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A Comparative Study on Super-Saturated Designs

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Abstract

Supersaturated Design is a factorial design having the more number of factors when compared with the number of design points. Several methods for constructing and analyzing two, multi or mixed level supersaturated designs have been proposed in recent literature. This paper presents a review of the available literature on supersaturated designs and its construction. Each method is illustrated with suitable example.

Keywords: Placket and Burman designs, Factorial designs, BIBD, Orthogonal designs, $E(s^2)$- optimality.

Introduction

In most practical situations of the Design and Analysis of Experiments, it is observed that out of a large number of potential factors, relatively few of the factors are actually effective. Such effects are called the effects of sparsity. The basic approach is to identify these few factors in an efficient way. In this situation, knowledge of each and every main effect is not useful since insignificant factors are not usually of interest.

Hence, the experimenter has to minimize the number of design points to identify the active factors for efficient utilization of resources and minimization of cost and time. If, the number of design points is more, then the problem of reduction of dimensionality occurs. In this aspect, instead of reduction of design points, some optimum design constructions may be proposed. In these situations, a super-saturated design plays a key role which reduces the experimental cost, time significantly.

Literature Review on Super-Saturated Designs

Satterthwaite[1] initially made an attempt to construct saturated designs randomly and suggested random balance designs. Booth and Cox[2] initially proposed a systematic method for the construction of super-saturated designs, which are factorial designs in which the number of factors exceeds the number of design points and also computed $E(s^2)$ criterion. After Booth and Cox[2], no attempts were made till Lin[3]. Later Several researchers made attempts on the construction of super-saturated designs with their $E(s^2)$ optimality. Authors who worked in this direction are: Lin[3], Nguyen[4] , Tang and Wu[5], Deng, Lin and Wang[6] , Fang, Lin and Ma[7], Liu and Zhang[8], Lu and Sun[9], Butler[10], and Yamada and Lin[11], Liu and Liu[12], Liu and Dean[13], Fang, Ma and Liu[14], Aggarwal and Gupta[15], Xu and Wu[16], Koukouvinov, Mantos and Mylona[17], Jones, Lin and Nachtsheim[18] , Nguyen and Cheng[19], Sun, Lin and Liu[20] etc.

In this paper, detailed procedures for the construction of super-saturated designs proposed by different authors are presented. Each method is illustrated with suitable example.

Methods of Construction of Super-Saturated Designs

Method 3.1: This method gives a systematic super-saturated design constructed by Booth & Cox[2]. Consider a trial set of vectors and compute the maximum cross-product ‘m’. Usually, several pairs of vectors have this maximum product. Select a final pair of vector for improvement. The maximum cross-product and the number of pairs having this are then punched out.

Consider, a suitable trial vector (i.e. one having an equal number of +1’s and -1’s), then calculate the cross product and substitute this for each vector of the pair in turn, if, this substitution produces a reduction in the cross-product of the pair, then check the cross-product of the new vector with all other members of the set ‘m’, equality only being admitted if the original pair of vectors also had cross-product ‘m’. If, these conditions are satisfied, the trial vector is substituted permanently and its value and location in the set is punched out. Return to the starting point and recalculate the maximum cross-product, etc. If, the conditions are not satisfied, a new trial vector is calculated and the process is repeated until an improvement is affected. Punch out the complete new set of vectors in a form suitable for re-input. This method is illustrated in the example 3.1.

Example 3.1: Consider the supersaturated design with 16 factors and 12 design points which is presented in Table-1.

Method 3.2: This method of construction was proposed by Wu[21] using Hadamard matrix. An Hadamard matrix is an N x N orthogonal matrix of +1 and -1, where N is multiple of four. One of its columns consists of all +1's. This column is removed from the matrix resulting in an N x (N-1) matrix. Denote the N-1 columns by $C_1, C_2, C_3, \ldots, C_{N-1}$. A super-saturated design X with N factors and N-1 design points can be constructed with elements as C_{ij} where, interaction column C_{ij} is defined as the entry-wise product of C_i and C_j. This method is illustrated in the example 3.2.

Example 3.2: Consider a Hadamard matrix of order 12 x 12. The resulting super-saturated design is presented in Table-2.

