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Numerical Solution of Singular Perturbation problems via deviating Argument through the Numerical methods

Parcha.Kalyani and Patibanda. S. Rama Chandra Rao
Kakatiya Institute of Technolog and Science, warangal- 506015, INDIA

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Abstract

An attempt is made in this article to obtain the numerical solution of singularly perturbed two point boundary value problems. To achieve this singular perturbation problem is reduced to first order differential equation by taking a small deviating argument. The Simpson's 3/8 rule is employed to get the equation in y (x_i). Hermite interpolation is used to obtain the value of y at the intermediate points of the boundary, finally yielding to a tridiagonal system of equations. The discrete invariant imbedding method is used to obtain the solution of system of equations. four linear singular perturbation problems of which two are with constant coefficients and two are with variable coefficients are solved to test the applicability and competence of the proposed method. The numerical results obtained by the proposed method are compared with the exact solution and also with the results obtained using Simpson's 1/3 rule. It is observed that the numerical results are very near to the exact solution.

Keywords: Singular perturbation problem, boundary layer, deviating argument, discrete invariant imbedding method.

Introduction

A problem is known to be perturbed problem in which the highest order derivative term is multiplied by a small parameter and the parameter is known as the perturbation parameter. Perturbation problems arise in a variety of fields in applied science, aerodynamics, elasticity quantum mechanics, engineering, and chemical reactor theory. Boundary layer problems; convective heat transport problems and the modeling of viscous flow problems with large Reynolds number are some of examples of perturbation problems. Number of researchers has employed various methods for solving singular perturbation problems. A detailed analysis on singular perturbation problems has been given by Bellman[1], Bender and Orsag[2], O'Malley[3], Eckhanus[4], Keverkian and Cole[5], Nayfen[6], Van Dyke[7], Bush[8], Holmes[9], Murdock[10]. Niijima[11] has given uniformly second order accurate difference scheme for reaction-diffusion equations, where as Miller[12] has given sufficient condition for the uniform first order convergence to a general three-point difference scheme. Spline approximation method has been investigated by Kadalbajoo and K.C. Patidar[13] to solve self-adjoint singular perturbation problems on non-uniform grids. Theory and discussions on perturbation problems has been presented by Elsgolts[14] and Reddy[15]. Reddy and Chakravarthy[16] constructed an exponentially fitted finite difference method for obtaining the solution of these problems. Kadalbajoo and Vikas gupta[17] presented a brief survey on numerical methods to solve singularly perturbed problems. Rashidinia et al.[18] used spline in compression to develop the numerical methods for singularly perturbed two-point boundary value problem and shown that the accuracy of proposed methods are of second order and fourth order. These methods are applicable for both singular and non-singular problems. Reddy[19] has employed numerical integration method to solve perturbation problems.

The objective of the work carried out in this paper is about obtaining numerical solution of singular perturbation problems. The singular perturbation problem is reduced to a differential equation of first order with a small deviating argument. The Simpson's 3/8 rule is employed to get the equation in y (xi). Hermite interpolation is used to obtain the value of y at the intermediate points of the boundary, finally yielding to a tridiagonal system of equations. The discrete invariant imbedding method is implemented to solve the system of equations. Numerical examples are given to show the applicability of our method. The results obtained by using this method are compared with the results obtained using Simpson's 1/3 rule. It is observed that the values obtained by this method are near to the exact solution in comparison with the solution obtained by employing Simpson's 1/3 rule.

Description of the Method

Left end boundary layer problem: We consider a class of singular perturbation problem

$$\in y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad x \in [0,1] \tag{1}$$

$$y(0) = \alpha \;\; and \;\; y(1) = \beta \tag{2}$$

Here $\in$ is a small positive parameter ($0<\in<<1$) and α, β are given constants. We assume that $a(x), b(x), f(x)$ to be sufficiently continuously differentiable functions in [0, 1]. In addition we assume that for some positive constant M, $b(x) \le 0$ *and* $a(x) \ge M > 0$ on [0, 1]. The assumptions entail that the singular perturbation problem (1) - (2) has a unique solution $y(x)$, which exhibits a boundary layer of width $o(\in)$ at $x = 0$ for small value of $\in$.

