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Inclusion Relation and Neighbour Properties of Univalent Functions Associated with Subordination

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Abstract

There are many subclasses of univalent functions. The objectives of this paper is to introduce new classes and we have attempted to obtain Inclusion relation and Neighbour properties for the classes $\mathcal{H}(A,B,\alpha)$ and $K\mathcal{H}(A,B,\alpha)$

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Introduction

Let Tdenote the class of functions $f(w)$ of the form

$$f(w) = w - \sum_{k=2}^{\infty} a_k w^k , \qquad a_k \geq 0 \tag{1}$$

which are univalent in the unit disc
$U = \{w : w \in \mathbb{C} and\ |w| < 1\}$

Definition 1.1: A function $f(w) \in T$ is said to be close to convex of order μ $(0 \leq \mu < 1)$ if

$Re\{f'(w)\} > \mu$ for all $w \in U$

A function $f(w) \in T$ is said to be in the subclass $H(\mu)$ of starlike function if

$$Re\left(\frac{wf'(w)}{f(w)}\right) > \mu ,\ w \in U \ \ 0 \leq \mu < 1$$

Definition 1.2: A function $f(w) \in T$is said to be in the subclass $G(\mu)$ of convex function if

$$Re\left(1 + \frac{wf'(w)}{f(w)}\right) > \mu , \qquad w \in U$$

Definition 1.3: Let $f(w) = w - \sum_{k=2}^{\infty} a_k w^k$, $g(w) = w - \sum_{k=2}^{\infty} a_k w^k$, $a_k \geq 0 , b_k \geq 0$ then the convolution is defined as

$$f(w) * g(w) = w - \sum_{k=2}^{\infty} a_k b_k w^k \tag{2}$$

Definition 1.4 : If f and g are regular in U , we say that f is subordinate to g , denoted by $f \prec g$ or $f(w) \prec g(w)$, if there exist a Schwarz function w, which is regular in U with $h(0) = 0$ and $|h(w)| < 1$,$w \in U$ such that $f(w) = g(h(w))$, $w \in U$. In particular if g is univalent in$U$, we have the equivalence $f(w) \prec g(w)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$

Definition 1.5: We say that a function $f(w) \in T$ is in the class $\mathcal{H}(A,B,\alpha)$if it satisfy

$$\frac{wf'(w) + \alpha w^2 f''(w)}{\alpha wf'(w) + (1-\alpha)f(w)} \prec \frac{1+Aw}{1+Bw} \tag{3}$$

for $0 < \alpha \leq 1$, $-1 \leq B < A \leq 1$

Furthermore a function $f(w) \in T$ is said to belong to the class $K\mathcal{H}(A,B,\alpha)$ if and only if $wf'(w) \in \mathcal{H}(A,B,\alpha)$.

Theorem 1.1: A function $f(w) = w - \sum_{k=2}^{\infty} a_k w^k$, $a_k \geq 0$ is in $\mathcal{H}(A,B,\alpha)$ if and only if

$$\sum_{k=2}^{\infty}\{k + \alpha k^2 - 2\alpha k - (1-\alpha) - [B(k + \alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\}a_k \leq (A - B)$$

Proof: Suppose $f(w)$ is in$\mathcal{H}(A,B,\alpha)$

Therefore from (2) we have

$$p(w) = \frac{wf'(w) + \alpha w^2 f''(w)}{\alpha wf'(w) + (1-\alpha)f(w)} \prec \frac{1+Aw}{1+Bw}$$

$$p(w) = \frac{1+Ah(w)}{1+Bh(w)}$$

$|h(w)| < 1$

$$\left|\frac{\left[\frac{wf'(w) + \alpha w^2 f''(w)}{\alpha wf'(w) + (1-\alpha)f(w)}\right] - 1}{A - B\left\{\frac{wf'(w) + \alpha w^2 f''(w)}{\alpha wf'(w) + (1-\alpha)f(w)}\right\}}\right| < 1$$

