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Revised Cramer's rule for solving linear systems

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Abstract

In this paper, it has been tried to revise the Cramer's rule for solving systems of linear equations and a new version, called revised Cramer's rule for solving linear systems is formulated. The revised Cramer's rule is formulated by starting with basic ideas of Cramer's rule and combining them with the transpose of the coefficient matrices. While Cramer's rule is based on column wise replacement of the coefficient matrix by the column vector of the right side constants, the revised Cramer's rule is based on the row replacement of the transpose of the coefficient matrix by the transpose of the column vector of the right side constants. The proof of the revised Cramer's rule for solving linear systems is also attempted and the working rule for the revised Cramer's rule is given. Numerical solution is obtained for the new version and its application to Electrical networks is incorporated. The result yielded that the revised Cramer's rule can be used for solving systems of linear equations as another method.

Keywords: Linear systems, revised, Cramer's rule, solving, systems of linear equations, working rules.

Introduction

It is often seen that data in business, Mathematics and science is arranged in rows and columns so as to obtain rectangular arrangement called matrix[1]. Matrix usually comes out as a table of numerical data which may start from physical and environmental observation; however it arises in diverse Mathematical settings as well[1].

Most of applied mathematics reduces to a set of linear equations called a linear system of the form $\mathbf{Ax} = \mathbf{b}$ with the coefficient matrix $\boldsymbol{A}$ and the vector $\mathbf{b}$ are given and the vector $\mathbf{x}$ is to be determined. Extensive set of algorithms have been developed for solving the linear systems. In this paper it is tried to revise the Cramer's rule for solving systems of linear equations and numerical example with application is solved.

Some Preliminary Concepts

A finite set of linear equations is said to be a system of linear equations or it is simply called a linear system[1] which is usually given in a matrix form as $\mathrm{Ax} = \mathrm{b}$ where A is linear system's coefficient matrix and b is column vector of the right side constants.

Theorem 1: If A is a square matrix of order n, then $\det(A^T) = \det(A)$[1-3].

Theorem 2: Assume that A is a square matrix. Then A is nonsingular if and only if $\det(A) \neq 0$[3,4].

Theorem 3: Let A and B be square matrices of the same order. Then $(AB)^T = B^T A^T$[4,5].

Theorem 4: For a non-singular matrix A of order η, we have A^T is non-singular and hence $(A^T)^{-1} = (A^{-1})^T$[1,6].

Proof: Since $\det(A^T) = \det(A) \neq 0$, , the matrix A^T is non-singular and hence $(A^T)^{-1}$ exists. Now let us consider $AA^{-1} = A^{-1}A = I_n$. By taking the transpose of both sides of $AA^{-1} = A^{-1}A = I_n$ we get

$(AA^{-1})^T = (A^{-1}A)^T = {I_n}^T = I_n$. This yields that

$$(AA^{-1})^T = (A^{-1})^T A^T = {I_n}^T = I_n$$

$$(A^{-1}A)^T = A^T (A^{-1})^T = {I_n}^T = I_n$$

Therefore, we obtain $(A^{-1})^T A^T = A^T (A^{-1})^T = I_n$ which shows that $(A^T)^{-1} = (A^{-1})^T$.

Theorem 5: If A is an invertible matrix of order n, then for each n × 1 matrix b, the system of equations Ax = b has exactly one solution which is given by $x = A^{-1}b$[1,2,3,7].

In the following theorem, the inverse of a non-singular matrix is used so as to formulate Cramer's rule. It helps us in finding solutions for linear systems given by Ax = b with n equations in n variables provided that A is nonsingular (or given that $\det(A) \neq 0$).

Theorem 6 (Cramer's Rule): For a linear system Ax = b with n equations in n variables, where: $\det(A) \neq 0$, there is unique solution which is given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)},$$

Where: A_i is obtained from A by substituting the i^{th} column of by A the column vector $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$[1,2,4,8].

Proposition: If $A = [a_{ij}]$ is a square matrix of order n, then $(\mathrm{adj}(A))^T = \mathrm{adj}(A^T)$[1,8,9,10].