Table-1
Factors

Design Points ↓	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	-1	-1	-1	-1	-1
2	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1	-1	-1	-1	-1	-1	-1
3	-1	+1	+1	+1	-1	-1	-1	+1	-1	-1	+1	+1	+1	-1	+1	+1
4	+1	+1	+1	-1	-1	-1	+1	-1	-1	+1	-1	-1	+1	+1	+1	+1
5	+1	+1	-1	-1	-1	+1	-1	-1	+1	-1	+1	+1	+1	-1	+1	-1
6	+1	-1	-1	-1	+1	-1	-1	+1	-1	+1	+1	+1	+1	+1	-1	+1
7	-1	-1	-1	+1	-1	-1	+1	-1	+1	+1	+1	+1	-1	+1	+1	+1
8	-1	-1	+1	-1	-1	+1	-1	+1	+1	+1	-1	+1	-1	+1	-1	+1
9	-1	+1	-1	-1	+1	-1	+1	+1	+1	-1	-1	+1	+1	+1	+1	-1
10	+1	-1	-1	+1	-1	+1	+1	+1	-1	-1	-1	-1	+1	+1	-1	-1
11	-1	-1	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1	-1	-1	+1	+1
12	-1	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1	-1	-1	-1	-1	-1

For the above design, $E(s^2)=7.06$

Table-2
Design Points

Factors											
↓	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	+1	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1
2	-1	+1	+1	-1	+1	+1	+1	-1	-1	-1	+1
3	+1	-1	+1	+1	-1	+1	+1	+1	-1	-1	-1
4	-1	+1	-1	+1	+1	-1	+1	+1	+1	-1	-1
5	-1	-1	+1	-1	+1	+1	-1	+1	+1	+1	-1
6	-1	-1	-1	+1	-1	+1	+1	-1	+1	+1	+1
7	+1	-1	-1	-1	+1	-1	+1	+1	-1	+1	+1
8	+1	+1	-1	-1	-1	+1	-1	+1	+1	-1	+1
9	+1	+1	+1	-1	-1	-1	+1	-1	+1	+1	-1
10	-1	+1	+1	+1	-1	-1	-1	+1	-1	+1	+1
11	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1	+1
12	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

For the above design, $E(s^2)=1$

Method 3.3: Lin[3] proposed the construction of super-saturated designs through Placket and Burman designs. The detailed method is presented below.

Step 1: Consider a Plackett and Burman design matrix of order ‘n’ and split the design matrix into two half fractions according to a specific branching column with signs equal +1 or -1.

Step 2: Select the rows that have +1 in the branching column. The remaining (n-2) columns other than the branching column will form a super-saturated design with n/2 design points in (n-2).

This method is illustrated in the example 3.3.

Example 3.3: Consider the Placket and Burman designs of order 12 as in Table-3.

Let, the branching column selected is column 11. Then, split the 12 rows into two groups, such that the first group (Group-I) contains the rows with the plus (+) sign in column 11 i.e., rows with numbers {2, 3, 5, 6, 7, 11} and the other group (Group-II) contains the rows with minus sign in column 11 i.e., rows with numbers {1, 4, 8, 9, 10,12}. Then, delete the column 11 from Group-I and Group II, which results in two super-saturated designs with 6 design points in 10 factors. The resulting super-saturated designs are

The Super-saturated design from Group-I: {2, 3, 5, 6, 7, 11} is (Table-4).

The Super-saturated design from Group-II: {1, 4, 8, 9, 10, 12} is (Table-5).

Table-3
Placket and Burman designs of order 12

	I	1	2	3	4	5	6	7	8	9	10	11
1	+1	+1	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1
2	+1	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1	+1
3	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1	+1	+1
4	+1	+1	+1	+1	-1	-1	-1	+1	-1	+1	+1	-1
5	+1	+1	+1	-1	-1	-1	+1	-1	+1	+1	-1	+1
6	+1	+1	-1	-1	-1	+1	-1	+1	+1	-1	+1	+1
7	+1	-1	-1	-1	+1	-1	+1	+1	-1	+1	+1	+1
8	+1	-1	-1	+1	-1	+1	+1	-1	+1	+1	+1	-1
9	+1	-1	+1	-1	+1	+1	-1	+1	+1	+1	-1	-1
10	+1	+1	-1	+1	+1	-1	+1	+1	+1	-1	-1	-1
11	+1	-1	+1	+1	-1	+1	+1	+1	-1	-1	-1	+1
12	+1	+1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

Table-4
Super-saturated design from Group-I

	I	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	+1	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1
3	+1	-1	+1	+1	+1	-1	-1	-1	+1	-1	+1
5	+1	+1	+1	-1	-1	-1	+1	-1	+1	+1	-1
6	+1	+1	-1	-1	-1	+1	-1	+1	+1	-1	+1
7	+1	-1	-1	-1	+1	-1	+1	+1	-1	+1	+1
11	+1	-1	+1	+1	-1	+1	+1	+1	-1	-1	-1

The $E(s^2)$ value of the design is 3.22.

