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Numerical Solution of Three-Parameter Eigenvalue Problems Using Kronecker Product Method

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Abstract

This paper discusses the decouple of three- parameter Eigenvalue problems in matrix form using Kronecker product and the implications of using this method.

Keywords: Multiparameter, Eigenvalue, Eigenvector, Kronecker product.

Introduction

Multiparameter Eigenvalue problems are generalization of one-parameter Eigenvalue problems and occur quite naturally when the method of separation of variables is applied to certain boundary value problems associated with partial differential equations. Numerical solution of MEP for matrices arises in the discretization of Multiparameter Sturm-Liouville Eigenvalue problems in ordinary differential equations[1]. Since Eigenvalues and eigenvectors give valuable information about the behavior and properties of a matrix; therefore research works on matrix Eigenvalue problems has its immense significance. Matrix Eigenvalue problems[2] involving several parameters comes from a large number of areas, such as chemistry, mechanics, dynamical systems, Markov chains, magneto-hydrodynamics, oceanography and economics.

Typical examples are provided, for example by a vibrating membrane[3] and a dynamical problem of homogeneous beam loaded by a vertical load[4].

In recent years the theory of multiparameter Eigenvalue problems has been investigated by many authors, and as number of works treats the subjects. However, few papers consider the numerical treatment of such problems. Examples are by Binding and Browne[5] which describes some variational methods for the solution of multiparameter Eigenvalue problems, Blum and Chang which uses a generalization of Rayleigh quotient iterative method, and Browne and Sleeman which uses gradient method, etc.

Three-parameter Eigenvalue problem and its reduction to a system of one-parameter problems: A three-parameter Eigenvalue problems in matrix form is as follows

$$A_1 x = \lambda B_{11} x + \mu B_{12} x + \vartheta B_{13} x$$
$$A_2 y = \lambda B_{21} y + \mu B_{22} y + \vartheta B_{23} y$$
$$A_3 z = \lambda B_{31} z + \mu B_{32} z + \vartheta B_{33} z \tag{1}$$

Where $\lambda, \mu, \vartheta \in \mathbb{C}$, and

$x \in \mathbb{C}^n \setminus \{0\}, \mathrm{A}_1, B_{11}, B_{12}, B_{13} \in \mathbb{C}^{n\times n}$

$y \in \mathbb{C}^m \setminus \{0\}, \mathrm{A}_2, B_{21}, B_{22}, B_{23} \in \mathbb{C}^{m\times m}$

$z \in \mathbb{C}^p \setminus \{0\}, \mathrm{A}_3, B_{31}, B_{32}, B_{33} \in \mathbb{C}^{p\times p}$

Where λ, μ, ϑ are called the Eigenvalues and x, y, z are called eigenvectors of the problem.

Problem (1) can be reduced to a system of three one-parameter problems:

$$\Delta_1 u = \lambda \Delta_0 u$$
$$\Delta_2 u = \mu \Delta_0 u \tag{2}$$
$$\Delta_3 u = \vartheta \Delta_0 u$$

Where: $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ are $(\mathrm{mnp})\times(\mathrm{mnp})$ dimensional matrices defined as

$$\begin{aligned}\Delta_0 = B_{11} \otimes B_{22} \otimes B_{33} - B_{11} \otimes B_{23} \otimes B_{32}\\ + B_{12} \otimes B_{23} \otimes B_{31} - B_{12} \otimes B_{21} \otimes B_{33}\\ + B_{13} \otimes B_{21} \otimes B_{32} - B_{13} \otimes B_{22} \otimes B_{31}\end{aligned} \tag{3}$$

$$\begin{aligned}\Delta_1 = A_1 \otimes B_{22} \otimes B_{33} - A_1 \otimes B_{23} \otimes B_{32} + B_{12} \otimes B_{23} \otimes A_3 - B_{12} \otimes A_2 \otimes B_{33}\\ + B_{13} \otimes A_2 \otimes B_{32} - B_{13} \otimes B_{22} \otimes A_3\end{aligned} \tag{4}$$

$$\begin{aligned}\Delta_2 = B_{11} \otimes A_2 \otimes B_{33} - B_{11} \otimes B_{23} \otimes A_3 + A_1 \otimes B_{23} \otimes B_{31} - A_1 \otimes B_{21} \otimes B_{33}\\ + B_{13} \otimes B_{21} \otimes A_3 - B_{13} \otimes A_2 \otimes B_{31}\end{aligned} \tag{5}$$

$$\Delta_3 = B_{11} \otimes B_{22} \otimes A_3 - B_{11} \otimes A_2 \otimes B_{32} + B_{12} \otimes A_2 \otimes B_{31} - B_{12} \otimes B_{21} \otimes A_3$$
$$+A_1 \otimes B_{21} \otimes B_{32} - A_1 \otimes B_{22} \otimes B_{31} \quad (6)$$

And
$u = x \otimes y \otimes z$
Where $\otimes$ is the Kronecker product of two matrices discussed in (1.3).

