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# The properties of Topp Leone exponentiated weibull distribution with application to survival data

**Sule Ibrahim***
Faculty of Physical Sciences, Department of Statistics, Ahmadu Bello University, Zaria, Nigeria
ibrahimsule76@yahoo.com

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## Abstract

*In this study, a new four-parameter lifetime distribution called the Topp Leone exponentiated Weibull distribution was introduced. The model includes several important sub-models as special cases such as Topp Leone Weibull, exponentiated Weibull and Weibull distributions. A linear representation for the probability distribution function was carried out. Some mathematical properties of the distribution were presented such as moments, moment generating function, quantile function, survival function, hazard function, reversed hazard function and odd function. The distribution of order statistic was obtained. Estimation of the parameters by maximum likelihood method was discussed. Two real-life application of the distribution was presented and the analysis showed the fit and flexibility of the new distribution over some lifetime models considered. The analysis showed that the model is effective in fitting survival data.*

**Keywords:** Exponentiated Weibull distribution, linear representation, real-life, reversed-J shaped, reliability function, symmetrical distributions, Topp Leone exponentiated-G family.

## Introduction

Extensive review of literature reveals that several continuous univariate distributions have been used for modeling data in many areas of human endeavors such as economics, engineering, biological studies and environmental sciences. Although, applied areas such as finance, lifetime analysis and insurance clearly require extended forms of these continuous univariate distributions. So, several families of continuous distributions have been developed by extending well-known continuous distributions. These generalized distributions give more flexibility by adding one or more parameter(s) to the baseline model.

The Weibull distribution is a very popular model and it has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. It is generally adequate for modeling monotone hazard rates.

Some of the generalizations of the Weibull distribution available inliterature are extensively studied by Silver et al.[1], Cordeiro et al.[2], Aryal and Tsokos[3], Shahbaz et al. [4]Cordeiro et al.[5], Merovci and Elbatal[6], Hanook et al.[7], Elbatal and Aryal[8], Cordeiro et al.[9], Cordeiro et al.[10], Afify et al.[11], Nofal et al.[12], Nofal et al.[13], Aryal et al.[14], Afify et al.[15], Afify et al.[16]

Recently, Ibrahim et al.[17] proposed a new family of continuous distributions called the Topp Leone exponentiated-G (TLEx-G) family with two extra shape parameters $\alpha$ and$\theta$. For an arbitrary baseline cumulative distribution function (cdf) $H(x,\varphi)$, the TLEx-G family with two extra positive shape parameters $\alpha$ and $\theta$ has cdf and probability density function (pdf) for (x > 0) given by

$$F(x;\alpha,\theta,\varphi) = \{1-[1-H(x,\varphi)^{\alpha}]^2\}^{\theta} \tag{1}$$

And

$$f(x;\alpha,\theta,\varphi) = 2\alpha\theta h(x;\ \varphi)H(x;\ \varphi)^{\alpha-1}[1-H(x;\ \varphi)^{\alpha}]\{1-[1-H(x;\ \varphi)^{\alpha}]^2\}^{\theta-1} \tag{2}$$

$x>0,\quad \alpha,\quad \theta,\quad \varphi>0$
respectively.

Parameters$\alpha$ and $\theta$ are positive shape parameters and $\varphi$ is the vector of parameters. The cdf and pdf of the Weibull distribution are given by

$$H(x;\ \beta,\lambda) = 1-e^{-(\beta x)^{\lambda}} \tag{3}$$

$$h(x;\ \beta,\lambda) = \lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}} \tag{4}$$

$x>0,\quad \beta,\quad \lambda>0$

The motivation behind proposing the new TLExW distribution by inducting two extra shape parameters to the Weibull model is to obtain a more flexible modelthat will have an improved goodness-of-fit to real data over the competing models.

