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The beta transmuted dagum distribution: theory and applications

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Abstract

In this paper, we introduce a new family of continuous distributions called the beta transmuted Dagum distribution which extends the beta and transmuted family. The density function, hazard function, shape of the density and hazard functions, moments, moment generating function, quantiles and stress-strength of the beta transmuted Dagum distribution(BTD) are provided and discussed in detail. We discuss the maximum likelihood estimation of the model parameters. We assess the performance of the maximum likelihood estimators in terms of biases, standard errors, and mean square of errors by means of simulation studies. The usefulness of the new model is illustrated through an application to survival dataset.

Keywords: Dagum distribution, transmutation map, parameter estimation, moments, power series distribution.

Introduction

Dagum realized that a further parameter was needed to such distribution which led to the Dagum type I and generalizations with three-parameter and four-parameter distributions[1,2]. More details can be found in Dagum[3,4]. Costa[5] used the Dagum distribution for human capital, Pérez and Alaiz[6], Ivana[7] and Lukasiewicz *et al.*[8] worked on the Dagum model to explain changes in personal income distribution.

Binoti *et al*[9] and Alwan *et al* [10] investigated the reliability measurement for mixed mode failures of electric power distribution stations and for describing diameter in teak stands subjected to thinning at different ages using Dagum distribution. Kleiber and Kotz[11], Shehzad and Asghar[12] and Pant and Headrick[13]studied the Dagum distribution.

Domma[14] obtained the asymptotic distribution of the maximum likelihood estimators of the parameters of the right-truncated Dagum distribution. Pollastri and Zambruno[15] described procedure for estimation the Dagum distribution of the ratio of two independent random variables. Domma *et al.* [16] described various properties of the Dagum distribution in reliability analysis. Domma *et al.*[17] estimated the parameters of the Dagum distribution using maximum likelihood method based on censored data.Domma *et al.* [18] obtained the Fisher information matrix based on various left and right-censored sample from a Dagum distribution. Domma[19] studied, through the kurtosis diagram proposed by Zenga[20] and Polisicchio and Zenga[21], the kurtosis of the log-Dagum distribution. Domma and Perri[22]gave some properties of the log-Dagum distribution and parameters estimation.

The cumulative distribution function of the Dagum (Type-I) distribution is given by

$$G(x) = \left(1+\alpha x^{-\theta}\right)^{-\beta}, x>0, \quad (1)$$

the density function for the Dagum distribution obtained by differentiating (1) with respect to x is

$$g(x) = \alpha\theta\beta x^{-\theta-1}\left(1+\alpha x^{-\theta}\right)^{-\beta-1}, \text{ for } \alpha,\theta,\beta>0, \quad (2)$$

Recently, Ibrahim and Gokarna[23] proposed a four parameter generalized Dagum distribution that is called the transmuted Dagum distribution. The cdf of the transmuted Dagum distribution is given by

$$\mathrm{G(x)} = \left(1+\alpha x^{-\theta}\right)^{-\beta}\left(1+\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right), \quad (3)$$

And the probability density function for the transmuted Dagum distribution is

$$\mathrm{g(x)} = \alpha\theta\beta x^{-\theta-1}\left(1+\alpha x^{-\theta}\right)^{-\beta-1}\left(1+\lambda-2\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right), \quad (4)$$

where λ is the transmuted parameter.

Eugene *et al.*[24] and Jones[25] gave a class of generalized distribution defined as

$$\mathrm{F(x)} = \mathrm{I}_{\mathrm{G(x)}}(\mathrm{a,b}) = \frac{1}{\mathrm{B(a,b)}}\int_0^{\mathrm{G(x)}} \mathrm{t}^{\mathrm{a}-1}\,(1-\mathrm{t})^{\mathrm{b}-1}\mathrm{dt}, \quad (5)$$

And

$$\mathrm{f(x)} = \frac{1}{\mathrm{B}(a,\mathrm{b})}[\mathrm{G}(x)]^{\mathrm{a}-1}[1-\mathrm{G}(x)]^{\mathrm{b}-1}\mathrm{g(x)}, \quad 0<a,b<\infty. \quad (6)$$

This paper is organized as follows. In section 2, we define the cumulative, density and hazard functions of the BTD distribution. In Section 3 we derive the mixture representation of the BTD distribution. Moments, quantile and stress-strength reliability are derived in section 4. The estimation of the unknown parameters of the BTD is discussed in Section 5. An application of the BTD model in the analysis of real datain section6. Some Conclusion are presented in Section 7.

