distribution was introduced by Mudasir et al.[2]. The density function of this distribution is given by

$$f(x;\theta)=\frac{2k^{k+\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)}\theta^{-\left(k+\frac{1}{2}\right)}x^{2k}e^{-\frac{k}{\theta}x^2}\quad ;x>0,k>0,\theta>0 \tag{1}$$

where θ and k are called scale and shape parameter respectively.

The joint density function or likelihood function of (1) is given by

$$f(\underline{x};\theta)=\left(\frac{2k^{k+\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)}\right)^n\theta^{-n\left(k+\frac{1}{2}\right)}\left(\prod_{i=1}^{n}x_i^{2k}\right)e^{-\frac{k}{\theta}\sum_{i=1}^{n}x_i^2} \tag{2}$$

Taking log of equation (2), we have

$$\log f(\underline{x};\theta)=n\log\left(\frac{2k^{k+\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)}\right)-n\left(k+\frac{1}{2}\right)\log\theta+\log\left(\prod_{i=1}^{n}x_i^{2k}\right)-\frac{k}{\theta}\sum_{i=1}^{n}x_i^2 \tag{3}$$

Differentiating (3) with respect to θ and equating to zero, we get

$$\hat{\theta}=\frac{k\sum_{i=1}^{n}x_i^2}{n\left(k+\frac{1}{2}\right)} \tag{4}$$

Bayesian Approach of Estimation

In this method generally we consider the squared error loss function (SELF)

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2. \tag{5}$$

The Bayes estimator under SELF, say, $\hat{\theta}_s$ is given by

$$\hat{\theta}_S=E(\theta). \tag{6}$$

Zellner[3] and Basu & Ebrahimi[4] used the asymmetric loss function. Norstrom[5] introduced new loss function known as precautionary which is given by

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}} \tag{7}$$

Under Precautionary loss function the Bayes estimator of θ is denoted by $\hat{\theta}_P$ and is obtained as

$$\hat{\theta}_P=\left[E\left(\theta^2\right)\right]^{1/2} \tag{8}$$

Weighted loss function[6] is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\theta} \tag{9}$$

Under weighted loss function the Bayes estimator of θ is denoted by $\hat{\theta}_W$ and is obtained as

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right) \right]^{-1} \qquad (10)$$

Now, consider the following prior density which will be used to estimate θ.

Quasi-prior: For the situation where we have no prior information about the parameter θ, we may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d} \; ; \theta > 0, \; d \geq 0, \qquad (11)$$

where *d = 0* leads to a diffuse prior and *d = 1*, a non-informative prior.

Inverted gamma prior: Generally, for the parameter θ, this density is used as prior distribution given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta} \; ; \theta > 0. \qquad (12)$$

Bayes Estimators under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (2), is given by

$$f(\theta/\underline{x}) = \frac{\left(\frac{2k^{k+\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)} \right)^n \theta^{-n\left(k+\frac{1}{2}\right)} \left(\prod_{i=1}^n x_i^{2k} \right) e^{-\frac{k}{\theta}\sum_{i=1}^n x_i^2} \theta^{-d}}{\int_0^\infty \left(\frac{2k^{k+\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)} \right)^n \theta^{-n\left(k+\frac{1}{2}\right)} \left(\prod_{i=1}^n x_i^{2k} \right) e^{-\frac{k}{\theta}\sum_{i=1}^n x_i^2} \theta^{-d} d\theta}$$

$$= \frac{\theta^{-\left(nk+\frac{n}{2}+d\right)} e^{-\frac{k}{\theta}\sum_{i=1}^n x_i^2}}{\int_0^\infty \theta^{-\left(nk+\frac{n}{2}+d\right)} e^{-\frac{k}{\theta}\sum_{i=1}^n x_i^2} d\theta}$$

$$= \frac{\left(k\sum_{i=1}^n x_i^2 \right)^{nk+\frac{n}{2}+d-1}}{\Gamma\left(nk+\frac{n}{2}+d-1\right)} \theta^{-\left(nk+\frac{n}{2}+d\right)} e^{-\frac{k}{\theta}\sum_{i=1}^n x_i^2} \qquad (13)$$