Theorem 1.2.3: Let $(\lambda, \mu, \vartheta)$ be an Eigenvalue and $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ a corresponding eigenvector of the system (1) then $(\lambda, \mu, \vartheta)$ is an Eigenvalue of the system (2) and $u = x \otimes y \otimes z$ is the corresponding eigenvector.

The Kronecker Product

Definition 1.3.1: The Kronecker product $(.\otimes.): \mathbb{C}^{m\times n} \times \mathbb{C}^{p\times q} \to \mathbb{C}^{mp\times nq}$ is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$$

Where $(A)_{ij} = a_{ij}$

Model problem

Here we shall construct a model problem. For simplicity we will consider a diagonal matrix.

Consider the following system of equations

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \vartheta \begin{pmatrix} 7 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} 8 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \mu \begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \vartheta \begin{pmatrix} 14 & 0 \\ 0 & 13 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} 30 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mu \begin{pmatrix} 75 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \vartheta \begin{pmatrix} 57 & 0 \\ 0 & 14 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

We have

$$x_1 = (3\lambda + 5\mu + 7\vartheta)\,\mathrm{x}_1$$
$$2x_2 = (4\lambda + 6\mu + 8\vartheta)\,\mathrm{x}_2$$
$$y_1 = (8\lambda + 10\mu + 14\vartheta)\,\mathrm{y}_1$$
$$y_2 = (9\lambda + 15\mu + 13\vartheta)\,\mathrm{y}_2$$
$$3z_1 = (30\lambda + 75\mu + 57\vartheta)\,\mathrm{z}_1$$
$$2z_2 = (6\lambda + 16\mu + 14\vartheta)\,\mathrm{z}_2$$

So, $x_1 \neq 0 \Rightarrow 3\lambda + 5\mu + 7\vartheta = 1$ (7)

$$x_2 \neq 0 \Rightarrow 2\lambda + 3\mu + 4\vartheta = 1 \quad (8)$$
$$y_1 \neq 0 \Rightarrow 4\lambda + 5\mu + 7\vartheta = 1 \quad (9)$$
$$y_2 \neq 0 \Rightarrow 9\lambda + 15\mu + 13\vartheta = 1 \quad (10)$$
$$z_1 \neq 0 \Rightarrow 10\lambda + 25\mu + 19\vartheta = 1 \quad (11)$$
$$z_2 \neq 0 \Rightarrow 3\lambda + 8\mu + 7\vartheta = 1 \quad (12)$$

Note $x_i \neq 0, y_i \neq 0 \Rightarrow x_j = 0, y_j = 0$ for $i \neq j$, else $z_1 = 0, \& z_2 = 0$ so we would have z=0.

Similarly $y_i \neq 0, z_i \neq 0 \Rightarrow y_j = 0, \mathrm{z}_j = 0$
and
$z_i \neq 0, x_i \neq 0 \Rightarrow z_j = x_j = 0$

$x_i \neq 0, \mathrm{y}_i \neq 0, \mathrm{z}_i \neq 0 \Rightarrow x_j = y_j = z_j = 0$

Without loss of generality we consider x_i, y_i and $z_i = 1$ in the corresponding eigenvectors. Hence we have.

$$x, y, z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (13)$$
$$x, y, z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (14)$$
$$x, y, z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (15)$$
$$x, y, z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (16)$$
$$x, y, z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (17)$$
$$x, y, z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (18)$$
$$x, y, z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (19)$$
$$x, y, z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (20)$$

So in case (13) we have the equations (7), (9), (11). Hence these imply that

$$\lambda = 0, \mu = -\frac{3}{20}, \vartheta = \frac{1}{4}$$

In case of (14) equations (7), (9), (12) correspond to an Eigenvalue $(\lambda, \mu, \vartheta) = (0, 0, \frac{1}{7})$.

In case of (15) equations (7), (10), (11) correspond to an Eigenvalue $(\lambda, \mu, \vartheta) = (0, -\frac{3}{20}, \frac{1}{4})$.