Proof: If $A = [a_{ij}]$, then we have $A^T = [a_{ij}]^T = [a_{ji}]$. Now the cofactor associated with each entry a_{ij} of the matrix A is given by c_{ij} for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Thus, the Adjoint of A is given by $\mathrm{adj}(A) = [c_{ij}]^T = [c_{ji}]$. This shows that $(\mathrm{adj}(A))^T = [c_{ji}]^T = [c_{ij}]$.

Furthermore, the cofactor associated with each entry a_{ji} of the matrix A^T is given by c_{ji} for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Hence, the Adjoint of A^T is $\mathrm{adj}(A^T) = [c_{ji}]^T = [c_{ij}]$.

Therefore, we observe that $(\mathrm{adj}(A))^T = \mathrm{adj}(A^T)$.

The Revised Cramer's Rule

The revised Cramer's rule helps us in finding the solutions to linear systems given by $Ax = b$ with n equations in n variables provided that transpose of coefficient matrix A^T is nonsingular (or while $\det(A^T) \neq 0$).

The Revised Cramer's Rule: If $Ax = b$ is a system of n linear equations in n variables such that $\det(A^T) \neq 0$, then the system has a unique solution. This unique solution is given by $x_1 = \frac{\det(A_1)}{\det(A^T)}$, $x_2 = \frac{\det(A_2)}{\det(A^T)}$, …, $x_n = \frac{\det(A_n)}{\det(A^T)}$, where A_i is the matrix obtained from A^T by replacing the entries in the i^{th} row ofA^T by the entries in the matrix

$$b^T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}^T = [b_1 \quad b_2 \quad \ldots \quad b_n].$$

Proof: If $\det(A^T) \neq 0$, then A^T is invertible and the unique solution of $Ax = b$ can be given by $x^T = b^T(A^{-1})^T = b^T(A^T)^{-1}$.

Thus it follows that

$$x^T = b^T(A^T)^{-1}$$

$$= b^T\left(\frac{1}{\det(A^T)}\mathrm{adj}(A^T)\right)$$

$$= \frac{1}{\det(A^T)}[b_1 \quad b_2 \quad \ldots \quad b_n]\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{bmatrix}.$$

Now by multiplying the matrices we get

$$x^T = \frac{1}{\det(A^T)}[b_1c_{11} + b_2c_{21} + \cdots + b_nc_{n1} \quad b_1c_{12} + b_2c_{22} + \cdots + b_nc_{n2} \quad \ldots \quad b_1c_{1n} + b_2c_{2n} + \cdots + b_nc_{nn}]$$

Therefore, the entry in the k^{th} column of x^T is obtained to be

$$x_k = \frac{b_1c_{1k} + b_2c_{2k} + \cdots + b_nc_{nk}}{\det(A^T)}.$$

Now let $A_k = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1(k-1)} & a_{2(k-1)} & \ldots & a_{n(k-1)} \\ b_1 & b_2 & \ldots & b_n \\ a_{1(k+1)} & a_{2(k+1)} & \ldots & a_{n(k+1)} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{bmatrix}$.

Since A_k differs from A^T only in the k^{th} row, it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_k are the same as the cofactors of the corresponding entries in the k^{th} row of A^T. Therefore, the cofactor expansion of $\det(A_k)$ along the k^{th} row is given by

$$\det(A_k) = b_1c_{1k} + b_2c_{2k} + \cdots + b_nc_{nk}.$$

Substituting this result in to $x_k = \frac{b_1c_{1k}+b_2c_{2k}+\cdots+b_nc_{nk}}{\det(A^T)}$, we obtain $x_k = \frac{\det(A_k)}{\det(A^T)}$.

The working rules of revised Cramer's rule

The working rules/steps for solving linear systems by using the revised Cramer's rule are described as follows:

Step 1: Write the transpose of the coefficient matrix of the linear system and then check that $\det(A^T) \neq 0$.