Theorem 1: Under SELF, the Bayes estimate of θ is obtained as

$$\hat{\theta}_S = \frac{k\sum_{i=1}^n x_i^2}{nk+\frac{n}{2}+d-2} \qquad (14)$$

Proof: From equation (6), on using (13),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x}) \, d\theta$$

$$= \frac{\left(k\sum_{i=1}^n x_i^2 \right)^{nk+\frac{n}{2}+d-1}}{\Gamma\left(nk+\frac{n}{2}+d-1\right)} \int_0^\infty \theta^{-\left(nk+\frac{n}{2}+d-1\right)} e^{-\frac{k}{\theta}\sum_{i=1}^n x_i^2} d\theta$$

$$= \frac{\left(k\sum_{i=1}^n x_i^2 \right)^{nk+\frac{n}{2}+d-1}}{\Gamma\left(nk+\frac{n}{2}+d-1\right)} \frac{\Gamma\left(nk+\frac{n}{2}+d-2\right)}{\left(k\sum_{i=1}^n x_i^2 \right)^{nk+\frac{n}{2}+d-2}}$$

or, $$\hat{\theta}_S = \frac{k\sum_{i=1}^n x_i^2}{nk+\frac{n}{2}+d-2} .$$

Theorem 2: Under precautionary loss, $\hat{\theta}_P$ is the Bayes estimate of θ, which is given as

$$\hat{\theta}_P = \frac{k\sum_{i=1}^n x_i^2}{\left[\left(nk+\frac{n}{2}+d-2\right)\left(nk+\frac{n}{2}+d-3\right) \right]^{\frac{1}{2}}} \qquad (15)$$

Proof: From equation (8), on using (13),

$$\left(\hat{\theta}_P\right)^2 = E\left(\theta^2\right) = \int \theta^2 f(\theta/\underline{x}) d\theta$$

$$= \frac{\left(k\sum_{i=1}^n x_i^2 \right)^{nk+\frac{n}{2}+d-1}}{\Gamma\left(nk+\frac{n}{2}+d-1\right)} \int_0^\infty \theta^{-\left(nk+\frac{n}{2}+d-2\right)} e^{-\frac{k}{\theta}\sum_{i=1}^n x_i^2} d\theta$$

$$=\frac{\left(k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+d-1}}{\Gamma\left(nk+\frac{n}{2}+d-1\right)}\frac{\Gamma\left(nk+\frac{n}{2}+d-3\right)}{\left(k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+d-3}}$$

$$=\frac{\left(k\sum_{i=1}^{n}x_i^2\right)^{2}}{\left(nk+\frac{n}{2}+d-2\right)\left(nk+\frac{n}{2}+d-3\right)}$$

$$\Rightarrow \qquad \hat{\theta}_P=\frac{k\sum_{i=1}^{n}x_i^2}{\left[\left(nk+\frac{n}{2}+d-2\right)\left(nk+\frac{n}{2}+d-3\right)\right]^{\frac{1}{2}}}$$

Theorem 3: Under weighted loss $\hat{\theta}_W$ is defined as the Bayes estimate of θ given by

$$\hat{\theta}_W=\frac{k\sum_{i=1}^{n}x_i^2}{nk+\frac{n}{2}+d-1} \tag{16}$$

Proof: From equation (10), on using (13),

$$\hat{\theta}_W=\left[E\left(\frac{1}{\theta}\right)\right]^{-1}=\left[\int\frac{1}{\theta}f\left(\theta/\underline{x}\right)d\theta\right]^{-1}$$

$$=\left[\frac{\left(k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+d-1}}{\Gamma\left(nk+\frac{n}{2}+d-1\right)}\int_0^\infty\theta^{-\left(nk+\frac{n}{2}+d+1\right)}e^{-\frac{k}{\theta}\sum_{i=1}^{n}x_i^2}d\theta\right]^{-1}$$

$$=\left[\frac{\left(k\sum_{i=1}^{n}x_i^2\right)^{n\left(k+\frac{1}{2}\right)+d-1}}{\Gamma\left(n\left(k+\frac{1}{2}\right)+d-1\right)}\times\frac{\Gamma\left(n\left(k+\frac{1}{2}\right)+d\right)}{\left(k\sum_{i=1}^{n}x_i^2\right)^{n\left(k+\frac{1}{2}\right)+d}}\right]^{-1}$$

