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A Renewal Risk Model with Dependence between Claim Sizes and Claim Intervals

T.M. Sajithamony[1] and K.K. Thampi[2]
[1]Bharathiar University, Coimbatore, Tamilnadu, INDIA
[2]SNM College, Maliankara, Eranakulam, Kerala, INDIA

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Abstract

This paper considers an application of probability to an insurance portfolio where the claim inter-arrival time depends on the previous claim size and follows Erlang (2) distribution. An explicit solution is derived for the crucial parameter of insurance companies, the probability of survival, using Laplace transform. The results are illustrated with examples.

Keywords: Survival probability, Laplace Transform, Dependent risk model, Erlang distribution.

Introduction

The classical and renewal risk models are widely investigated in recent times. But, most of them assume that the inter-arrival time between two successive claims and the claim amounts are independent. However, when modelling natural events, this assumption is very restrictive. To avoid this restriction Albrecher and Boxmaconsidered a model in which the distribution of inter-arrival time depends on previous claim size and was extended to a semi-Markovian risk model[1,2]. In the study by Boudreault, Cossette, Landriault, and Marceau the claim size depends on the inter-claim time[3]. Albrecher and Teugels considered a dependence structure through copula[4]. Kwan and H. Yang proposed another dependent structure where claim size distribution depends on inter-claim time[5]. Himanshu, Ashivani, and Vivek studied a probability model of continuous fertility[6]. Himanshu and Ashivani developed a probability model for the child mortality under the assumption that the families under consideration have one birth prior to the study[7]. Kusum and Srivastava derived p.d.f of Inverse Maxwell distribution which is suitable for survival models[8]. Meng, Zhang, and Guo considered a dependent setting where the time between two claims determines the distribution of the next claim size[9]. Asimit and Badescu introduced a general dependence structure of heavy tailed claim sizes in the presence of a constant force of interest[10]. Dhanesh studied a single server dependent queueing model where the arrival rate depends on time and service rate is constant[11]. Hamid, Asghar and Mostafa the probability of occurrence of earthquakes in induced landslide based on slop, material, precipitation, fault data and weight composition using ArcGIS[12]. Chen and Yuen constructed a dependent structure via the conditional distribution of the inter-arrival time given the subsequent claim size being large[13]. Xie and Zou described a dependence model of inter-arrival time, premium size and claim size[14]. Albrecher, Boxma, and Ivanovs extended the study by Kwan and H. Yang to a generalised risk model with phase type distribution[15,5]. Chadjiconstantinidis and S. Vrontos studied a dependent renewal risk model under Farlie-Morgenstern copula[16].

Motivated by the work of Albrecher and Boxmain this paper a generalization of the dependent model to a renewal process is considered, where the inter-arrival time of two consecutive claims follows Erlang[1]. For this renewal model, we derive explicit solutions for the probability of survival via Laplace-Stieltjes transform.

Preliminaries

Consider a risk process where the claims occur as an ordinary renewal process. Let $\{Ti\}_{i=1}^{\infty}$ be a sequence of strictly positive, independent and identically distributed random variables, Ti denotes the inter-arrival time between the (i-1)th and ith claims. We assume Ti follows Erlang (n, λ) distribution with probability distribution function.

$$k(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \text{ for } t > 0$$

where n is a positive integer. In this paper, we illustrate ideas by restricting our attention to the case in which n=2. Let m1 denote the mean of this distribution. Let $\{Xi\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables where Xi denote the ith claim size. Let f(x) denotes the p.d.f, μ the average and $\tilde{\chi}(s)$ the Laplace-Stieltjes transform (LST).

Consider the following surplus process $U_t(u)$ of an insurance portfolio.

$$U_t(u) = u + ct - \sum_{k=1}^{N(t)} X_{k,}$$

where u>0 is the initial surplus, c>0 is the constant rate of premium, and N(t) is the number of claims up to time t. Let us assume the following Markovian model for the claim occurrence process: if a claim size Xi is greater than a threshold Ai then the time until the next claim follows Erlang(2) distribution with parameter β_1, otherwise it follows an Erlang (2) distribution with another parameter β_2. The quantities Ai are assumed to be i.i.d random variable with distribution function F(A).