### The Topp Leone Exponentiated Weibull (TLExW) Distribution

This section discusses the TLExW model. The TLExWcdf is given by

$$F(x;\alpha,\theta,\beta,\lambda)=\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta} \quad (5)$$

The pdf corresponding to (5) is

$$f(x;\alpha,\theta,\beta,\lambda)=\ 2\alpha\theta\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta-1} \quad (6)$$

where $x>0,\beta>0$ is the scale parameter and$\alpha,\lambda,\theta>0$ are the shape parameters.

The graphs of the pdf of TLExW distribution are presented below with different parameter values.

**Infinite mixture representation:** Using binomial expansion for cdf in Equation (5), we have

$$\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta+1)}{i!\,\Gamma(\theta+1-i)}\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2i}$$

$$\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2i}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(2i+1)}{j!\,\Gamma(2i+1-j)}\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha j}$$

$$\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha j}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\alpha j+1)}{k!\,\Gamma(\alpha j+1-k)}\left(e^{-(\beta x)^{\lambda}}\right)^{k}$$

Then, the expansion for the cdf in Equation (5) is

$$F(x;\alpha,\theta,\beta,\lambda)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta+1)\Gamma(2i+1)\Gamma(\alpha j+1)}{i!j!k!\Gamma(\theta+1-i)\Gamma(2i+1-j)\Gamma(\alpha j+1-k)}\left(e^{-(\beta x)^{\lambda}}\right)^{k}. \quad (7)$$

Also, we derive the expansion for the pdf in Eq. (6) by using binomial expansion on the last term

$$\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta)}{i!\Gamma(\theta-i)}\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2i} \quad (8)$$

Substituting Equation (8) in to Equation (6) we have

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\lambda\beta^{\lambda}\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta)}{i!\Gamma(\theta-i)}x^{\lambda-1}e^{-(\beta x)^{\lambda}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2i+1}} \quad (9)$$

Using binomial expansion on the last term in Equation (9), we have

$$\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2i+1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(2(i+1))}{j!\Gamma(2(i+1)-j)}\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha j} \quad (10)$$

Substituting Equation (10) into Equation (9), we have

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\lambda\beta^{\lambda}\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta)}{i!\Gamma(\theta-i)}\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(2(i+1))}{j!\Gamma(2(i+1)-j)}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha(j+1)-1} \quad (11)$$

Again using binomial expansion on the last term in Equation (11), we have

$$\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha(j+1)-1}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\alpha(j+1))}{k!\Gamma(\alpha(j+1)-k)}\left(e^{-(\beta x)^{\lambda}}\right)^{k} \quad (12)$$

Substituting Equation (12) into Equation (11), we have

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\lambda\beta^{\lambda}\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta)}{i!\Gamma(\theta-i)}\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(2(i+1))}{j!\Gamma(2(i+1)-j)}\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\alpha(j+1))}{k!\Gamma(\alpha(j+1)-k)}x^{\lambda-1}\left(e^{-(\beta x)^{\lambda}}\right)^{k+1} \quad (13)$$

The expansion for the pdf in Equation (6) is

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\lambda\beta^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))}{i!j!k!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)}x^{\lambda-1}\left(e^{-(\beta x)^{\lambda}}\right)^{k+1} \quad (14)$$

Where $(1-y)^{b-1}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta)}{i!\Gamma(\theta-i)}y^{i}$ and $\Gamma(.)$ is the gamma function defined by $\Gamma(b)=\int_{0}^{\infty}y^{b-1}e^{-y}dy$

Equation (14) is the expansion for pdf of the TLExW which can be used to derive some of its properties like mean, moment and moment generating function.

## Some properties of the TLExW distribution

**Quantile function:** The Quantile function is given by;

$$Q(u)=F^{-1}(u) \quad (15)$$

$$x=Q(u)=\frac{1}{\beta}\left(-log\left\{1-\left[1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\alpha}}\right\}\right)^{\frac{1}{\lambda}} \quad (16)$$

For $u=0.5$, the median is given as

$$x_{m}=Q(0.5)=\frac{1}{\beta}\left(-log\left\{1-\left[1-\left[1-0.5^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\alpha}}\right\}\right)^{\frac{1}{\lambda}} \quad (17)$$

The random sample can also be easily generated from Eq. (17) by using $u$ as uniform random number.