Taking the expansion in the neighborhood of a point by Taylor's series. we have

$$y'(x-\delta) = y'(x) - \delta\, y''(x), \tag{3}$$

$$y''(x) = \frac{1}{\delta}\left(y'(x) - y'(x-\delta)\right) \tag{4}$$

Here δ is a small positive deviating argument $(0<\delta<<1)$.
Substituting (4) in (1)

$$\in\left(\frac{y'(x)-y'(x-\delta)}{\delta}\right) + a(x)y'(x) + b(x)y(x) = f(x)$$

$$y'(x) = p(x)y'(x-\delta) + q(x)y(x) + r(x), \tag{5}$$

where

$$p(x) = \frac{\in}{\in + \delta a(x)} \qquad q(x) = \frac{-\delta b(x)}{\in + \delta a(x)} \qquad r(x) = \frac{\delta f(x)}{\in + \delta a(x)} \tag{6}$$

For $\delta \le x < 1$. We take the mesh size h i.e., $h = \frac{1}{N}$ and $xi = ih,\ i = 0, 1, \ldots\ldots\ldots N$ by dividing [0, 1] into N equal parts. Now integrating (5) in $[x_i,\ x_{i+1}]$ $\quad (i = 1, 2, \ldots\ldots N-1)$ we get

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} \left(p(x)\,y'(x-\delta) + q(x)y(x) + r(x)\right)dx$$

$$= P_{i+1}\,y\left(x_{i+1}-\delta\right) - p_i\,y(x_i-\delta) + \int_{x_i}^{x_{i+1}} \left(-p'(x)\,y(x-\delta) + q(x)y(x) + r(x)\right)dx \tag{7}$$

To evaluate the integral approximately we use Simpson's 3/8 rule.

$$y_{i+1} - y_i = P_{i+1}y\left(x_{i+1}-\delta\right) - P_i y\left(x_i - \delta\right)$$
$$+\frac{h}{8}\Big[-P'(x_i)\,y(x_i-\delta) + q(x_i)\,y(x_i) + r(x_i)$$
$$+3\left[-P\left(\frac{2x_i+x_{i+1}}{3}\right)y\left(\frac{2x_i+x_{i+1}}{3}-\delta\right)+q\left(\frac{2x_i+x_{i+1}}{3}\right)y\left(\frac{2x_i+x_{i+1}}{3}\right)+r\left(\frac{2x_i+x_{i+1}}{3}\right)\right]$$
$$+3\left[-P\left(\frac{x_i+2x_{i+1}}{3}\right)y\left(\frac{x_i+2x_{i+1}}{3}-\delta\right)+q\left(\frac{x_i+2x_{i+1}}{3}\right)y\left(\frac{x_i+2x_{i+1}}{3}\right)+r\left(\frac{x_i+2x_{i+1}}{3}\right)\right]$$
$$-P'\left(x_{i+1}\right)y\left(x_{i+1}-\delta\right) + q\left(x_{i+1}\right)y\left(x_{i+1}\right) + r\left(x_{i+1}\right)\Big] \tag{8}$$

To get the solution of (1)-(2) we need to approximate some of the terms of (8). By Taylor's series we have

$$y(x_i-\delta) = y(x_i) - \delta y'(x_i) = y_i - \delta\left(\frac{y_i - y_{i-1}}{h}\right)$$

$$y(x_i-\delta) = \left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i-1} \tag{9}$$

$$y\left(x_{i+1}-\delta\right) = \left(1-\frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i \tag{10}$$

and

$$y\left(\frac{2x_i + x_{i+1}}{3} - d\right) = y\left(x_{i+1/3} - d\right) =$$

$$y_{i+1/3} - \frac{d}{h}y_{i+1} + \frac{d}{h}y_i \tag{11}$$

$$y\left(\frac{x_i+2x_{i+1}}{3} - \delta\right) = y_{i+2/3} - \frac{\delta}{h}y_{i+1} + \frac{\delta}{h}y_i \tag{12}$$

From hermite interpolation we have

$$y_{i+\frac{1}{3}} = \frac{20y_i}{27} + \frac{7y_{i+1}}{27} + \frac{2h}{27}\left[2y'_i - y'_{i+1}\right] \tag{13}$$

$$y_{i+\frac{2}{3}} = \frac{7y_i}{27} + \frac{20y_{i+1}}{27} + \frac{2h}{27}\left[y'_i - 2y'_{i+1}\right] \tag{14}$$