Table-5
Super-saturated design from Group-II

	I	1	2	3	4	5	6	7	8	9	10
1	+1	+1	+1	-1	+1	+1	+1	-1	-1	-1	+1
4	+1	+1	+1	+1	-1	-1	-1	+1	-1	+1	+1
8	+1	-1	-1	+1	-1	+1	+1	-1	+1	+1	+1
9	+1	-1	+1	-1	+1	+1	-1	+1	+1	+1	-1
10	+1	+1	-1	+1	+1	-1	+1	+1	+1	-1	-1
12	+1	+1	-1	-1	-1	-1	-1	-1	-1	-1	-1

For the above design is $E(s^2) = 3.490$

Method 3.4: Lin[22] proposed method for generating the Systematic Super-saturated Designs which is presented below

Step 1: Let n be the number of design points to be constructed for the super-saturated design. If, n is even, then consider half of n values as +1 and another half of n values as -1. If, n is odd, then consider half of (n+1)/2 values as +1 and another half of (n-1) /2 values as -1.

Step 2: Permute these signs in all possible ways.

Step 3: Specify the value of correlation coefficient (r) for checking the orhogonality. Then, measure the degree of non-orthogonality between two columns C_i and C_j with their correlation $r_{ij} = (C_i' C_j) / n$

Step 4: Compare each column randomly with first column and calculate correlation coefficient value(r). If, the maximum correlation is less than the pre-specified correlation coefficient value(r) then go to next column. Otherwise, drop the selected column.

Step 5: Pick up the columns that are most nearly to orthogonal. These columns form a super-saturated design.

This method is illustrated in the example 3.3.4.

Example 3.4: If the number of design points chosen is 12, then the number of factors is 66. Choose specified correlation as $r \leq 0.33$.The corresponding design X'X is evaluated and its $E(s^2)$ value is $E(s^2)=10.3$

Method 3.5: Nguyen[4] (1996) proposed the construction of super-saturated designs from BIBD for a series of parameters v=2t-1, b=4t-2, r=2t-2, k=t-1 and λ=t-2. The detailed method is presented below.

Step 1: Construct a BIBD with parameters v=2t-1, b=4t-2, r=2t-2, k=t-1 and λ=t-2 using two initial blocks and assign +1's to all the treatments in the initial block and assign -1's to all other treatments.

Step 3: Generate the first (2t-1) columns by cyclic permutation from the first initial block and generate the remaining (2t-1) columns from the second initial block

Step 3: Augment one principal row I with all +1's. The resulting design will be a super-saturated design with 2t design points and (4t-2) treatments.
The method is illustrated through the example 3.5.

Example 3.5: Consider a BIBD with parameters v=7, b=14, r=6, k=3 and λ=2 and initial blocks for this design are (2, 3, 7) and (2, 3, 5). Generate the columns 2 to 7 from the first column i.e. first initial block and generate the columns 9 to 14 from the 8^{th} column i.e. second initial block. Obtain the super-saturated design by augmenting one row with all +1's as in Table-6

Method 3.6: Liu and Zhang[8] proposed a method to generate a super-saturated design using cyclic BIBD. The procedure is presented below.

Step 1: Consider two initial blocks of size k with v treatments to construct a Cyclic BIBD. Assign +1's to all the treatments in the initial blocks and assign -1's to all other treatments.

Step 2: Permute these initial blocks to generate an array of (2*v* – 1) columns.

Step 3: Augment a principal block containing all +1's to generate a super saturated design with (v+1) design points in 2v factors.

This method is illustrated in the example 3.6.

Example 3.6: Generate a cyclic BIBD with parameters v=5, b=10, r=2, k=2 and λ=1 through two initial blocks as (1, 2) and (1, 3). Assign +1 and –1 signs for the treatments of the two initial block vectors as (+1, +1, -1 , -1 , -1) and (+1, -1 , +1, -1 , -1) . A super-saturated design of size (6, 10) obtained using the above procedure is given in Table-7.

Table-6
Obtain the super-saturated design

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
I	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1
1	-1	+1	-1	-1	-1	+1	+1	-1	-1	-1	+1	-1	+1	+1
2	+1	-1	+1	-1	-1	-1	+1	+1	-1	-1	-1	+1	-1	+1
3	+1	+1	-1	+1	-1	-1	-1	+1	+1	-1	-1	-1	+1	-1
4	-1	+1	+1	-1	+1	-1	-1	-1	+1	+1	-1	-1	-1	+1
5	-1	-1	+1	+1	-1	+1	-1	+1	-1	+1	+1	-1	-1	- 1
6	-1	-1	-1	+1	+1	-1	+1	-1	+1	-1	+1	+1	-1	-1
7	+1	-1	-1	-1	+1	+1	-1	-1	-1	+1	-1	+1	+1	-1

With $E(s^2) = 4.923$.