Step 2: Find A_k by replacing the k^{th} row of A^T by b^T for each $k = 1, 2, 3, \ldots, n$.

Step 3: Compute $\det(A_k)$ for each $k = 1, 2, 3, \ldots, n$.

Step 4: Finally, use $x_k = \frac{\det(A_k)}{\det(A^T)}$ so as to obtain the solution of the given linear system for each $k = 1, 2, 3, \ldots, n$.

Example: Use the revised Rule to solve a linear system of equations given by $\begin{cases} x_1 + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30. \\ -x_1 - 2x_2 + 3x_3 = 8 \end{cases}$

Solution: In this Example, the coefficient matrix of the given system of linear equations is $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$.

Thus, we have $A^T = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 4 & -2 \\ 2 & 6 & 3 \end{bmatrix}$ and $b^T = [6 \quad 30 \quad 8]$.

Now $\det(A^T) = \begin{vmatrix} 1 & -3 & -1 \\ 0 & 4 & -2 \\ 2 & 6 & 3 \end{vmatrix} = 24 + 3(4) - (-8) = 44.$

Since $\det(A^T) = 44 \neq 0$, the system of linear equations has a unique solution which is given by $x_1 = \frac{\det(A_1)}{\det(A^T)}$, $x_2 = \frac{\det(A_2)}{\det(A^T)}$ and $x_3 = \frac{\det(A_3)}{\det(A^T)}$, where $A_1 = \begin{bmatrix} 6 & 30 & 8 \\ 0 & 4 & -2 \\ 2 & 6 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -3 & -1 \\ 6 & 30 & 8 \\ 2 & 6 & 3 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 4 & -2 \\ 6 & 30 & 8 \end{bmatrix}$.

Now $\det(A_1) = \begin{vmatrix} 6 & 30 & 8 \\ 0 & 4 & -2 \\ 2 & 6 & 3 \end{vmatrix} = 6(24) - 30(4) + 8(-8) = -40,$

$$\det(A_2) = \begin{vmatrix} 1 & -3 & -1 \\ 6 & 30 & 8 \\ 2 & 6 & 3 \end{vmatrix} = 42 + 3(2) + 24 = 72 \text{ and}$$

$$\det(A_3) = \begin{vmatrix} 1 & -3 & -1 \\ 0 & 4 & -2 \\ 6 & 30 & 8 \end{vmatrix} = 92 + 3(12) - (-24) = 152.$$

Therefore, it follows that

$$x_1 = \frac{\det(A_1)}{\det(A^T)} = \frac{-40}{44} = -\frac{10}{11}, \; x_2 = \frac{\det(A_2)}{\det(A^T)} = \frac{72}{44} = \frac{18}{11} \text{ and}$$

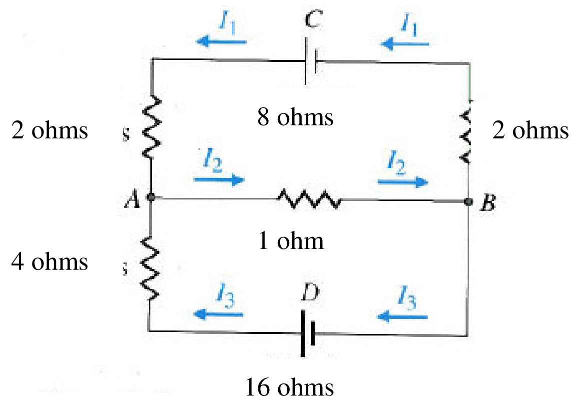
$$x_3 = \frac{\det(A_3)}{\det(A^T)} = \frac{152}{44} = -\frac{38}{11}.$$

This shows that $x_1 = -\frac{10}{11}$, $x_2 = \frac{18}{11}$ and $x_3 = -\frac{38}{11}$ is the solution of the given linear system.

Application to Electrical networks

Electrical networks are types of networks where analysis is usually made[1]. An analysis of such systems uses two properties of electrical networks called Kirchhoff's Laws mentioned below: i. All the current flowing into a junction must flow out of it. ii. The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage in the path.