Putting $x = x_{i+1}$ and $x = x_i$ in (5)

$$y'\left(x_{i+1}\right) = y'_{i+1} = P_{i+1}y'\left(x_{i+1}-\delta\right) + q_{i+1}y_{i+1} + r_{i+1} \tag{15}$$

$$y'_i = P_i y'\left(x_i - \delta\right) + q_i y_i + r_i \tag{16}$$

Also

$$y'\left(x_{i+1}-\delta\right) = \frac{y\left(x_{i+1}-\delta\right) - y\left(x_i-\delta\right)}{h} = \frac{1}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \left(\frac{2\delta}{h}-1\right)y_i - \frac{\delta}{h}y_{i-1}\right] \tag{17}$$

$$y^i\left(x_i-\delta\right) = \frac{y\left(x_{i+1}-\delta\right) - y\left(x_i-\delta\right)}{h}$$

$$= \frac{1}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \left(\frac{2\delta}{h}-1\right)y_i - \frac{\delta}{h}y_{i-1}\right] \tag{18}$$

From (15) and (16)

$$2y'_i - y'_{i+1} = \left(2P_i y'\left(x_i-\delta\right) + 2q_i y_i + 2r_i\right) - P_{i+1}y'\left(x_{i+1}-\delta\right) - q_{i+1}y_{i+1} - r_{i+1}$$

$$= \frac{\left(2P_i - P_{i+1}\right)}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \left(\frac{2\delta}{h}-1\right)y_i - \frac{\delta}{h}y_{i-1}\right] + 2q_i y_i - q_{i+1}y_{i+1} + 2r_i - r_{i+1} \tag{19}$$

$$y_i' - 2y_{i+1}' = \frac{(P_i - 2P_{i+1})}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \left(\frac{2\delta}{h}-1\right)y_i - \frac{\delta}{h}y_{i-1}\right] + q_i y_i - 2q_{i+1}y_{i+1} + r_i - 2r_{i+1} \quad (20)$$

Substituting (19) in (13) and then in (11) we get

$$y\left(\frac{2x_i + x_{i+1}}{3} - \delta\right) = \frac{20y_i}{27} + \frac{7y_{i+1}}{27} + \frac{2h}{27}\left[\left(\frac{2Pi - P_{i+1}}{h}\right)\left[1-\frac{\delta}{h}\right]y_{i+1} + \left(\frac{2\delta}{h}-1\right)y_i - \frac{\delta}{h}y_{i-1}\right] + \frac{2h}{27}[2q_i y_i - q_{i+1}y_{i+1} + 2r_i - r_{i+1} - \frac{\delta}{h}y_{i+1} + \frac{\delta}{h}y_i]$$
(21)

Substituting (20) in (14) and then (12) we get

$$y\left(\frac{x_i + 2x_{i+1}}{3} - \delta\right) = \frac{7y_i}{27} + \frac{20y_{i+1}}{27} + \frac{2h}{27}\left(\frac{P_i - 2P_{i+1}}{h}\right)\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \left(\frac{2\delta}{h}-1\right)y_i - \frac{\delta}{h}y_{i-1}\right] + \frac{2h}{27}\left[+q_i y_i - 2q_{i+1}y_{i+1} + r_i - 2r_{i+1}\right] - \frac{\delta}{h}y_{i+1} + \frac{\delta}{h}y_i \quad (22)$$

Now substituting (9), (10), (21) and (22) in (8)

$$y_{i+1} - y_i = P_{i+1}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i\right] - P_i\left[\left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i-1}\right] + \frac{h}{8}\left[-P_i'\left[\left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i-1}\right] + q_i y_i + r_i - 3\left(P'_{i+\frac{1}{3}} - q_{i+1/3}\right)\right.$$

$$\left[\frac{20y_i}{27} + \frac{7y_{i+1}}{27} + \frac{2}{27}(2P_i - P_{i+1})\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \left(\frac{2\delta}{h}-1\right)y_i - \frac{\delta}{h}y_{i-1}\right.\right.$$

$$\left.+\frac{4h}{27}q_i y_i - \frac{2h}{27}q_{i+1}y_{i+1} + \frac{4h}{27}r_i - \frac{2h}{27}r_{i+1}\right] - 3P'_{i+1/3}\left(-\frac{\delta}{h}y_{i+1} + \frac{\delta}{h}y_i\right) + 3r_{i+1/3} - 3\left(P'_{i+2/3} - q_{i+2/3}\right)$$