$$\left|\frac{wf'(w)+\alpha w^2 f''(w)-\alpha wf'(w)-(1-\alpha)f(w)}{A[\alpha wf'(w)+(1-\alpha)f(w)]-B[wf'(w)+\alpha w^2 f''(w)]}\right| < 1 \quad (4)$$

$$wf'(w)+\alpha w^2 f''(w)-\alpha wf'(w)-(1-\alpha)f(w)$$

$$= -\sum_{k=2}^{\infty}[k+\alpha k^2-2\alpha k-(1-\alpha)]\, a_k w^k$$

$$A[\alpha wf'(w)+(1-\alpha)f(w)]-B[wf'(w)+\alpha w^2 f''(w)]$$

$$=(A-B)w+\sum_{k=2}^{\infty}\left[-A(\alpha k+1-\alpha)+B\big(k+\alpha k(k-1)\big)\right] a_k w^k$$

From (4) we have

$$\left|\frac{-\sum_{k=2}^{\infty}[k+\alpha k^2-2\alpha k-(1-\alpha)]\, a_k w^k}{(A-B)w+\sum_{k=2}^{\infty}\left[-A(\alpha k+1-\alpha)+B\big(k+\alpha k(k-1)\big)\right] a_k w^k}\right| < 1$$

Since Re $(w) < |w|$. We obtain after choosing the values of w on real axis and letting $w \to 1$
we get

$$\sum_{k=2}^{\infty}\left\{k+\alpha k^2-2\alpha k-(1-\alpha)-\left[B\big(k+\alpha k(k-1)\big)-A(\alpha k+1-\alpha)\right]\right\}a_k \le (A-B)$$

Corollary 1.1 If $f(w) \in \mathcal{H}(A,B,\alpha)$then

$$a_k \le \frac{(A-B)}{k+\alpha k^2-2\alpha k-(1-\alpha)-\left[B\big(k+\alpha k(k-1)\big)-A(\alpha k+1-\alpha)\right]}$$

and the equality holds for

$$f(w) = w-\frac{(A-B)}{k+\alpha k^2-2\alpha k-(1-\alpha)-\left[B\big(k+\alpha k(k-1)\big)-A(\alpha k+1-\alpha)\right]}w^k$$

Theorem 1.2: A function $f(w) = w-\sum_{k=2}^{\infty} a_k w^k$, $a_k \ge 0$ is in $K\mathcal{H}(A, B, \alpha)$ if and only if

$$\sum_{k=2}^{\infty}\{[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]-A[-\alpha k^2-k(1-\alpha)]-B[k^2+\alpha k^2(k-1)]\}a_k \le (A-B)$$

Proof: Suppose $f(w)$ is in $K\mathcal{H}(A,B,\alpha)$

If and only if $wf'(w)$ is in $\mathcal{H}(A,B,\alpha)$

Let $g(w) = wf'(w)$

Therefore from (1.1) we have

$$\left|\frac{wg'(w)+\alpha w^2 g''(w)-\alpha wg'(w)-(1-\alpha)g(w)}{A[\alpha wg'(w)+(1-\alpha)g(w)]-B[wg'(w)+\alpha w^2 g''(w)]}\right| < 1$$

$$\left|\frac{-\sum_{k=2}^{\infty}[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]a_k w^k}{(A-B)w+\sum_{k=2}^{\infty}\{A[-\alpha k^2-k(1-\alpha)]+B[k^2+\alpha k^2(k-1)]\}a_k w^k}\right| < 1$$

Since Re $(w) < |w|$. We obtain after choosing the values of w on real axis and letting $w \to 1$ we get

$$\sum_{k=2}^{\infty}\{[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]-A[-\alpha k^2-k(1-\alpha)]-B[k^2+\alpha k^2(k-1)]\}a_k \le (A-B)$$