The net profit condition is

$$\mu < 2c\left(\frac{P(X \geq A)}{\beta_1} + \frac{P(X < A)}{\beta_2}\right)$$

Let $T = \inf\{t > 0; U(t) < 0\}$ be the time of ruin, $\emptyset_i(u) = P\{T = \infty / U(0) = u, T_i \sim \text{Erlang}(2, \beta_i)\}$ be the ultimate survival probability and $\Psi_i(u) = 1 - \emptyset_i(u)$ be the ultimate ruin probability.

Integro-Differential Equation and Laplace Transform

In this section we show that Φ satisfies an integro-differential equation. This equation will be the basis for our explicit solution for Φ.

Theorem 1. The ultimate survival probability Φ_i (u) (i = 1; 2) satisfies the following integro-differential equation

$$c^2 \frac{d}{d_u}\emptyset_i(u) - c\,\beta_i\emptyset_i(u) = -\beta_i^2 \int_u^\infty e^{\beta_i\left(\frac{s-u}{c}\right)} \int_0^s [P(A \leq x)\, f(x)\emptyset_1(s - x) + P(A > x)f(x)\emptyset_2(s - x)]\, d_x d_s \quad (1)$$

Proof. By conditioning on the time and the amount of the first claim and for i=1.

Let $k_1(t) = \beta_1^2 te^{-\beta_1 t}$

$$\emptyset_1(u) = \int_0^\infty k_1(t) \int_0^{u+ct} [P(A \leq x)f(x)\emptyset_1(u + ct - x) + P(A > x)f(x)\emptyset_2(u + ct - x)]d_x d_t$$

Put s = u + ct

$$c\emptyset_1(u) = \int_u^\infty k_1\left(\frac{s-u}{c}\right) \int_0^s [P(A \leq x)f(x)\emptyset_1(s - x) + P(A > x)f(x)\emptyset_2(s - x)]d_x d_s$$

Differentiating with respect to u we get

$$c\frac{d}{d_u}\emptyset_1(u) - \beta_1\emptyset_1(u) = \frac{-1}{c}\int_u^\infty \beta_1^2 e^{-\beta_1\left(\frac{s-u}{c}\right)} \int_0^s [P(A \leq x)f(x)\emptyset_1(s - x) + P(A > x)f(x)\emptyset_2(s - x)]d_x d_s$$

Similarly we can prove the result for i = 2.

Theorem 2 Let $\tilde{\emptyset_i}(s)$ be the Laplace transform of $\emptyset_1(u)$ (i = 1, 2) and Re s ≥ 0. Define

$$\chi_1(s) = \int_0^\infty e^{-sx}A(x)f(x)dx$$

$$\chi_2(s) = \int_0^\infty e^{-sx}(1 - A(x))f(x)dx$$

Then

$$\tilde{\emptyset_1}(s) = \frac{c^2\beta_1^2\chi_2(s)\frac{d}{du}\emptyset_2(0) + c^2\begin{bmatrix} c^2s^2 - 2c\beta_2 s \\ +\beta_2^2(1-\chi_2(s))\end{bmatrix}\frac{d}{du}\emptyset_1(0) + \beta_1^2\chi_2(s)\emptyset_2(0)\begin{bmatrix} c^2s \\ -2c\beta_2\end{bmatrix} + \begin{bmatrix} c^2s^2 - 2c\beta_2 s \\ +\beta_2^2(1-\chi_2(s))\end{bmatrix}\begin{bmatrix} c^2s \\ -2c\beta_1\end{bmatrix}\emptyset_1(0)}{\begin{bmatrix} c^2s^2 - 2c\beta_1 s \\ +\beta_1^2(1-\chi_1(s))\end{bmatrix}\begin{bmatrix} c^2s^2 - 2c\beta_2 s \\ +\beta_2^2(1-\chi_2(s))\end{bmatrix} - \beta_1^2\beta_2^2\chi_1(s)\chi_2(s)} \quad (2)$$

$$\tilde{\emptyset_2}(s) = \frac{c^2\beta_2^2\chi_1(s)\frac{d}{du}\emptyset_1(0) + c^2\begin{bmatrix} c^2s^2 - 2c\beta_1 s \\ +\beta_1^2(1-\chi_1(s))\end{bmatrix}\frac{d}{du}\emptyset_2(0) + \beta_2^2\chi_1(s)\emptyset_1(0)\begin{bmatrix} c^2s \\ -2c\beta_1\end{bmatrix} + \begin{bmatrix} c^2s^2 - 2c\beta_1 s \\ +\beta_1^2(1-\chi_1(s))\end{bmatrix}[c^2s - 2c\beta_2]\emptyset_2(0)}{\begin{bmatrix} c^2s^2 - 2c\beta_1 s \\ +\beta_1^2(1-\chi_1(s))\end{bmatrix}\begin{bmatrix} c^2s^2 - 2c\beta_2 s \\ +\beta_2^2(1-\chi_2(s))\end{bmatrix} - \beta_1^2\beta_2^2\chi_1(s)\chi_2(s)} \quad (3)$$