The Beta Transmuted Dagum Distribution

If $G(x)$ is the cdf of the transmuted Dagum distribution given by (3), then equation (5) gives the cdf of the BTD distribution as follows

$$F(x) = I_{(1+\alpha x^{-\theta})^{-\beta}\left(1+\lambda-\lambda(1+\alpha x^{-\theta})^{-\beta}\right)}(a,b),$$

$$= \frac{1}{B(a,b)}\int_0^{(1+\alpha x^{-\theta})^{-\beta}\left(1+\lambda-\lambda(1+\alpha x^{-\theta})^{-\beta}\right)} t^{a-1}(1-t)^{b-1}dt, \qquad (7)$$

where $x > 0, \alpha > 0, \theta > 0, \beta > 0, |\lambda| \leq 1$ and $a > 0, b > 0$.

Using the hypergeometric function see Cordeiro and Nadarajah[26], we can express the cdfas follows:

$$F(x) = \frac{\left[(1+\alpha x^{-\theta})^{-\beta}\left(1+\lambda-\lambda(1+\alpha x^{-\theta})^{-\beta}\right)\right]^{a}}{aB(a,b)}.\,2F1\left(a, 1-b; a+1; (1+\alpha x^{-\theta})^{-\beta}\left(1+\lambda-\lambda(1+\alpha x^{-\theta})^{-\beta}\right)\right), \qquad (8)$$

Where

$2F1(c,d;e;z) = \sum_{k=0}^{\infty}\frac{(c)_k(d)_k}{(e)_k}\frac{(z^k)}{k!}$, is the hypergeometric function,$(c)_k$ is

$$(c)_k = \begin{cases} c(c+1)(c+2)\ldots.(c+k-1) & k = 1,2,3,\ldots \\ 1 & k = 0. \end{cases}$$

The pdf $f(x)$isgiven by

$$f(x) = \frac{\alpha\theta\beta}{x^{\theta+1}B(a,b)}(1+x^{-\theta}\alpha)^{-\beta-1}\left(1+\lambda-2\lambda(1+x^{-\theta}\alpha)^{-\beta}\right)[(1+x^{-\theta}\alpha)^{-\beta}]^{a-1}\left[\left(1+\lambda(1+x^{-\theta}\alpha)^{-\beta}\right)\right]^{a-1}\left[1-(1+x^{-\theta}\alpha)^{-\beta}\left(1+\lambda(1+x^{-\theta}\alpha)^{-\beta}\right)\right]^{b-1}, \qquad (9)$$

The BTD distribution has several models as special cases, it is easy to see that:

when $\lambda = 0$, we obtained beta Dagum distribution from the BTD distribution. when $a = b = 1$, we obtained transmuted Dagum distribution from the BTD distribution.

When $a = b = 1$and $\lambda = 0$, we obtained Dagum distribution from the BTD distribution.

Plots of the function (9) for some special value of θ, λ and a with $\alpha = 1.5,\ \beta = 1.5\ and\ b = 1.$ is given in Figure-1.

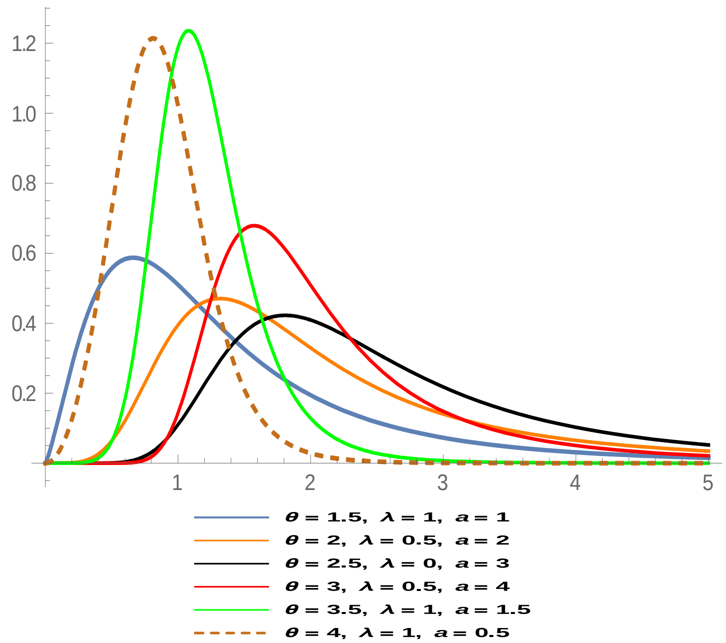

Figure-1: Pdf of BTD distribution for different values.