$$\left[\frac{7y_i}{27} + \frac{20y_{i+1}}{27} + \frac{2}{27}(P_i - 2P_{i+1})\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \left(\frac{2\delta}{h}-1\right)y_i - \frac{\delta}{h}y_{i-1}\right.\right.$$

$$\left.+\frac{2h}{27}q_i y_i - \frac{4h}{27}q_{i+1}y_{i+1} + \frac{2h}{27}r_i - \frac{4h}{27}r_{i+1}\right] - 3P'_{i+2/3}\left(-\frac{\delta}{h}y_{i+1} + \frac{\delta}{h}y_i\right) + 3r_{i+2/3}$$

$$\left.- P'_{i+1}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i\right] + q_{i+1}y_{i+1} + r_{i+1}\right] \quad (23)$$

Rearranging the terms, we get the following 3 term recurrence relation

$$A_i y_{i-1} - B_i y_i + C_i y_{i+1} = D_i \quad (24)$$

for $i = 1, 2 .... N-1$ .Where

$$A_i = P_i\frac{\delta}{h} + \frac{\delta}{8}P_i' + \frac{\delta}{36}\left(P'_{i+1/3} - q_{i+1/3}\right)(P_{i+1} - 2P_i) + \frac{\delta}{36}\left(P'_{i+2/3} - q_{i+2/3}\right)(2P_{i+1} - P_i)$$

$$B_i = 1 + P_{i+1}\frac{\delta}{h} - P_i\left(1-\frac{\delta}{h}\right) - \frac{h}{8}P_i^1\left(1-\frac{\delta}{h}\right) + \frac{h}{8}q_i - \frac{h}{9}\left(P'_{i+1/3} - q_{i+1/3}\right)$$
$$\left[\frac{5}{2} + \frac{1}{4}(2P_i - P_{i+1})\left(\frac{2\delta}{h} - 1\right) + \frac{h}{2}q_i\right] - \frac{h}{9}\left(P'_{i+2/3} - q_{i+2/3}\right)$$
$$\left[\frac{7}{8} + \frac{1}{4}(P_i - 2P_{i+1})\left(\frac{2\delta}{h} - 1\right) + \frac{h}{4}q_i\right] - \frac{3\delta}{8}\left(P'_{i+1/3} + P'_{i+2/3}\right) - P'_{i+1}\frac{\delta}{8}$$

$$C_i = 1 - P_{i+1}\left(1 - \delta/h\right) + \frac{h}{9}\left(P'_{i+1/3} - q_{i+1/3}\right)\left[\frac{7}{8} + \frac{1}{4}(2P_i - P_{i+1})\left(1-\frac{\delta}{h}\right) - \frac{h}{4}q_{i+1}\right]$$
$$+\frac{h}{9}\left(P'_{i+2/3} - q_{i+2/3}\right)\left[\frac{5}{2} + \frac{1}{4}(P_i - 2P_{i+1})\left(1-\frac{\delta}{h}\right) - \frac{h}{2}q_{i+1}\right] - \frac{3\delta}{8}\left(P'_{i+1/3} + P'_{i+2/3}\right)$$
$$+\frac{h}{8}P'_{i+1}\left(1-\frac{\delta}{h}\right) - \frac{h}{8}q_{i+1}$$

$$D_i = \frac{h}{8}(r_i + r_{i+1}) + \frac{3h}{8}\left(r_{i+1/3} + r_{i+2/3}\right) - \frac{h^2}{36}\left(P'_{i+1/3} - q_{i+1/3}\right)(2r_i - r_{i+1})$$
$$-\frac{h^2}{36}\left(P'_{i+2/3} - q_{i+2/3}\right)(r_i - 2r_{i+1}) \quad (25)$$

From equation (24) we get a system of equations with (N+1) unknowns y_0 to y_N Solving these equations by using the given boundary conditions (2) we obtain the solution (y_0 to y_N) of the boundary value problem (1)-(2).