Proof. Differentiating (1) with respect to u for i=1 we get

$$c^2\frac{d^2}{d_u^2}\emptyset_1(u) - 2c\beta_1\frac{d}{d_u}\emptyset_i(u) + \beta_1^2\emptyset_1(u) = \beta_1^2\int_0^u [P(A \leq x)f(x)\emptyset_1(u - x) + P(A > x)f(x)\emptyset_2(u - x)]d_x$$

Multiplying by e^{-su} and integrating from 0 to ∞

$$c^2\left[-\frac{d}{du}\emptyset_1(0) - s\emptyset_1(0) + s^2\tilde{\emptyset_1}(s)\right] - 2c\beta_1[-\emptyset_1(0) + s\tilde{\emptyset_1}(s)] + \beta_1^2\tilde{\emptyset_1}(s) = \beta_1^2[\tilde{\emptyset_1}(s)\chi_1(s) + \tilde{\emptyset_2}(s)\chi_2(s)]$$

=>

$$c^2\frac{d}{du}\emptyset_1(0) + [c^2s - 2\beta_1 c]\emptyset_1(0) = \tilde{\emptyset_1}(s)\left[c^2s^2 - 2\beta_1 cs + \beta_1^2\left(1 - \chi_1(s)\right)\right] - \beta_1^2\chi_2(s)\tilde{\emptyset_2}(s)$$

Similarly for i=2

$$c^2\frac{d}{du}\emptyset_2(0) + [c^2s - 2\beta_2 c]\emptyset_2(0) = \tilde{\emptyset_2}(s)\left[c^2s^2 - 2\beta_2 cs + \beta_2^2\left(1 - \chi_2(s)\right)\right] - \beta_2^2\chi_1(s)\tilde{\emptyset_1}(s)$$

The result is obtained by solving the above simultaneous equations for $\tilde{\emptyset_1}(s)$ and $\tilde{\emptyset_2}(s)$

Inversion of Laplace Transform

After getting the formula of $\tilde{\emptyset_1}(s)$ and $\tilde{\emptyset_2}(s)$ next we want to determine $\Phi_1(0)$ and $\Phi_2(0)$. Since $\lim_{x\to\infty}\emptyset i(x) = 1$ we have $\lim_{s\to 0} s\tilde{\emptyset_i}(s) = 1$; i=1, 2.

Using the above result w.l.o.g in (2), we obtain

$$1=\lim_{s\to 0}\frac{\begin{array}{c}c^2\beta_1^2\chi_2(s)\frac{d}{du}\emptyset_2(0)\\ +c^2\begin{bmatrix}c^2s^2-2c\beta_2 s\\ +\beta_2^2\left(1-\chi_2(s)\right)\end{bmatrix}\frac{d}{du}\emptyset_1(0)\\ +\beta_1^2\chi_2(s)\emptyset_2(0)\left[c^2s-2c\beta_2\right]\\ +\begin{bmatrix}c^2s^2-2c\beta_2 s\\ +\beta_2^2\left(1-\chi_2(s)\right)\end{bmatrix}\left[c^2s-2c\beta_1\right]\emptyset_1(0)\end{array}}{\begin{array}{c}\frac{1}{s}\begin{bmatrix}c^2s^2-2c\beta_1 s\\ +\beta_1^2\left(1-\chi_1(s)\right)\end{bmatrix}\begin{bmatrix}c^2s^2-2c\beta_2 s\\ +\beta_2^2\left(1-\chi_2(s)\right)\end{bmatrix}\\ -\beta_1^2\beta_2^2\chi_1(s)\chi_2(s)\end{array}}$$