Mixture representation

Useful expansions can be derived using the concept of power series and the integral in (5), we have

$$\frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\frac{[G(x)]^{a+i}}{(a+i)},$$

where, $\binom{b-1}{i} = \frac{\Gamma(b)}{\Gamma(b-i)\ i!}$.

Then

$$F(x) = \sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\frac{\left[(1+\alpha x^{-\theta})^{-\beta}(1+\lambda\ (1+\alpha x^{-\theta})^{-\beta})\right]^{a+i}}{B(a,b)(a+i)},$$

$$= \sum_{i,k,l=0}^{\infty}(-1)^{i+k}\binom{b-1}{i}\binom{a+i}{k}\binom{a+i}{l}\lambda^l\frac{(G_1(x;\alpha,\theta(k+l),\beta))}{B(a,b)(a+i)}, \qquad (10)$$

where $G_1(x;\alpha,\theta(k+l),\beta)$ is the Dagum cdf with scale parameterα and shape parameters $\theta(k+l)\ and\ \beta$.

Differentiating (10) with respect to x

$$f(x) = \sum_{k,l=0}^{\infty}w_{kl}\ g(x;\alpha,\theta(k+l),\beta), \qquad x > 0, \qquad (11)$$

where

$$w_{kl} = \sum_{i=0}^{\infty}(-1)^{i+k+l}\binom{b-1}{i}\binom{a+i}{k}\binom{a+i}{l}\frac{\lambda^l}{B(a,b)(a+i)}$$

and $g(x;\alpha,\theta(k+l),\beta)$ is the Dagum pdf with scale α and shape $\theta(k+l)and\ \beta$parameters.

Moments and Moments Generating Function

Some of the most important features and characteristics of a distribution can be studied through moments such as, mean, variance, skewness and kurtosis. If $X\sim BTD(\alpha,\theta,\beta,\lambda,a,b)$ then the r^{th} moment of X are given by the following

$$E(X^r)=\int_0^\infty x^r f(x)dx=\alpha^{\frac{r}{\theta}}\,B\left(\beta+\frac{r}{\theta},\ 1-\frac{r}{\theta}\right)\sum_{k,l=0}^{\infty} w_{kl}\,(k+l)^{\frac{r}{\theta}}. \quad (12)$$

The r^{th} moment will be defined only when $\theta>r$. In particular,

$$E(X)=\alpha^{\frac{1}{\theta}}\,B\left(\beta+\frac{1}{\theta},1-\frac{1}{\theta}\right)\sum_{k,l=0}^{\infty} w_{kl}\,(k+l)^{\frac{1}{\theta}},\quad \text{if } \theta>1,$$

The variance, skewness and kurtosis of the BTD distribution can be obtained from (12) as below.

$$\mathrm{Var}(X)=\alpha^{\frac{2}{\theta}}\frac{\Gamma(\beta)\Gamma\left(\beta+\frac{2}{\theta}\right)\Gamma\left(\frac{-2+\theta}{\theta}\right)-\Gamma\left(\beta+\frac{1}{\theta}\right)^2\Gamma\left(\frac{-1+\theta}{\theta}\right)^2}{\Gamma(\beta)^2}\sum_{k,l=0}^{\infty} w_{kl}\,(k+1)^{\frac{2}{\theta}},$$