In case of (16) equations (8), (9), (11) correspond to an Eigenvalue $(\lambda, \mu, \vartheta) = (-\frac{18}{11}, -\frac{3}{11}, \frac{14}{11})$

In case of (17) equations (7), (10), (12) correspond to an Eigenvalue $(\lambda, \mu, \vartheta) = (-\frac{1}{4}, 0, \frac{1}{4})$.

In case of (18) equations (8), (10), (11) correspond to an Eigenvalue $(\lambda, \mu, \vartheta) = (-\frac{18}{97}, -\frac{33}{97}, \frac{58}{97})$.

In case of (19) equations (8), (9), (12) correspond to an Eigenvalue $(\lambda, \mu, \vartheta) = (-\frac{9}{5}, -\frac{3}{5}, \frac{8}{5})$

In case of (20) equations (8), (10), (12) correspond to an Eigenvalue $(\lambda, \mu, \vartheta) = (-\frac{9}{38}, -\frac{6}{19}, \frac{23}{38})$.

Numerical Example

In this section we shall use the previously constructed model example from 2.1 and show what the corresponding Kronecker product form is.

Here we consider

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} B_{11} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} B_{12} = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix} B_{13} = \begin{pmatrix} 7 & 0 \\ 0 & 8 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} B_{21} = \begin{pmatrix} 8 & 0 \\ 0 & 9 \end{pmatrix} B_{22} = \begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix} B_{23} = \begin{pmatrix} 14 & 0 \\ 0 & 13 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} B_{31} = \begin{pmatrix} 30 & 0 \\ 0 & 6 \end{pmatrix} B_{32} = \begin{pmatrix} 75 & 0 \\ 0 & 16 \end{pmatrix} \quad B_{33} = \begin{pmatrix} 57 & 0 \\ 0 & 14 \end{pmatrix}$$

So using the formula for Δ_0 given by equation (3), we have

$$\Delta_0 = \begin{pmatrix} 480 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 84 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 144 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 264 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 40 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 582 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 152 \end{pmatrix}$$

Similarly, using (4) we get

$$\Delta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -432 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -72 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -108 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -36 \end{pmatrix}$$

Using (5), we get

$$\Delta_2 = \begin{pmatrix} -72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -90 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -72 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -198 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -48 \end{pmatrix}$$

And using (6), we get

$$\Delta_3 = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 150 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 336 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 348 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 92 \end{pmatrix}$$

So in Kronecker product system it can be written as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -432 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -72 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -108 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -36 \end{pmatrix} u = \lambda \begin{pmatrix} 480 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 84 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 144 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 264 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 40 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 582 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 152 \end{pmatrix} u \qquad (21)$$

$$\begin{pmatrix} -72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -90 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -72 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -198 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -48 \end{pmatrix} u = \mu \begin{pmatrix} 480 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 84 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 144 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 264 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 40 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 582 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 152 \end{pmatrix} u \qquad (22)$$

$$\begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 150 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 336 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 348 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 92 \end{pmatrix} u = \vartheta \begin{pmatrix} 480 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 84 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 144 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 264 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 40 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 582 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 152 \end{pmatrix} u \qquad (23)$$

From the Eigenvalues of (2.1) we have

(13) corresponds to the eigenvector $u=(1,0,0,0,0,0,0,0)^T$ and Eigenvalue (0,-0.15,0.25)

(14) corresponds to the eigenvector $u=(0,1,0,0,0,0,0,0)^T$ and Eigenvalue (0,0,0.1428)

(15) corresponds to the eigenvector $u=(0,0,1,0,0,0,0,0)^T$ and Eigenvalue (0, -0.15,0.25).

(16) corresponds to the eigenvector $u=(0,0,0,0,1,0,0,0)^T$ and Eigenvalue (-1.64, -.27, 1.27)

(17) corresponds to the eigenvector $u=(0,0,0,1,0,0,0,0)^T$ and Eigenvalue (-.25,0,.25).

(18) corresponds to the eigenvector $u=(0,0,0,0,0,0,1,0)^T$ and Eigenvalue (-.18,-.34,.59).

(19) corresponds to the eigenvector $u=(0,0,0,0,0,1,0,0)^T$ and Eigenvalue (-1.8, -.6, 1.6).

(20) corresponds to the eigenvector $u=(0,0,0,0,0,0,0,1)^T$ and Eigenvalue (-.23,-.31,.60).

Conclusion

From the above it can be conclude that the Eigenvalue obtained from Kronecker product form correspond exactly with the Eigen values given in the model problem. In case of some three parameter Eigenvalue problems it is not easy or impossible to find out the Eigenvalues, in that case Kronecker product form will definitely help us to find out the Eigenvalues.

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