**Moment**

Some of the interesting characteristics and features of a distribution can be studied through its moments (e.g. measure of tendency, measure dispersion, skewness and kurtosis). Therefore, it is good to derive the moments when a new model is proposed.

$$E(X^{r})=\int_{0}^{\infty}x^{r}f(x)dx \quad (18)$$

$$E(X^{r})=2\alpha\theta\lambda\beta^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))}{i!j!k!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)}\int_{0}^{\infty}x^{r+\lambda-1}\left(e^{-(\beta x)^{\lambda}}\right)^{k+1}dx \quad (19)$$

Let $y = (k+1)(\beta x)^{\lambda} \Rightarrow x = y^{\frac{1}{\lambda}}(k+1)^{-\frac{1}{\lambda}}\beta^{-1}$

$$\frac{dy}{dx} = \lambda(k+1)\beta^{\lambda}x^{\lambda-1}$$

$$\int_0^{\infty}\left(y^{\frac{1}{\lambda}}(k+1)^{-\frac{1}{\lambda}}\beta^{-1}\right)^{r+\lambda-1} e^{-y}\,\frac{dy}{\lambda(k+1)\beta^{\lambda}\left(y^{\frac{1}{\lambda}}(k+1)^{-\frac{1}{\lambda}}\beta^{-1}\right)^{\lambda-1}}$$

$$\int_0^{\infty}\left(y^{\frac{1}{\lambda}}\right)^{r} e^{-y}\,\frac{dy}{\lambda(k+1)\beta^{\lambda}}$$

$$\frac{\left((k+1)^{-\frac{1}{\lambda}}\beta^{-1}\right)^{r}}{\lambda(k+1)\beta^{\lambda}}\int_0^{\infty} y^{\frac{r}{\lambda}}e^{-y}dy$$

$$\int_0^{\infty} y^{\frac{r}{\lambda}}e^{-y}dy = \Gamma\left(\frac{r}{\lambda}+1\right)$$

$$\frac{\left((k+1)^{-\frac{1}{\lambda}}\beta^{-1}\right)^{r}}{\lambda(k+1)\beta^{\lambda}}\Gamma\left(\frac{r}{\lambda}+1\right)$$

$$E(X^r) = 2\alpha\theta\lambda\beta^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))}{i!j!k!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)}\frac{\left((k+1)^{-\frac{1}{\lambda}}\beta^{-1}\right)^{r}}{\lambda(k+1)\beta^{\lambda}}\Gamma\left(\frac{r}{\lambda}+1\right)$$

$$E(X^r) = 2\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))\Gamma\left(\frac{r}{\lambda}+1\right)}{i!j!k!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)(k+1)^{\frac{r}{\lambda}+1}\beta^{r}} \quad (20)$$

**Moment generating function**

$$E(e^{tx}) = \int_0^{\infty} x^r f(x)dx \quad (21)$$

$$e^{tx} = \sum_{m=0}^{\infty}\frac{t^m x^m}{m!} \quad (22)$$

Using the expansion in Equation (22), the MGF of TLExW is given as

$$E(e^{tx}) = 2\alpha\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))t^m\Gamma\left(\frac{m}{\lambda}+1\right)}{i!j!k!m!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)(k+1)^{\frac{m}{\lambda}+1}\beta^{m}} \quad (23)$$

**Survival function:** The survival function is also known as reliability function, is the probability of an item not failing prior to some time. It can be defined as

$$S(x) = 1 - F(x) \quad (24)$$

$$S(x) = 1 - \left\{1 - \left[1 - \left(1 - e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^2\right\}^{\theta} \quad (25)$$