$$\mathrm{Skewness}(X)=\sum_{k,l=0}^{\infty} w_{kl}\,(k+1)^{\frac{3}{\theta}}\alpha^{\frac{3}{\theta}}\Bigg[\Bigg(\Gamma(\beta)^2\Gamma\left(\beta+\frac{3}{\theta}\right)\Gamma\left(\frac{-3+\theta}{\theta}\right)-3\Gamma(\beta)\Gamma\left(\beta+\frac{1}{\theta}\right)\Gamma\left(\beta+\frac{2}{\theta}\right)\Gamma\left(\frac{-2+\theta}{\theta}\right)\Gamma\left(\frac{-1+\theta}{\theta}\right)+2\,\Gamma\left(\beta+\frac{1}{\theta}\right)^2\Gamma\left(\frac{-1+\theta}{\theta}\right)^3\Bigg)\Big/\Gamma(\beta)^3\left(\frac{\Gamma(\beta)\Gamma\left(\beta+\frac{2}{\theta}\right)\Gamma\left(\frac{-2+\theta}{\theta}\right)-\Gamma\left(\beta+\frac{1}{\theta}\right)^2\Gamma\left(\frac{-1+\theta}{\theta}\right)^2}{\Gamma(\beta)^2}\right)^{\frac{3}{2}}\Bigg],$$

$$\mathrm{Kurtosis}(X)=\sum_{k,l=0}^{\infty} w_{kl}\Bigg[-3\Bigg(\Gamma(\beta)^2\Bigg(\Gamma(\beta)\Gamma\left(\beta+\frac{4}{\theta}\right)\Gamma\left(\frac{-4+\theta}{\theta}\right)+3\Gamma\left(\beta+\frac{2}{\theta}\right)^2\Gamma\left(\frac{-2+\theta}{\theta}\right)^2-4\Gamma\left(\beta+\frac{1}{\theta}\right)\Gamma\left(\beta+\frac{3}{\theta}\right)\Gamma\left(\frac{-3+\theta}{\theta}\right)\Gamma\left(\frac{-1+\theta}{\theta}\right)\Bigg)\Bigg)\Big/\left(\Gamma(\beta)\Gamma\left(\beta+\frac{2}{\theta}\right)\Gamma\left(\frac{-2+\theta}{\theta}\right)-\Gamma\left(\beta+\frac{1}{\theta}\right)^2\Gamma\left(\frac{-1+\theta}{\theta}\right)^2\right)^2\Bigg],$$

and the moment generating function (MGF)of the BTD is defined by

$$M_X(t)=\sum_{k,l=0}^{\infty} w_{kl}\sum_{r=0}^{\infty}\frac{t^r}{r!}\alpha^{\frac{r}{\theta}}\,B\left(\beta(k+l)+\frac{r}{\theta},1-\frac{r}{\theta}\right),\quad \theta>r. \quad (13)$$

Quantiles: The quantile of a BTD distribution with cdf (5) is

$$X=Q(u)=\alpha^{\frac{1}{\theta}}\left[-1+\left(\frac{(1+\lambda)+\sqrt{(1+\lambda)^2-4\lambda\left(I_u^{-1}(a,b)\right)}}{2\lambda}\right)^{\frac{-1}{\beta}}\right]^{\frac{-1}{\theta}}, \quad (14)$$

where $I_u^{-1}(a,b)$ is the inverse of the incomplete beta function which can be found in Zea *et al.*[27]

Stress-strength model

In stress-strength modeling, we use $R=Pr\,(X_1>X_2)$ as a measure of reliability with random strength X_1 and is subjected to a random stress X_2. Consider $X_1$and $X_2$to be independently distributed, with $X_1\sim BTD(\alpha_1,\theta,\beta,\lambda_1,a_1,b_1)$ and $X_2\sim BTR\,(\alpha_2,\theta,\beta,\lambda_2,a_2,b_2)$. The cdf $F_1$of X_1 and pdf f_2 of X_2 obtained from (10) and (11), respectively. Then,

$$R=Pr(X_1>X_2)=\int_0^\infty f_2(y)[1-F_1(y)]dy$$

$$=1+\sum_{k,l=0}^{\infty} w_{kl}^{(1)}\int_0^\infty f_2(y)(1+\alpha(k+l)x^{-\theta})^{-\beta}dy$$

$$=\sum_{k,l=0}^{\infty} w_{kl}^{(1)}A(k,l),$$

Where

$$w_{kl}^{(i)}=\sum_{j=0}^{\infty}(-1)^{j+k+l}\binom{b_i-1}{j}\binom{a_i+j}{k}\binom{a_i+j}{l}\frac{\lambda^l}{B(a,b)(a_i+j)}\qquad i=1,2,$$