$$=\frac{\begin{array}{c}c^2\beta_1^2\chi_2(0)\frac{d}{du}\emptyset_2(0)\\ +c^2\beta_2^2\left(1-\chi_2(0)\right)\frac{d}{du}\emptyset_1(0)\\ -\beta_1^2\chi_2(0)\emptyset_2(0)2c\beta_2\\ -\beta_2^2\left(1-\chi_2(0)\right)2c\beta_1\emptyset_1(0)\end{array}}{\begin{array}{c}-2c\beta_1\beta_2^2+2c\beta_1\beta_2^2\chi_2(0)-2c\beta_1^2\beta_2\\ -\beta_1^2\beta_2^2\chi_2'(0)+\beta_1^2\chi_1(0)2c\beta_2-\beta_1^2\beta_2^2\chi_1'(0)\end{array}}$$

We have $\chi_2(0)=P(x<A),\chi_1(0)=P(x\geq A)$ and $\chi_1(0)+\chi_2(0)=1$.Also $(X/x<A)=-\chi_2'(0),E(X/x\geq A)=-\chi_1'(0)$.

So $\mu=-\chi_1'(0)-\chi_2'(0)$
$$c^2\beta_1^2P(x\leq A)\frac{d}{du}\emptyset_2(0)+c^2\beta_2^2P(x>A)\frac{d}{du}\emptyset_1(0)+2c\beta_1^2\beta_2P(x\leq A)\left(1-\emptyset_2(0)\right)+2c\beta_1\beta_2^2\left(1-\emptyset_1(0)\right)P(x>A)=\beta_1^2\beta_2^2\mu \quad (4)$$

Since r1 is a zero of denominators of (2) and (3) it must also be a zero of the numerators, giving
$$c^2\beta_1^2\chi_2(r1)\frac{d}{du}\emptyset_2(0)+c^2\left[c^2r_1^2-2\beta_2cr_1+\beta_2^2\left(1-\chi_2(r_1)\right)\right]\frac{d}{du}\emptyset_1(0)+\beta_1^2\chi_2(r_1)\emptyset_2(0)\left[c^2r_1-2\beta_2c\right]+\left[c^2r_1^2-2\beta_2cr_1+\beta_2^2\left(1-\chi_2(r_1)\right)\right]\left[c^2r_1-2\beta_1c\right]\emptyset_1(0)=0 \quad (5)$$

$$c^2\beta_2^2\chi_1(r1)\frac{d}{du}\emptyset_1(0)+c^2\left[c^2r_1^2-2\beta_1cr_1+\beta_1^2\left(1-\chi_1(r_1)\right)\right]\frac{d}{du}\emptyset_2(0)+\beta_2^2\chi_1(r_1)\emptyset_1(0)\left[c^2r_1-2\beta_1c\right]+\left[c^2r_1^2-2\beta_1cr_1+\beta_1^2\left(1-\chi_1(r_1)\right)\right]\left[c^2r_1-2\beta_2c\right]\emptyset_2(0)=0 \quad (6)$$

Solving (5) and (6) we get
$$\frac{d}{du}\emptyset_2(0)=-\left[c^2r1-2\beta_2c\right]\frac{\emptyset_2(0)}{c^2} \quad (7)$$

$$\frac{d}{du}\emptyset_1(0)=-\left[c^2r1-2\beta_1c\right]\frac{\emptyset_1(0)}{c^2} \quad (8)$$

Substituting in (4)

$$\beta_1^2\beta_2^2\mu-2c\beta_1^2\beta_2P(x\leq A)-2c\beta_2^2\beta_1P(x>A)+\beta_1^2P(x\leq A)c^2r1\,\emptyset_2(0)+\beta_2^2P(x>A)c^2r1\emptyset_1(0)=0 \quad (9)$$

Then (2) and (3) becomes
$$\tilde{\emptyset_1}(s)=\frac{\beta_1^2\chi_2(s)\emptyset_2(0)\left[c^2s-c^2r_1\right]+\left[c^2s^2-2\beta_2cs+\beta_2^2\left(1-\chi_2(s)\right)\right]\emptyset_1(0)\left[c^2s-c^2r_1\right]}{\left[c^2s^2-2c\beta_1s+\beta_1^2\left(1-\chi_1(s)\right)\right]\left[c^2s^2-2c\beta_2s+\beta_2^2\left(1-\chi_2(s)\right)\right]-\beta_1^2\beta_2^2\chi_1(s)\chi_2(s)} \quad (10)$$