## Hazard rate function

The hazard rate function is given as

$$\tau(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \quad (26)$$

$$\tau(x) = \frac{2\alpha\theta\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^2\right\}^{\theta-1}}{1-\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^2\right\}^{\theta}} \quad (27)$$

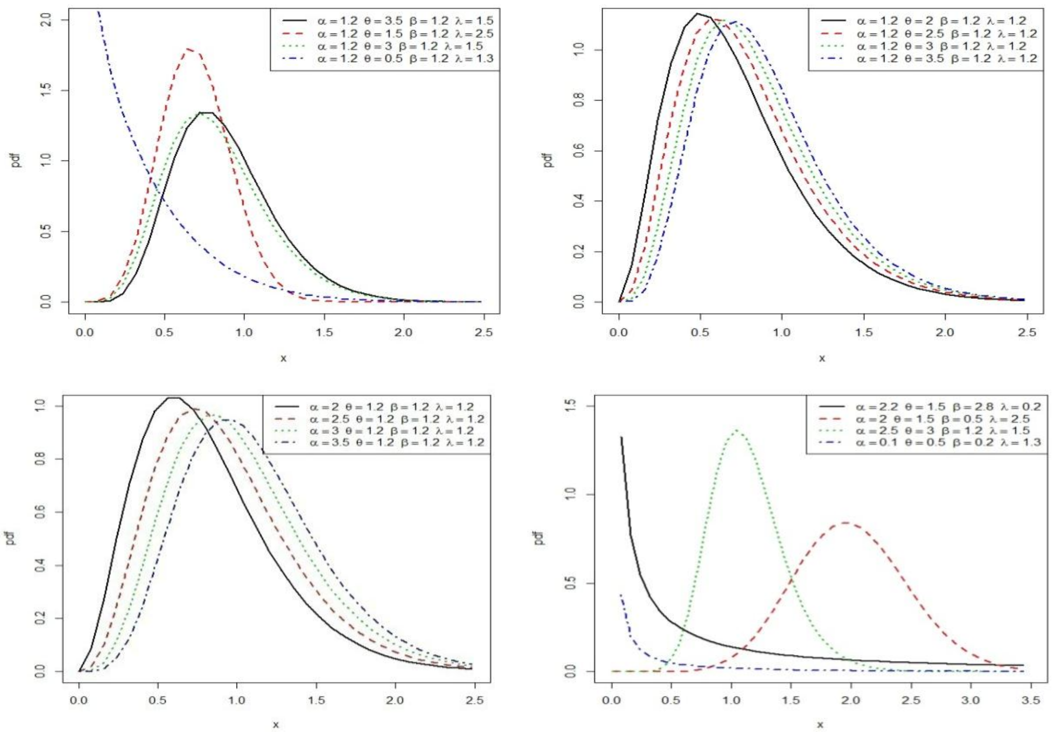


**Figure-1:** Plots of pdf of the TLExW distribution with different parameter value.

The graphs of the hazard rate function with different parameter values are presented in Figure-2.

**Odds function:** The odds function is obtained using the relation

$$Q(x) = \frac{F(x)}{S(x)} \tag{28}$$

$$Q(x) = \frac{\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta}}{1-\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta}} \tag{29}$$

**Reversed hazard rate function:** The reverse hazard rate function of the extended generalized inverse exponential distribution is given as

$$\emptyset(x) = \frac{f(x)}{F(x)} \tag{30}$$

$$\emptyset(x) = \frac{2\alpha\theta\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}}{\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta}} \tag{31}$$

**Order Statistic:** Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be n independent random variables from the TLExW distribution and let $X_{(1)} \leq X_{(2)} \leq, \dots, \leq X_{(n)}$ be their corresponding order statistic.

Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1, 2, 3, \dots, n$ denote the cdf and pdf of the $r^{th}$ order statistic $X_{r:n}$ respectively. The pdf of $X_{r:n}$ is given as

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \tag{32}$$

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^{i} f(x)[F(x)]^{r+i-1} \tag{33}$$

Inserting Equation (5) and Equation (6) into Equation (33), we have

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^{i} e^{-(\beta x)^{\lambda}}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta(r+i-1)}$$

$$f_{r:n}(x) = \frac{2\alpha\theta\lambda\beta^{\lambda}x^{\lambda-1}}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^{i} e^{-(\beta x)^{\lambda}}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-e^{-(\beta x)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta(r+i)-1}$$

$$f_{r:n}(x) = \frac{2\alpha\theta\lambda\beta^{\lambda}x^{\lambda-1}}{B(r, n-r+1)} \sum_{i=0}^{n-r}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^{i+j+k+l} \binom{\theta(r+i)-1}{j}\binom{2j+1}{k}\binom{\alpha(k+1)-1}{l}\left(e^{(-\beta x)^{\lambda}}\right)^{l+1} \tag{34}$$

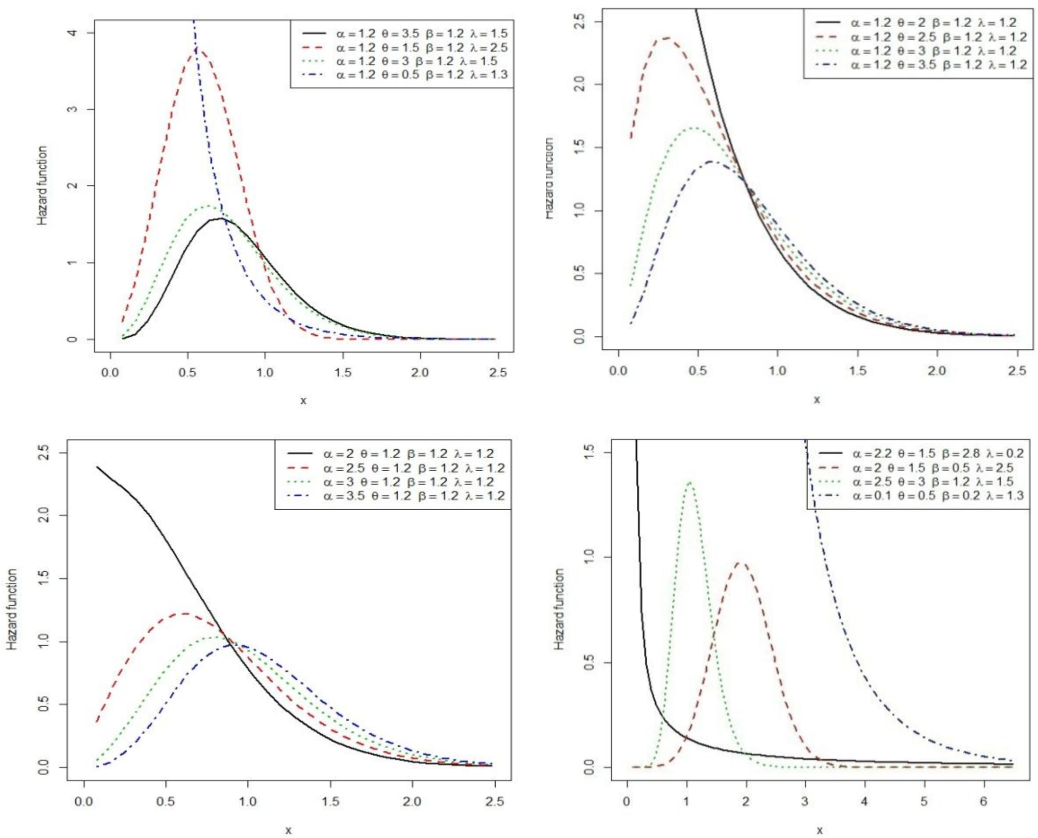


**Figure-2:** Plots of hazard function for the TLExW distribution with different parameter value.

Equation (34) is the $r^{th}$ order statistic of the TLExW distribution.