Corollary 1.2: If $f(w) \in K\mathcal{H}(A,B,\alpha)$ then

$$a_k \le \frac{(A-B)}{[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]-A[-\alpha k^2-k(1-\alpha)]-B[k^2+\alpha k^2(k-1)]}$$

and the equality holds for

$$f(w)=w-\frac{(A-B)}{[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]-A[-\alpha k^2-k(1-\alpha)]-B[k^2+\alpha k^2(k-1)]}w^k$$

Inclusion Relation and Neighbour Properties

Definition-2.1: Let $\mu \ge 0$ and $f(w) = w-\sum_{k=2}^{\infty} a_k w^k$ be the function in the (k,ξ) −neighborhood of a function $f(w)$ defined as

$$N_k^{\xi}(f) = \left\{g \in T : g(w) = w-\sum_{k=2}^{\infty} b_k w^k \text{and} \sum_{k=2}^{\infty} k|a_k-b_k| \le \xi\right\}$$

For the identity function $I(w) = w$, we have

$$N_k^{\xi}(I) = \left\{g \in T : g(w) = w-\sum_{k=2}^{\infty} b_k w^k \text{and} \sum_{k=2}^{\infty} k|b_k| \le \xi\right\}$$

Definition 2.2: The function $f(w) = w-\sum_{k=2}^{\infty} a_k w^k$ is said to be a member of the subclass

$\mathcal{H}(A,B,\alpha)$ if there exist a function $g \in \mathcal{H}(A,B,\alpha)$ such that

$$\left|\frac{f(w)}{g(w)}-1\right| \le 1-\theta \ ,0 \le \theta < 1$$

Theorem 2.1: Let $0 < \alpha \le 1$ **,** $-1 \le B < A \le 1$. Then

$\mathcal{H}(A,B,\alpha) \subseteq \mathcal{H}(A_1,B_1,0)$ where $-1 \le B_1 < A_1 \le 1$ and $\frac{A_1-n}{1-n} \ge B_1$ and $A_1 \ge 2n-1$

Where:

$$n = \frac{A-B}{\{(1+\alpha)(1-2B+A)\}-(A-B)}$$

Proof: Let $f(w) = w-\sum_{k=2}^{\infty} a_k w^k$ be in the class $\mathcal{H}(A,B,\alpha)$

Therefore by **Theorem 1.1:**

$$\sum_{k=2}^{\infty}\frac{\left\{k+\alpha k^2-2\alpha k-(1-\alpha)-\left[B\big(k+\alpha k(k-1)\big)-A(\alpha k+1-\alpha)\right]\right\}}{(A-B)}a_k \le 1 \quad (5)$$

Now we want to find the values of A_1 , B_1 such that $-1 \le B_1 < A_1 \le 1$, and

$$\sum_{k=2}^{\infty} \frac{\{k-1-[B_1 k - A_1]\}}{(A_1 - B_1)} a_k \le 1 \qquad (6)$$

The inequality (5) imply (6) if

$$\frac{\{k-1-[B_1 k - A_1]\}}{(A_1 - B_1)} \le \frac{\{k+\alpha k^2 - 2\alpha k - (1-\alpha) - [B(k+\alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\}}{(A-B)} = \delta \qquad (7)$$

Simplifying (7) we get

$$\frac{B_1 - A_1}{B_1 - 1} \ge \frac{k-1}{\delta - 1} \quad , \; k \ge 2 \,, \qquad (8)$$

It is clear that the right hand side of (8) decreases as k increases and maximum for $k = 2$

Thus (8) is satisfied provided

$$\frac{B_1 - A_1}{B_1 - 1} \ge \frac{A-B}{\{(1+\alpha)(1-2B+A)\} - (A-B)} = n \qquad (9)$$

fixing A_1 in (9), we get

$$\frac{A_1 - n}{1-n} \ge B_1$$

For $-1 \le B_1$, we have $A_1 \ge 2n - 1$

The proof of theorem is complete.