$$=\left[\frac{nk+\frac{n}{2}+d-1}{k\sum_{i=1}^{n}x_i^2}\right]^{-1}$$

or, $$\hat{\theta}_W=\frac{k\sum_{i=1}^{n}x_i^2}{nk+\frac{n}{2}+d-1}.$$

Bayes Estimators under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (2), is obtained as

$$f\left(\theta/\underline{x}\right)=\frac{\left(\frac{2k^{k+\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)}\right)^{n}\theta^{-n\left(k+\frac{1}{2}\right)}\left(\prod_{i=1}^{n}x_i^{2k}\right)e^{-\frac{k}{\theta}\sum_{i=1}^{n}x_i^2}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta}}{\int_0^\infty\left(\frac{2k^{k+\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)}\right)^{n}\theta^{-n\left(k+\frac{1}{2}\right)}\left(\prod_{i=1}^{n}x_i^{2k}\right)e^{-\frac{k}{\theta}\sum_{i=1}^{n}x_i^2}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta}d\theta}$$

$$=\frac{\theta^{-\left(nk+\frac{n}{2}+\alpha+1\right)}e^{-\frac{1}{\theta}\left(\beta+k\sum_{i=1}^{n}x_i^2\right)}}{\int_0^\infty\theta^{-\left(nk+\frac{n}{2}+\alpha+1\right)}e^{-\frac{1}{\theta}\left(\beta+k\sum_{i=1}^{n}x_i^2\right)}d\theta}$$

$$=\frac{\theta^{-\left(nk+\frac{n}{2}+\alpha+1\right)}e^{-\frac{1}{\theta}\left(\beta+k\sum_{i=1}^{n}x_i^2\right)}}{\Gamma\left(nk+\frac{n}{2}+\alpha\right)\Big/\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha}}$$

$$=\frac{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha}}{\Gamma\left(nk+\frac{n}{2}+\alpha\right)}\theta^{-\left(nk+\frac{n}{2}+\alpha+1\right)}e^{-\frac{1}{\theta}\left(\beta+k\sum_{i=1}^{n}x_i^2\right)} \tag{17}$$

Theorem 4: Assuming the squared error loss function, the Bayes estimate of the parameter θ, is of the form

$$\hat{\theta}_S=\frac{\beta+k\sum_{i=1}^{n}x_i^2}{nk+\frac{n}{2}+\alpha-1} \tag{18}$$

Proof: From equation (6), on using (17),

$$\hat{\theta}_S=E(\theta)=\int\theta f\left(\theta/\underline{x}\right)d\theta$$

$$=\frac{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha}}{\Gamma\left(nk+\frac{n}{2}+\alpha\right)}\int_0^\infty \theta^{-\left(nk+\frac{n}{2}+\alpha\right)} e^{-\frac{1}{\theta}\left(\beta+k\sum_{i=1}^{n}x_i^2\right)}d\theta$$

$$=\frac{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha}}{\Gamma\left(nk+\frac{n}{2}+\alpha\right)}\frac{\Gamma\left(nk+\frac{n}{2}+\alpha-1\right)}{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha-1}}$$

or, $$\hat{\theta}_S=\frac{\beta+k\sum_{i=1}^{n}x_i^2}{nk+\frac{n}{2}+\alpha-1}.$$

Theorem 5: Assuming precautionary loss, $\hat{\theta}_P$ is defined as the Bayes estimate of the parameter given by

$$\hat{\theta}_P=\frac{\beta+k\sum_{i=1}^{n}x_i^2}{\left[\left(nk+\frac{n}{2}+\alpha-1\right)\left(nk+\frac{n}{2}+\alpha-2\right)\right]^{\frac{1}{2}}} \tag{19}$$