Therefore, the pdf of the minimum and maximum order statistic of the TLExW distribution are obtained by setting $r = 1$ and $r = n$ respectively in Equation (34).

**Estimation:** In this section we estimate the parameters of the TLExW distribution using the maximum like hood estimation (MLE). In this study, we discuss the MLE for the parameters of the TLExW distribution for complete samples. For a random sample, $X_1, X_2, X_3, \dots, X_n$ of size n from TLExW$(\alpha, \beta, \theta, \lambda)$, the log-likelihood function $L(\alpha, \beta, \theta, \lambda)$ is given as

$$logL(x) = nlog2 + nlog\alpha + nlog\theta + nlog\lambda + n\lambda log\beta + (\lambda - 1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}(\beta x_i)^{\lambda} + (\alpha - 1)\sum_{i=1}^{n} log\,(A) + \sum_{i=1}^{n} log\,[1 - A^{\alpha}] + (\theta - 1)\sum_{i=1}^{n} log\{1 - [1 - A^{\alpha}]^2\} \quad (35)$$

Where $A = 1 - e^{-(\beta x_i)^{\lambda}}$

We take the first derivative of the log-likelihood equation with respect to each parameter and equate to zero respectively.

$$\frac{\partial logL(x)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\,(A) - \sum_{i=1}^{n}\frac{A^{\alpha}\log(A)}{1-A^{\alpha}} + (\theta - 1)\sum_{i=1}^{n}\frac{2A^{\alpha}(1-A)\log(A)}{1-(1-A^{\alpha})^2} = 0 \quad (36)$$

$$\frac{\partial logL(x)}{\partial \beta} = \frac{n\lambda}{\beta} - \lambda\sum_{i=1}^{n}(\beta x_i)^{\lambda} + (\alpha - 1)\sum_{i=1}^{n}\frac{\lambda(\beta x_i)^{\lambda}(1-A)}{A} - \sum_{i=1}^{n}\frac{\alpha\lambda(\beta x_i)^{\lambda}(1-A)A^{\alpha}}{(1-A^{\alpha})} + (\theta - 1)\sum_{i=1}^{n}\frac{2\alpha\lambda(\beta x_i)^{\lambda}(1-A)A^{\alpha}(1-A^{\alpha})}{1-(1-A^{\alpha})^2} = 0 \quad (37)$$

$$\frac{\partial logL(x)}{\partial \lambda} = \frac{n}{\lambda} + nlog\beta - \sum_{i=1}^{n}(\beta x_i)^{\lambda}\log(\beta x_i) + \sum_{i=1}^{n}\log(x_i) + (\alpha - 1)\sum_{i=1}^{n}\frac{(\beta x_i)^{\lambda}(1-A)\log(\beta x_i)}{A} - \sum_{i=1}^{n}\frac{\alpha(\beta x_i)^{\lambda}(1-A)A^{\alpha}\log(\beta x_i)}{1-A^{\alpha}} + (\theta - 1)\sum_{i=1}^{n}\frac{2\alpha(\beta x_i)^{\lambda}(1-A)(1-A^{\alpha})A^{\alpha}\log(\beta x_i)}{1-(1-A^{\alpha})^2} = 0 \quad (38)$$

$$\frac{\partial logL(x)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} log\{1 - [1 - A^{\alpha}]^2\} = 0 \quad (39)$$

Since Equations (36), (37), (38) and (39) are in the complex form, therefore the exact solution of ML estimator for unknown parameters is not possible. So it is convenient to use nonlinear Newton Raphson algorithm for exact numerically solution to maximize the above likelihood function.

## Application to real-life data set

In this section, we present some applications of the TLExW distribution using different data sets to demonstrate the flexibility of the distribution to model these real data sets. The data are fitted to the TLExW distribution and other distributions such as the Topp Leone Weibull, Exponentiated Weibull and Weibull distributions.