Right end boundary layer problems: Consider a singular perturbation problem

$$\in y''(x) + a(x)y'(x) + b(x)y(x) = f(x); x \in [0,1] \quad (26)$$

With $y(0) = \alpha$ & $y(1) = \beta$ (27)

We suppose that $a(x), b(x), f(x)$ to be sufficiently continuously differentiable functions in [0, 1]. In addition we assume that for some negative constant M, $a(x) \le M < 0$ in $[0,1]$.This assumptions entail that the boundary layer will be in the neighborhood of $x = 1$. By Taylors series expansion we have

$$y'(x+\delta) = y'(x) + \delta y''(x) \quad (28)$$

$$\Rightarrow y''(x) = \frac{y'(x+\delta) - y'(x)}{\delta}$$

Where δ is a small positive deviating argument (0< δ <<1). Substituting in (26) and rearranging

$$y'(x) = P(x)y'(x+\delta) + q(x)y(x) + r(x) \quad (29)$$

where $P(x) = \frac{-\in}{\delta a(x) - \in}, q(x) = \frac{-\delta b(x)}{\delta a(x) - \in}, r(x) = \frac{\delta f(x)}{\delta a(x) - \in}$ (30)

Dividing the interval $[0,1]$ in to N equal parts with mesh size h; i.e. $h=\frac{1}{N}, x_i = ih, i = 0,1....N$, integrating (29) in $[x_{i-1}, x_i]$ $(i = 1,2 .... N - 1)$.

We get

$$y(x_i) - y(x_{i-1}) = \int_{x_{i-1}}^{x_i} \left(P(x)y'(x+\delta) + q(x)y(x) + r(x)\right)dx$$

$$= P(x_i)y(x_i+\delta) - P(x_{i-1})y(x_{i-1}+\delta) + \int_{x_{i-1}}^{x_i}\left(-P'(x)y(x+\delta) + q(x)y(x) + r(x)\right)dx \quad (31)$$

We use Simpson's $3/8$ rule for evaluating the integral approximately $y_i - y_{i-1} = P_i y(x_i+\delta) - P_{i-1}y(x_{i-1}+\delta)$

$$+ \frac{h}{8}\Big[-P'_{i-1}y(x_{i-1}+\delta) + q_{i-1}y_{i-1} + r_{i-1} + 3\Big[-P^1\left(\frac{2x_{i-1}+x_i}{3}\right)y\left(\frac{2x_{i-1}+x_i}{3}+\delta\right) + q_{i-2/3}\,y\left(\frac{2x_{i-1}+x_i}{3}\right) + r_{i-2/3}$$
$$- P'_{i-1/3}\,y\left(\frac{x_{i-1}+2x_i}{3}+\delta\right) + q_{i-1/3}\,y\left(\frac{x_{i-1}+2x_i}{3}\right) + r_{i-1/3}\Big] - P'_i y(x_i+\delta) + q_i y_i + r_i\Big] \quad (32)$$

By Taylors series we have

$$y(x_{i-1}+\delta) \cong y_{i-1} + \delta y'_{i-1} = \left(1-\frac{\delta}{h}\right)y_{i-1} + \frac{\delta}{h}y_i \quad (33)$$

$$y(x_i+\delta) = \left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i+1} \quad (34)$$

and

$$y\left(\frac{2x_{i-1}+x_i}{3}+\delta\right) = y\left(x_{i-2/3}+\delta\right) = y_{i-2/3} + \frac{\delta}{h}y_i - \frac{\delta}{h}y_{i-1} \quad (35)$$

$$y\left(\frac{x_{i-1}+2x_i}{3}+\delta\right) = y\left(x_{i-1/3}+\delta\right) = y_{i-1/3} + \frac{\delta}{h}y_i - \frac{\delta}{h}y_{i-1} \quad (36)$$

From hermite interpolation

$$y\left(x_{i-2/3}\right) = \frac{1}{27}\left[20y_{i-1} + 7y_i\right] + \frac{2h}{27}\left[2y'_{i-1} - y'_i\right] \quad (37)$$

$$y\left(x_{i-1/3}\right) = \frac{1}{27}\left[7y_{i-1} + 20y_i\right] + \frac{2h}{27}\left[y'_{i-1} - 2y'_i\right] \quad (38)$$