Theorem 2.2: Let $0 < \alpha \le 1, -1 \le B < A \le 1$. Then $K\mathcal{H}(A, B, \alpha) \subseteq K\mathcal{H}(A_1, B_1, 0)$ where $-1 \le B_1 < A_1 \le 1$ and $\frac{A_1 - n}{1-n} \ge B_1$ and $A_1 \ge 2n - 1$

Where

$$n = \frac{A-B}{\{(1+\alpha)(1-2B+A)\} - (A-B)}$$

Proof: Let $f(w) = w - \sum_{k=2}^{\infty} a_k w^k$ be in the class $\mathcal{H}(A, B, \alpha)$.

Therefore by **Theorem 1.2:**

$$\sum_{k=2}^{\infty} \frac{\{[k^2(1-2\alpha) + \alpha k^3 - k(1-\alpha)] - A[-\alpha k^2 - k(1-\alpha)] - B[k^2 + \alpha k^2(k-1)]\}}{(A-B)} a_k \le 1 \qquad (10)$$

Now we want to find the values of A_1 , B_1 such that $-1 \le B_1 < A_1 \le 1$, and

$$\sum_{k=2}^{\infty} \frac{k\{k-1-B_1 k + A_1\}}{(A_1 - B_1)} a_k \le 1 \qquad (11)$$

The inequality (10) imply (11) if

$$\frac{k\{k-1-[B_1 k - A_1]\}}{(A_1 - B_1)} \le \frac{\{[k^2(1-2\alpha) + \alpha k^3 - k(1-\alpha)] - A[-\alpha k^2 - k(1-\alpha)] - B[k^2 + \alpha k^2(k-1)]\}}{(A-B)} = \delta \qquad (12)$$

Simplifying (12) we get

$$\frac{A_1 - B_1}{1 - B_1} \ge \frac{k-1}{\delta - 1} \quad , \; k \ge 2 \,, \qquad (13)$$

Note that the right hand side of (13) decreases as k increases and maximum for $k = 2$

Thus (13) is satisfied provided

$$\frac{B_1 - A_1}{B_1 - 1} \ge \frac{A-B}{\{(1+\alpha)(1-2B+A)\} - (A-B)} = n \qquad (14)$$

fixing A_1 in (14), we get

$$\frac{A_1 - n}{1-n} \ge B_1$$

For $-1 \le B_1$, we have $A_1 \ge 2n - 1$

The proof of theorem is complete.

Theorem 2.3: Let

$$\xi = \frac{2(A-B)}{(1+\alpha)(1-2B+A)}$$

Then

$$\mathcal{H}(A, B, \alpha) \subset N_k^{\xi}(I)$$

Proof: Let $f(w) = w - \sum_{k=2}^{\infty} a_k w^k$ be in the class $\mathcal{H}(A, B, \alpha)$

Then

$$\sum_{k=2}^{\infty} \{k + \alpha k^2 - 2\alpha k - (1-\alpha) - [B(k + \alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\} a_k \le (A-B)$$

$$(1+\alpha)(1-2B+A) \sum_{k=2}^{\infty} a_k \le (A-B)$$

Therefore

$$\sum_{k=2}^{\infty} a_k \le \frac{(A-B)}{(1+\alpha)(1-2B+A)} \qquad (15)$$

Also for $|w| < r$

$$|f'(w)| \le 1 + |w| \sum_{k=2}^{\infty} k a_k \le 1 + r \sum_{k=2}^{\infty} k a_k$$

From (15) we have

$$|f'(w)| \le 1 + r \frac{2(A-B)}{(1+\alpha)(1-2B+A)}$$

From above inequalities we get

$$\sum_{k=2}^{\infty} k a_k \le \frac{2(A-B)}{(1+\alpha)(1-2B+A)} = \xi$$

Therefore $f \in N_k^{\xi}(I)$

Theorem 2.4: Let
$$\xi = \frac{(A-B)}{(1+\alpha)(1-2B+A)}$$

Then
$$K\mathcal{H}(A,B,\alpha) \subset N_k^{\xi}(I)$$

Proof: Let $f(w) = w - \sum_{k=2}^{\infty} a_k w^k$ be in the class $K\mathcal{H}(A,B,\alpha)$
Then
$$\sum_{k=2}^{\infty}\{[k^2(1-2\alpha)+\alpha k^3 - k(1-\alpha)] - A[-\alpha k^2 - k(1-\alpha)] - B[k^2 + \alpha k^2(k-1)]\}a_k \leq (A-B)$$