And

$$A(k,l)=\int_0^\infty f_2(y)(1+\alpha(k+l)x^{-\theta})^{-\beta}dy.$$

Now,

$$A(k,l)=\sum_{r,s=0}^{\infty} w_{rs}^{(2)}\int_0^\infty (r+s)\theta\beta\,\alpha_2 x^{-\theta-1}\left[\left\{\left(1+\left(\alpha_2(r+s)+\alpha_1(k+l)\right)x^{-\theta}\right)^{-\beta-1}\right\}\right]dy$$

$$A(k,l)=\sum_{r,s=0}^{\infty} w_{rs}^{(2)}\frac{(r+s)\alpha_2}{(k+l)\alpha_1+(r+s)\alpha_2}.$$

Hence,

$$R=1+\sum_{k,l=0}^{\infty} w_{kl}^{(1)}\sum_{r,s=0}^{\infty} w_{rs}^{(2)}\frac{(r+s)\alpha_2}{(k+l)\alpha_1+(r+s)\alpha_2},$$

$$=1+\sum_{k=0}^{\infty}\sum_{r=0}^{\infty} w_k^{*(1)}\,w_r^{*(2)}\frac{r\alpha_2}{k\alpha_1+r\alpha_2}, \quad (15)$$

Where

$$w_m^{*(i)} = \sum_{k,l:k+l=m} w_{kl}^{*(i)}, \quad i = 1,2.$$

Parameters Estimation

Consider a random sample $x_1, x_2, \ldots, x_n$ from $X \sim BTD(\alpha, \theta, \beta, \lambda, a, b)$ distribution.The log-likelihood function denoted by $l(\Theta)$

$$l(\Theta) = n\ln(\alpha) + n\ln(\theta) + n\log\beta - n\ln[B(a,b)] - (\theta+1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln\left[1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right] + (a-1)\left[\sum_{i=1}^{n}\ln\left[\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right] + \sum_{i=1}^{n}\ln\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\right] + (b-1)\sum_{i=1}^{n}\ln\left[1-\left(\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\right] \quad (16)$$

The entries of the score function are given by

$$\frac{\partial l(\Theta)}{\partial\alpha} = \frac{n}{\alpha} - \beta(a-1)\sum_{i=1}^{n}\frac{x_i^{-\theta}}{1+\alpha x_i^{-\theta}} + 2\beta\lambda\sum_{i=1}^{n}\frac{x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-(\beta+1)}}{(1+\lambda-2\lambda(1+\alpha x_i^{-\theta})^{-\beta})} + \beta\lambda(a-1)\sum_{i=1}^{n}\frac{x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{(1+\lambda-\lambda(1+\alpha x_i^{-\theta})^{-\beta})} - \beta(b-1)\sum_{i=1}^{n}\frac{\lambda x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-2\beta-1} + x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{1-(1+\alpha x_i^{-\theta})^{-\beta}(1+\lambda-\lambda(1+\alpha x_i^{-\theta})^{-\beta})} \quad (17)$$

$$\frac{\partial l(\Theta)}{\partial\theta} = \frac{n}{\alpha} - \sum_{i=1}^{n}ln(x_i) + \alpha\beta(a-1)\sum_{i=1}^{n}\frac{\ln x_i\ x_i^{-\theta}}{1+\alpha x_i^{-\theta}} - 2\beta\lambda\sum_{i=1}^{n}\frac{\log x_i x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} - \alpha\beta\lambda(a-1)\sum_{i=1}^{n}\frac{x_i^{-\theta}\ln x_i\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + \alpha\beta(b-1)\sum_{i=1}^{n}\frac{\lambda x_i^{-\theta}\ln x_i\left(1+\alpha x_i^{-\theta}\right)^{-2\beta-1} + x_i^{-\theta}\ln x_i\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}, \quad (18)$$

$$\frac{\partial l(\Theta)}{\partial\beta} = \frac{n}{\beta} - (a-1)\sum_{i=1}^{n}ln\left(1+\alpha x_i^{-\theta}\right) + 2\lambda\sum_{i=1}^{n}\frac{ln\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-\beta}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + \lambda(a-1)\sum_{i=1}^{n}\frac{ln\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-\beta}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + (b-1)\sum_{i=1}^{n}\frac{-\lambda\, ln\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-2\beta} + ln\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)},$$

$$\frac{\partial l(\Theta)}{\partial\lambda} = \sum_{i=1}^{n}\frac{1-2\left(1+\alpha x_i^{-\theta}\right)^{-\beta}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + (a-1)\sum_{i=1}^{n}\frac{\left(1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} - (b-1)\sum_{i=1}^{n}\frac{\left(1+\alpha x_i^{-\theta}\right)\left(1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right) + \ln\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{\left(1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}, \quad (20)$$