Topp Leone Weibull distribution

$$f(x) = 2\theta\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\left[1 - e^{-(\beta x)^{\lambda}}\right]\left\{1 - \left[1 - \left(1 - e^{-(\beta x)^{\lambda}}\right)\right]^2\right\}^{\theta-1}$$

Exponentiated Weibull distribution

$$f(x) = \alpha\lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}$$

Weibull distribution

$$f(x) = \lambda\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}$$

**Data set 1:** The first data set represents the survival times (in weeks) of 33 patients suffering from acute myelogeneous leukemia. These data have been studied by Feigl and Zelen[18]. The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

**Table-1:** Statistics for data set 1.

| Model | Goodness of fit (G-O-F) criteria | | | | |
|---|---|---|---|---|---|
| | BIC | AIC | CAIC | HQIC | $-\ell$ |
| TLExW | 218.9551 | 216.9410 | 218.3696 | 218.9551 | 104.4705 |
| TLW | 434.6316 | 430.1421 | 430.9697 | 431.6527 | 212.0710 |
| ExW | 357.7886 | 353.2990 | 354.1266 | 354.8096 | 173.6495 |
| W | 326.9911 | 323.9980 | 324.3980 | 325.0051 | 159.9990 |

**Table-2:** Maximum likelihood estimates of distributions considered for data set 1.

| Model | Estimates | | | |
|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $\hat{\theta}$ |
| TLExW | 1.5264 | 0.0039 | 0.2515 | 0.2406 |
| TLW | - | 0.0654 | 2.0290 | 4.1837 |
| ExW | 1.5991 | 0.0427 | 1.2864 | - |
| W | - | 0.0729 | 0.3733 | - |

**Table-3:** Statistics for data set 2.

| Model | Goodness of fit (G-O-F) criteria | | | | |
|---|---|---|---|---|---|
| | BIC | AIC | CAIC | HQIC | $-\ell$ |
| TLExW | 400.1795 | 392.5315 | 393.4203 | 395.4439 | 192.2657 |
| TLW | 568.9393 | 563.2032 | 563.7249 | 565.3875 | 278.6016 |
| ExW | 590.7134 | 584.9773 | 585.4991 | 587.1616 | 289.4887 |
| W | 326.9911 | 471.2030 | 475.0270 | 472.6592 | 233.6015 |

**Table-4:** Maximum likelihood estimates of distributions considered for data set 2.

| Model | Estimates | | | |
|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $\hat{\theta}$ |
| TLExW | 1.7274 | 0.0041 | 0.1288 | 0.3224 |
| TLW | - | 0.0336 | 1.7406 | 1.6758 |
| ExW | 0.4262 | 0.0309 | 1.5036 | - |
| W | - | 0.0274 | 0.4546 | - |

**Data set 2:** The second dataset consists of 50 failure times of devices analyzed by Aarset[19]. The data are: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

The maximum likelihood estimates of data set 1 and data set 2 are reported in Table-2 and Table-4 and the Information Criteria values for the fitted distributions are reported in Table-1 and Table-3. The results show that the TLExW distribution provides a significantly better fit than the other three models considered base on the values of AIC, CAIC, BIC and HQIC.

## Conclusion

We proposed a new distribution, called the Topp Leone exponentiated Weibull distribution which extends the Topp Leone Weibull distribution. An expansion for the density was derived which was now used to generate properties of the distribution, including the moments and moment generating function. Some other properties were derived such as the survival function, hazard function, quantile function, the median and order statistics. The estimation of parameters by the method of the maximum likelihood was carried out using a package in *R* known as *Adequacy Model*. Application of the Topp Leone exponentiated Weibull distribution to two real datasets showed from the Table-1 and Table-3 that the Topp Leone exponentiated Weibull distribution is quite effective and flexible model in fitting survival data.

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