That is
$$2(1+\alpha)(1-2B+A)\sum_{k=2}^{\infty} a_k \leq (A-B)$$

Therefore
$$\sum_{k=2}^{\infty} a_k \leq \frac{(A-B)}{2(1+\alpha)(1-2B+A)} \quad (16)$$

Also for $|z| < r$
$$|f'(w)| \leq 1 + |w|\sum_{k=2}^{\infty} k a_k \leq 1 + r\sum_{k=2}^{\infty} k a_k$$

From (16) we have
$$|f'(w)| \leq 1 + r\frac{2(A-B)}{2(1+\alpha)(1-2B+A)}$$

From above inequalities we get
$$\sum_{k=2}^{\infty} k a_k \leq \frac{(A-B)}{(1+\alpha)(1-2B+A)} = \xi$$

Therefore
$$f \in N_k^{\xi}(I)$$

Theorem 2.5: Let $g(w) = w - \sum_{k=2}^{\infty} b_k w^k$ be in the class $\mathcal{H}(A,B,\alpha)$ and

$$\beta = 1 - \frac{\xi}{2}\left(\frac{(1+\alpha)(1-2B+A)}{(1+\alpha)(1-2B+A)-(A-B)}\right)$$

Then $N_k^{\xi}(f) \subset \mathcal{H}(A,B,\alpha,\beta)$, $0 < \alpha \leq 1$ **,** $-1 \leq B < A \leq 1$, $0 \leq \beta < 1$

Proof: $f \in N_k^{\xi}(f)$ then by definition we have
$$\sum_{k=2}^{\infty} k|a_k - b_k| \leq \xi$$

Then
$$\sum_{k=2}^{\infty} |a_k - b_k| \leq \frac{\xi}{2}$$

Since $g \in \mathcal{H}(A,B,\alpha)$, we have
$$\sum_{k=2}^{\infty} a_k \leq \frac{(A-B)}{(1+\alpha)(1-2B+A)} \quad (17)$$

Therefore
$$\left|\frac{f(w)}{g(w)} - 1\right| < \frac{\sum_{k=2}^{\infty}|a_k - b_k|}{1 - \sum_{k=2}^{\infty} b_k}$$
$$\leq \frac{\xi}{2}\left(\frac{(1+\alpha)(1-2B+A)}{(1+\alpha)(1-2B+A)-(A-B)}\right) = 1-\beta$$

Then by DEFINITION 2.2, we get
$f \in \mathcal{H}(A,B,\alpha,\beta)$

Theorem 2.6: Let $g(w) = w - \sum_{k=2}^{\infty} b_k w^k$ be in the class $K\mathcal{H}(A,B,\alpha)$ and
$$\beta = 1 - \xi\left(\frac{(1+\alpha)(1-2B+A)}{2(1+\alpha)(1-2B+A)-(A-B)}\right)$$

Then $N_k^{\xi}(f) \subset K\mathcal{H}(A,B,\alpha,\beta)$, $0 < \alpha \leq 1$ **,** $-1 \leq B < A \leq 1$, $0 \leq \beta < 1$

Proof: Similar to THEOREM 2.5

Conclusion

Here we have defined two classes $\mathcal{H}(A,B,\alpha)$ and $K\mathcal{H}(A,B,\alpha)$. We have obtained coefficient estimate. With the help of Theorem 1.1 we have investigated Inclusion Relation and Neighbour Properties for these two classes.

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