Table-7
A super-saturated design of size (6, 10) obtained

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
I	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1
1	+1	-1	-1	-1	+1	+1	-1	-1	+1	-1
2	+1	+1	-1	-1	-1	-1	+1	-1	-1	+1
3	-1	+1	+1	-1	-1	+1	-1	+1	-1	-1
4	-1	-1	+1	+1	-1	-1	+1	-1	+1	-1
5	-1	-1	-1	+1	+1	-1	-1	+1	-1	+1

With $E(s^2) = 4$.

Method 3.7: Fang, Gennian and M Q[24] (2002) proposed a method of construction from resolvable BIBD. The procedure is presented below

Step 1: Construct an α-resolvable BIBD with parameters v, b, r, k, λ. and assign the numbers to the blocks as {1, 2, … b}.

Step 3: Construct a parallel class P_j, (j= 1, 2…m) containing the α- blocks from the α-resolvable BIBD.

Step 4: Each parallel class P_j, consisting α- blocks, is a super-saturated design with α- levels for v factors and r design points, corresponding to each factor k, with $x_{kj}=u$, if P_j containing the K^{th} factor is in the u^{th} block. Obtain such a super-saturated design with α-levels.

This method is illustrated through the example 3.7.

Example 3.7
Consider a resolvable BIBD with parameters v=10, b=45, r=9, k=2, λ=1. with α=10/2=5. (Table-8)

A α – level factorial design can be constructed by transforming the α-resolvable BIBD with r factors and v design points such that each block is transformed into a level and treatments in each block are transformed into design points then we get required super-saturated design. The super-saturated design of order $S(10, 5^9)$ is (Table-9).

Table-8
Parallel Classes

Block No.	Parallel Classes								
	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
1	{1,10}	{2,10}	{4,9}	{3,7}	{2,8}	{5,7}	{5,6}	{1,7}	{1,6}
2	{8,9}	{5,8}	{3,10}	{4,10}	{6,9}	{2,4}	{3,4}	{2,5}	{2,7}
3	{4,5}	{3,6}	{7,8}	{1,2}	{5,10}	{1,9}	{1,8}	{4,6}	{4,8}
4	{6,7}	{7,9}	{2,6}	{5,9}	{1,3}	{3,8}	{7,10}	{3,9}	{3,5}
5	{2,3}	{1,4}	{1,5}	{6,8}	{4,7}	{6,10}	{2,9}	{8,10}	{9,10}

Table-9
Factors

	Factors								
Design Points ↓	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	1	5	5	3	4	3	3	1	1
2	5	1	4	3	1	2	5	2	2
3	5	3	2	1	4	4	2	4	4
4	3	5	1	2	5	2	2	3	3
5	3	2	5	4	3	1	1	2	4
6	4	3	4	5	2	5	1	3	1
7	4	4	3	1	5	1	4	1	2
8	2	2	3	5	1	4	3	5	3
9	2	4	1	4	2	3	5	4	5
10	1	1	2	2	3	5	4	5	5

With $E(s^2) = 6097.97$

Method 3.8: Xuan Lu, Wenbino Hu and Yan Zhen[25] (2003) proposed a multilevel super-saturated design using α-resolvable BIBD. The detailed steps of construction are presented below with example.

Step1: Consider an α-resolvable BIBD with parameters v, b, r, k and λ and let $v = \alpha t$.

Step 2: Construct an α –level factorial design by transforming the α-resolvable BIBD with r factors, v design points such that each block is transformed into a level and treatments in each block are transformed in to design points.

Step 3: Compute the s_{ij}^2 values as $s_{ij}^2 = \text{Trace}(N_{ij}N_{ij}^T) - t^2$ and $N_{ij} = (Z_i^T Z_j)$ where Z is the incidence matrix of resolvable BIBD.

Step 4: Consider the 'm' permutation matrices $P_1, P_2, \dots, P_m$ of order 'n'. Construct $X_i = P_{i-1} X_1$ for i=1,2…,m+1. Combine X_i's together to obtain a super-saturated design with (m+1) r columns as $X = (X_1\ X_2\ \dots, X_{m+1})$ where, X is the incidence matrix of α-factorial design.