Example: Find the Currents I_1, I_2 and I_3 in the electrical network given below.



There are two batteries and four resistors in this electrical network. Moreover, Current I_1 flows through the top branch BCA, Current I_2 flows across the middle branch AB and Current I_3 flows through the top branch BDA. By the Current law we get $I_1 - I_2 + I_3 = 0$ at node A. We also obtain the same equation at node B.

By applying the Voltage law we obtain the following equations. For circuit CABC, the voltage drops at the resistors $2I_1$, I_2 and $2I_1$. Hence we obtain the equation $4I_1 + I_2 = 8$.

Similarly, for circuit DABD we get $I_2 + 4I_3 = 16$.

If we go against the flow, we observe that there is actually a 3rd circuit CADBC. Here we treat the voltages and the resistances on the reversed paths to be negative. By doing this we obtain $2I_1 + 2I_1 - 4I_3 = 8 - 16$ or $4I_1 - 4I_3 = -8$ which is the difference of the voltage equations for the other two circuits. So this equation can be omitted since it does not contribute any new information.

Thus, we get a system of three linear equations in three variables given by

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 4I_1 + I_2 = 8 \\ I_2 + 4I_3 = 16 \end{cases}.$$

Now we apply the revised Cramer's Rule for solving this linear system. Here the coefficient matrix of the system is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 8 \\ 16 \end{bmatrix}.$$

Thus, we have $A^T = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 4 \end{bmatrix}$ and $b^T = [0 \quad 8 \quad 16]$.

Now $\det(A^T) = \begin{vmatrix} 1 & 4 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 4 \end{vmatrix} = (4 - 0) - 4(-4 - 1) + 0(0 - 1) = 24.$

Since $\det(A^T) = 96 \neq 0$, the system of linear equations has a unique solution which is given by $x_1 = \frac{\det(A_1)}{\det(A^T)}$, $x_2 = \frac{\det(A_2)}{\det(A^T)}$ and $x_3 = \frac{\det(A_3)}{\det(A^T)}$, where $A_1 = \begin{bmatrix} 0 & 8 & 16 \\ -1 & 1 & 1 \\ 1 & 0 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 8 & 16 \\ 1 & 0 & 4 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 1 & 1 \\ 0 & 8 & 16 \end{bmatrix}$.

Now $\det(A_1) = \begin{vmatrix} 0 & 8 & 16 \\ -1 & 1 & 1 \\ 1 & 0 & 4 \end{vmatrix} = 0(4-0) - 8(-4-1) + 16(0-1) = 24$,

$\det(A_2) = \begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 16 \\ 1 & 0 & 4 \end{vmatrix} = (32-0) - 4(0-16) + 0(0-8) = 96$ and

$\det(A_3) = \begin{vmatrix} 1 & 4 & 0 \\ -1 & 1 & 1 \\ 0 & 8 & 16 \end{vmatrix} = (16-8) - 4(-16-0) + 0(-8-0) = 72$.

Therefore, it follows that

$x_1 = \frac{\det(A_1)}{\det(A^T)} = \frac{24}{24} = 1$, $x_2 = \frac{\det(A_2)}{\det(A^T)} = \frac{96}{24} = 4$ and $x_3 = \frac{\det(A_3)}{\det(A^T)} = \frac{72}{24} = 3$.

This shows that $x_1 = 1$, $x_2 = 4$ and $x_3 = 3$ is the solution of the given linear system.

Conclusion

In this paper, the Cramer's rule for solving linear systems is revised and a new version named revised Cramer's rule is formulated. This revised Cramer's rule is investigated by starting with basic ideas of Cramer's rule and combining with the transpose of the linear system's coefficient matrix. Proof of revised Cramer's rule is also attempted and numerical solution is obtained for conclusion. The result showed that the revised Cramer's rule for solving linear systems can be used for solving systems of linear equations as another additional solution method. Furthermore, the working rule for the revised Cramer's rule is given and its application to Electrical networks has been added.

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