Proof: From equation (8), on using (17),

$$\left(\hat{\theta}_P\right)^2=E\left(\theta^2\right)=\int\theta^2 f\left(\theta/\underline{x}\right)d\theta$$

$$=\frac{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha}}{\Gamma\left(nk+\frac{n}{2}+\alpha\right)}\int_0^\infty \theta^{-\left(nk+\frac{n}{2}+\alpha-1\right)} e^{-\frac{1}{\theta}\left(\beta+k\sum_{i=1}^{n}x_i^2\right)}d\theta$$

$$=\frac{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha}}{\Gamma\left(nk+\frac{n}{2}+\alpha\right)}\frac{\Gamma\left(nk+\frac{n}{2}+\alpha-2\right)}{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha-2}}$$

$$=\frac{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^2}{\left(nk+\frac{n}{2}+\alpha-1\right)\left(nk+\frac{n}{2}+\alpha-2\right)}$$

or, $$\hat{\theta}_P=\frac{\beta+k\sum_{i=1}^{n}x_i^2}{\left[\left(nk+\frac{n}{2}+\alpha-1\right)\left(nk+\frac{n}{2}+\alpha-2\right)\right]^{\frac{1}{2}}}.$$

Theorem 6: Assuming weighted loss, the Bayes estimate of θ is of the form

$$\hat{\theta}_W=\frac{\beta+k\sum_{i=1}^{n}x_i^2}{nk+\frac{n}{2}+\alpha} \tag{20}$$

Proof: From equation (10), on using (17),

$$\hat{\theta}_W=\left[E\left(\frac{1}{\theta}\right)\right]^{-1}=\left[\int\frac{1}{\theta}f\left(\theta/\underline{x}\right)d\theta\right]^{-1}$$

$$=\left[\frac{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha}}{\Gamma\left(nk+\frac{n}{2}+\alpha\right)}\int_0^\infty \theta^{-\left(nk+\frac{n}{2}+\alpha+2\right)} e^{-\frac{1}{\theta}\left(\beta+k\sum_{i=1}^{n}x_i^2\right)}d\theta\right]^{-1}$$

$$=\left[\frac{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha}}{\Gamma\left(nk+\frac{n}{2}+\alpha\right)}\frac{\Gamma\left(nk+\frac{n}{2}+\alpha+1\right)}{\left(\beta+k\sum_{i=1}^{n}x_i^2\right)^{nk+\frac{n}{2}+\alpha+1}}\right]^{-1}$$

$$=\left[\frac{nk+\frac{n}{2}+\alpha}{\beta+k\sum_{i=1}^{n}x_i^2}\right]^{-1}$$

or, $$\hat{\theta}_W=\frac{\beta+k\sum_{i=1}^{n}x_i^2}{nk+\frac{n}{2}+\alpha}.$$

Conclusion

In this paper, we have obtained a number of estimators of parameter of length biased Nakagami distribution. In equation (4) we have obtained the maximum likelihood estimator of the parameter. In equation (14), (15) and (16) we have obtained the Bayes estimators under different loss functions using quasi prior. In equation (18), (19) and (20) we have obtained the Bayes estimators under different loss functions using inverted gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

References

1. M. Nakagami (1960). The m-distribution – a general formula of intensity distribution of rapid fading in

Statistical Methods in Radio Wave Propagation. Proceedings of a Symposium Held at the University of California, Los Angeles, June 18-20, 1958, W.C. Hoffman, Ed., pp. 3-36, Pergamon Press, Oxford, UK.

2. Mudasir Sofi, Ahamad, S.P. and Ahamad, A. (2016). Bayesian Estimation of Length Biased Nakagami Distribution. *International Journal of Modern Mathematical Sciences*, 14(2), 147-159.

3. Zellner, A., (1986). Bayesian estimation and prediction using asymmetric loss functions. *Jour. Amer. Stat. Assoc.*, 91, 446-451.

4. Basu, A. P. and Ebrahimi, N. (1991). Bayesian approach to life testing and reliability estimation using asymmetric loss function. *Jour. Stat. Plann. Infer.*, 29, 21-31.

5. Norstrom, J. G. (1996). The use of precautionary loss functions in Risk Analysis. *IEEE Trans. Reliab.*, 45(3), 400-403.

6. Kaisar Ahmad, S. P. Ahmad, and A. Ahmad (2016). Classical and Bayesian Approach in Estimation of Scale Parameter of Nakagami distribution. *Journal of Probability and Statistics*, Article ID 7581918, 8 pages.