Step 5: For the given $k\alpha = v$ and r, choose r^* so that $r^* > r$ and r^* satisfying the condition $r^*(k-1) = \lambda(v-1)$. An optimal super-saturated design X^* of k^* columns can be constructed in the first three steps. Then, delete r^*-r columns from X^* to obtain a super-saturated design of r columns. $E(s^2)$ is given as

$$E(s^2) = \frac{[trace\{(Z^T Z)^2\} - tnk]}{k(k-1)}$$

This method is illustrated in the example 3.8.

Example 3.8: Consider an α-resolvable BIBD with parameters v=6, b=15, r=5, k=2, λ=1 and α = 3 and t =2. Let Z be its incidence matrix

$$Z = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$N = Z^T Z = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 5 \end{bmatrix}$$

then

$$NN^T == (ZZ^T)^2 = \begin{bmatrix} 30 & 14 & 14 & 14 & 14 & 14 \\ 14 & 30 & 14 & 14 & 14 & 14 \\ 14 & 14 & 30 & 14 & 14 & 14 \\ 14 & 14 & 14 & 30 & 14 & 14 \\ 14 & 14 & 14 & 14 & 30 & 14 \\ 14 & 14 & 14 & 14 & 14 & 30 \end{bmatrix}$$

The final super-saturated design can be obtained after the implementation of steps 4 and step5 iteratively with $E(s^2)$=2.

Method 3.9: Gupta and Rajender Prasad[26] generated efficient two-level super-saturated designs. The construction procedure is explain ed below

Step 1: Consider the design matrix X of order n x p with each factor at two levels +1 and -1 such that the number of +1's and -1's in each and every column is equal. Assume n-1 <p, n=2t (t is a positive integer).

Step 2: Compute X'X= (s_{ij}); i, j=1, 2 , ….m and also compute $E(s^2) = \left(\sum_{1 \le i \le j \le p} s_{ij}^2 \right) / (p_{C2})$ where s_{ij}^2 is sum of cross products between columns i and j of the design.

Step 3: The column with maximum value of S_j^2 is to be selected for modification. Where $S_j^2 = \sum_{1 \le i \prec j \le p} s_{ij}^2$. All possible coordinate exchange steps are implemented in this column and for each exchange, the value of $E(s^2)$ is computed.

Step 4: Repeat Step2 and Step3 until $E(s^2)$ becomes stable or $E(s^2)$ reaches its lower bound.

This method is illustrated in the example 3.9.

Example 3.9: Consider a design matrix $X_{8x11}$with 8 design points and 11 factors each column consisting of with four +1's and -1's.

$$X = \begin{bmatrix} -1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 \\ +1 & -1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 \\ +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 \\ -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 \\ -1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 \end{bmatrix}$$

From X`X matrix, the $S_j^2 = \sum_{i(\ne j)=1}^{p} s_{ij}^2$ = 112 +112 + 64 + 64 + 128 + 128 + 80 + 80 + 48 + 16 + 32 = 864. Here, S_5^2 and S_6^2 both are same equal to 128. Select column 6 for modification. All possible coordinate exchange steps are implemented in this column and for each exchange, the value $E(s^2)$ is computed. The best exchange is when rows 6 and 7 are exchanged in this column. After exchanging rows 6 and 7 $E(s^2)$ is reduced to 6.1091 then the modified design matrix is.

$$X = \begin{bmatrix} -1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 \\ +1 & -1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 \\ +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 \\ -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & +1 \\ -1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 \end{bmatrix}$$

Repeat the steps and identify the column with maximum S_j^2. From the above matrix, column 2 is selected for modification. All the possible coordinate exchange steps are implemented in column2 The best exchange is when rows 1 and 8 are exchanged in column 2 with $E(s^2)$=5.2364 After exchange of the rows, the new design becomes

$$X = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

Repeating the above procedure, by selecting column1 for modification, all the possible rows exchange steps are implemented in column1.The best exchange, is when rows 1and 3 are exchanged in column1 with $E(s^2)$=4.6545. After exchange of the rows, the new design is

$$X= \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

The $E(s^2)$ value is computed from the X`X matrix $\sum s_{ij}^2$ = 256 and m_{c_2} =55. The $E(s^2)$ = 4.654. It is observed that $E(s^2)$ attains the lower bound with efficiency 1. The procedure is stopped at this stage.

Conclusion

In this paper we have tried to give a laconic review of the construction and analysis of SSDs. Since no universal optimal method exists, either for the construction or analysis of SSDs. In conclusion we can say that one should be very cautious when using any method for constructing, analyzing or generally using SSDs.

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