$$\frac{\partial l(\Theta)}{\partial a} = -n[\psi(a) - \psi(a+b)] + \sum_{i=1}^{n}\ln\left(1+\alpha x_i^{-\theta}\right)^{-\beta} + \sum_{i=1}^{n}\ln\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right), \quad (21)$$

$$\frac{\partial l(\Theta)}{\partial b} = -n[\psi(b) - \psi(a+b)] + \sum_{i=1}^{n}\ln\left[1-\left(\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\right], \quad (22)$$

where $\Psi(x) = \frac{d}{dx}\ln\Gamma(x)$.The values $\alpha, \theta, \beta, \lambda, a\ and\ b$ are obtained by solving iteratively equations (17-22).

Asymptotic Confidence Intervals

Here, the maximum likelihood estimator (MLE)$\widehat{\Theta}$ is the point in the parameter space Ξ which maximizes the likelihood function. The maximum likelihood estimator is the solution of $\frac{\partial l(\Theta)}{\partial\Theta} = 0$.

Under very general conditions, the maximum likelihood estimator has an asymptotic normal distribution with mean equal to the parameter being estimated and covariance matrix given by the inverse of the observed information matrix. An estimator $\widehat{\Theta}$ such that

$$\sqrt{n}\left(\widehat{\Theta} - \Theta\right) \xrightarrow{d} N\left(0, \upsilon(\Theta)\right)$$

for some positive quantity $\upsilon(\theta)$ is said to be best asymptotically normal if $\upsilon(\Theta)$ is equal to $I(\Theta)^{-1}$, where $I(\Theta)$ is the Fisher information matrix defined as $I(\Theta) = E\left(\frac{\partial^2 l(\Theta)}{\partial\Theta\partial\Theta'}\right)$.

The asymptotic normal distribution can be used to construct confidence intervals and tests of hypothesis. A large sample $100(1-\gamma)\%$ confidence intervals for $\alpha, \theta, \beta, \lambda, a\ and\ b$ are:

$\widehat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\widehat{\Theta})}$, $\widehat{\theta} \pm Z_{\frac{\gamma}{2}}\sqrt{I_{\theta\theta}^{-1}(\widehat{\Theta})}$, $\hat{\beta} \pm Z_{\frac{\gamma}{2}}\sqrt{I_{\beta\beta}^{-1}(\widehat{\Theta})}$, $\hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{I_{\lambda\lambda}^{-1}(\widehat{\Theta})}$, $\hat{a} \pm Z_{\frac{\gamma}{2}}\sqrt{I_{aa}^{-1}(\widehat{\Theta})}$, $\hat{b} \pm Z_{\frac{\gamma}{2}}\sqrt{I_{bb}^{-1}(\widehat{\Theta})}$, respectively, where $I_{\alpha\alpha}^{-1}(\widehat{\Theta})$, $I_{\theta\theta}^{-1}(\widehat{\Theta})$, $I_{\beta\beta}^{-1}(\widehat{\Theta})$, $I_{\lambda\lambda}^{-1}(\widehat{\Theta})$, $I_{aa}^{-1}(\widehat{\Theta})$ and $I_{bb}^{-1}(\widehat{\Theta})$ are the diagonal elements of $I_n^{-1}(\widehat{\Theta})$, and $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}^{th}$ percentile of a standard normal distribution.

Simulation Study

We have simulated the MLEs for different sample sizes n = (10,50,100,150,200,300,500). The calculation of the estimation is based on 1000 simulated samples from the BTD. The bias and MSE are such useful measures. The bias and MSE are calculated by respectively. Table-1 shows the biases and mean squared errors (MSE).

$$\mathrm{Bias}\left(\widehat{\Theta}\right) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\widehat{\Theta}_i - \Theta\right)$$

and

$$\text{MSE}(\widehat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\widehat{\Theta}_i - \Theta\right)^2$$

From Table-1 it is clear that as the sample size increases, the biases and the mean squared errors (MSEs) decrease. This indicates the high efficiency of the estimates.