$$\tilde{\emptyset_1}(s)=\frac{\beta_2^2\chi_1(s)\emptyset_1(0)\left[c^2s-c^2r_1\right]+\left[c^2s^2-2\beta_1cs+\beta_1^2\left(1-\chi_1(s)\right)\right]\emptyset_2(0)\left[c^2s-c^2r_1\right]}{\left[c^2s^2-2c\beta_1s+\beta_1^2\left(1-\chi_1(s)\right)\right]\left[c^2s^2-2c\beta_2s+\beta_2^2\left(1-\chi_2(s)\right)\right]-\beta_1^2\beta_2^2\chi_1(s)\chi_2(s)} \quad (11)$$

Values of $\Phi_1(0)$ and $\Phi_2(0)$

Now we need the value of $\Phi_1(0)$ to substitute in equation (9) for getting the value of $\Phi_2(0)$. The renewal risk process by Dickson becomes dependent risk model only when the claim size goes beyond a threshold Ai[17]. So we make an assumption that the initial claims are less than Ai. Then we can adopt the formula of $\Phi_1(0)$ by Dickson and Hipp[18].

$$\emptyset_1(0)=\frac{2\beta_1c-\beta_1^2\mu}{c^2s_0} \quad (12)$$

where s_0 is solution of the equation
$c^2s^2-2c\beta_1cs+\beta_1^2(1-\chi)=0$
Substituting in (9) we get the value of $\emptyset_2(0)$.

On the other hand if the initial claims are larger than the threshold it becomes dependent model only when the claim size goes below the threshold Ai. So we can adopt the above method for finding $\Phi_2(0)$ using β_2 in equation (12) and substitute in (9) to get the value of $\emptyset_1(0)$.

Calculation of Survival Probabilities

Equations (12) and (9) gives the values of Φ1(0) and Φ2(0). Substituting in equations (10) and (11) and inverting them gives the survival probabilities with initial surplus u.

Example-1: For the special case let the initial claims occur according to Erlang(2) with parameter β_1. Also let X~ Erlang (2, 2), A~ Erlang (2, 2) $\beta_1=2$, $\beta_2=1$, c=2. The net profit condition is satisfied. Then by inverting the Laplace transforms and applying Lundberg's inequality we get

$$\emptyset_1(x)=1-0.0026e^{-4.6661x}+0.0444e^{-2.4931x}-0.3040e^{-1.1182x}$$
$$\emptyset_2(x)=1-0.0008e^{-4.6661x}+0.0151e^{-2.4931x}-0.1302e^{-1.1182x}$$

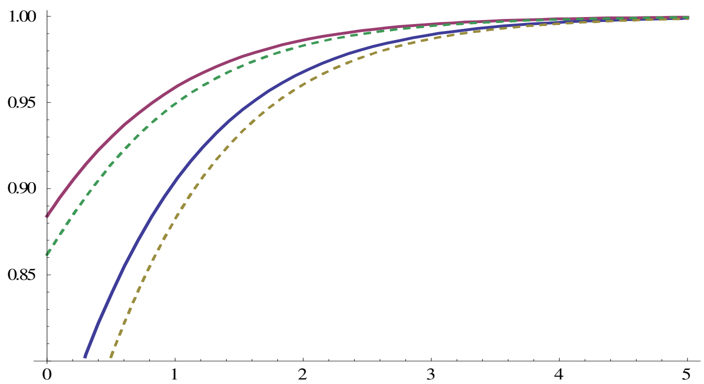
Example-2: In the above example let us assume the initial claims occur according to Erlang (2) with parameter β_2. Then the result becomes

$$\emptyset_1(x)=1+0.0138e^{-4.6661x}+0.0517e^{-2.4931x}-0.3734e^{-1.1182x}$$

$\emptyset_2(x) = 1 + 0.0042e^{-4.6661x} + 0.0175e^{-2.4931x} - 0.1599e^{-1.1182x}$

Conclusion

Figure 1 shows the results of example 1 and 2. We made the above assumption on the distribution of initial claims because of its analytical tractability. So we can use the very same method for finding the values of ruin and survival probabilities of dependent risk model when the distribution is Erlang.



Solid lines: Values of example 1. Dotted lines: Values of example 2.

Figure-1

Survival probability curve of example 1 and 2.

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