Applications

In this section, we use a real data to compare the fits of the BTD distribution with three generalized distributions: beta Dagum (BD), transmuted Dagum (TD) and Dagum (D) distributions. The data set is obtained from Lee and Wang[28] which consists of 128 bladder cancer patients. Summary of the data is given in Table-2.

To determine the optimum model, we also compute the estimated log-likelihood values $\hat{l}$, Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan- Quinn information criterion (HQIC), Anderson-Darling (A^*), Crameér–von Mises (W^*) and Kolmogorov Smirnov (K-S) to compare the four fitted models. The statistics $A^*$and $W^*$are described in details in Chen and Balakrishnan[29]. In general, the smaller the values of these statistics, the better the fit to the data. The values of the MSEs and the (standard errors in parentheses) of the BTD distribution are tabulated in Table-3.

Table-4 lists the goodness-of-fits statistics from the fitted models.

Table-1: Bias and MSE (in parentheses) for the BTD distribution.

N	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
10	0.59774 (0.3699)	1.58172 (2.45394)	0.28385 (0.74282)	0.99315 (0.95073)	0.52758 (0.27723)	0.48437 (0.19760)
50	0.56691 (0.32996)	1.56640 (2.33733)	0.26411 (0.74088)	0.97876 (0.93128)	0.50392 (0.25178)	0.43705 (0.17474)
100	0.53730 (0.29927)	1.52470 (2.12749)	0.24767 (0.73360)	0.96240 (0.91643)	0.47900 (0.23003)	0.40951 (0.14274)
150	0.50747 (0.26021)	1.50350 (1.86316)	0.22686 (0.72288)	0.93603 (0.85263)	0.450840 (0.21341)	0.34336 (0.10895)
200	0.48217 (0.23885)	1.41240 (1.59375)	0.21728 (0.70516)	0.90756 (0.83905)	0.42394 (0.17540)	0.30415 (0.05317)
300	0.45248 (0.20631)	1.26325 (1.04578)	0.17219 (0.67550)	0.90536 (0.80763)	0.36477 (0.14473)	0.24381 (0.01413)
500	0.41266 (0.17291)	1.05070 (0.85182)	0.15470 (0.45734)	0.75054 (0.63469)	0.26533 (0.05470)	0.14452 (0.01144)

Table-2: Summary statistics for data set.

N	Mean	Median.	SD	Variance	Skewness	Kurtosis	Min.	Max.
128	9.3656	6.395	10.5083	110.425	3.2866	18.4831	0.08	79.05

Table-3: MLEs and standard errors (given in parentheses).

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
BTD	1.27124 (0.01176)	0.19211 (0.33037)	1.22086 (0.64109)	0.60101 (0.42042)	0.89641 (0.04386)	0.79715 (0.03202)
BD	0.94883 (0.08572)	1.34553 (0.47009)	0.05548 (0.79009)	- -	0.65840 (0.07601)	0.57668 (0.06433)
TD	1.16397 (0.10300)	2.70655 (0.64010)	0.97023 (0.95317)	0.58200 (0.46134)	- -	- -

D	1.08554 (0.15830)	0.19354 (0.71774)	1.10559 (1.12413)	- -	- -	- -

Table-4: Summary of Fits to bladder cancer patients data.

Model	-2ℓ	AIC	CAIC	BIC	HQIQ	A^*	W^*	K-S
BTD	959.256	969.256	969.748	983.516	975.050	0.35581	0.05719	0.02558
BD	975.705	987.705	988.399	1004.817	994.658	0.50680	0.07437	0.05687
TD	1015.059	1007.059	1006.734	995.651	1002.424	0.84440	0.09424	0.08520
D	1076.765	1082.765	1082.959	1091.321	1086.242	1.32960	0.15359	0.13810

Table-4 shows that the BTD distribution could be chosen as the best model among the fitted models since these models have the lowest values of the -2ℓ, AIC, CAIC, BIC, HQIQ, A^*, $W^*$and KS statistics.

Conclusion

We proposed a new six-parameter distribution called the beta transmuted Dagum (BTD) distribution that includes the beta Dagum, transmuted Dagum and Dagum distribution. The statistical properties of the BTD distribution including the moments, moment generating functions, quantile function and Stress-strength modelare given Simulation study was conducted to examine the performance of maximum likelihood estimators through the measures, biases and mean square of errors for the BTD distribution. We also present application of the BTD distribution to a real data set in order to illustrate the usefulness of the distribution.

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