Substituting $x = x_{i+1}$ and $x = x_i$ in (28) we get

$$y'_{i-1} = P_{i-1}y'(x_{i-1}+\delta) + q_{i-1}y_{i-1} + r_{i-1} \quad (39)$$

$$y'_i = P_i y'(x_i+\delta) + q_i y_i + r_i \quad (40)$$

Also

$$y'(x_{i-1}+\delta) = \frac{y(x_i+\delta) - y(x_{i-1}+\delta)}{h} = \frac{1}{h}\left[\frac{\delta}{h}y_{i+1} + \left(1-\frac{2\delta}{h}\right)y_i - \left(1-\frac{\delta}{h}\right)y_{i-1}\right] \quad (41)$$

$$y'(x_i+\delta) = \frac{y(x_i+\delta) - y(x_{i-1}+\delta)}{h} = \frac{1}{h}\left[\frac{\delta}{h}y_{i+1} + \left(1-\frac{2\delta}{h}\right)y_i - \left(1-\frac{\delta}{h}\right)y_{i-1}\right] \quad (42)$$

From (39) and (40) we can evaluate the values of $2y'_{i-1} - y'_i$ and $y'_{i-1} - 2y'_i$.

Substituting these values in (35) we get

$$y\left(\frac{2x_{i-1}+x_i}{3}+\delta\right) = \frac{1}{27}\left[20y_{i-1} + 7y_i\right] + \frac{2h}{27}\Big[\frac{(2P_{i-1}-P_i)}{h}\left(\frac{\delta}{h}y_{i+1} + \left(1-\frac{2\delta}{h}\right)y_i - \left(1-\frac{\delta}{h}\right)y_{i-1}\right) + 2q_{i-1}y_{i-1} - q_i y_i + 2r_{i-1} - r_i\Big] + \frac{\delta}{h}y_i - \frac{\delta}{h}y_{i-1} \quad (43)$$

Substituting these values in (36) we get

$$y\left(\frac{x_{i-1}+2x_i}{3}+\delta\right) = \frac{1}{27}\left[7y_{i-1} + 20y_i\right] + \frac{2h}{27}\Big[\frac{(P_{i-1}-2P_i)}{h}\left(\frac{\delta}{h}y_{i+1} + \left(1-\frac{2\delta}{h}\right)y_i - \left(1-\frac{\delta}{h}\right)y_{i-1}\right) + q_{i-1}y_{i-1} - 2q_i y_i + r_{i-1} - 2r_i\Big] + \frac{\delta}{h}y_i - \frac{\delta}{h}y_{i-1} \quad (44)$$

Substituting (33), (34), (43) and (44) in (32)

$$y_i - y_{i-1} = P_i\left(\left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i+1}\right) - P_{i-1}\left[\left(1-\frac{\delta}{h}\right)y_{i-1} + \frac{\delta}{h}y_i\right]$$
$$+ \frac{h}{8}\Big[-P'_{i-1}\left[\left(1-\delta/h\right)y_{i-1} + \delta/h\, y_i\right] + q_{i-1}y_{i-1} + r_{i-1}$$
$$+ 3\Big[\left(-P'_{i-2/3} + q_{i-2/3}\right)\Big[\frac{(20y_{i-1}+7y_i)}{27} + \frac{2h}{27}\Big[\left(\frac{2P_{i-1}-P_i}{h}\right)\Big[\frac{\delta}{h}y_{i+1}$$
$$+ \left(1-\frac{2\delta}{h}\right)y_i - \left(1-\frac{\delta}{h}\right)y_{i-1}\Big] + 2q_{i-1}y_{i-1} - q_i y_i + 2r_{i-1} - r_i\Big]$$
$$+ \left[-P'_{i-2/3}\left(\frac{\delta}{h}y_i - \frac{\delta}{h}y_{i-1}\right)\right] + r_{i-2/3} + \left(-P'_{i-1/3} + q_{i-1/3}\right)\Big[\frac{7y_{i-1}+20y_i}{27} + \frac{2h}{27}$$
$$\Big[\frac{(P_{i-1}-2P_i)}{h}\Big(\frac{\delta}{h}y_{i+1}$$
$$+ \left(1-\frac{2\delta}{h}\right)y_i - \left(1-\frac{\delta}{h}\right)y_{i-1}\Big) + q_{i-1}y_{i-1} - 2q_i y_i + r_{i-1} - 2r_i\Big]$$
$$+ \left[-P'_{i-1/3}\left(\frac{\delta}{h}y_i - \frac{\delta}{h}y_{i-1}\right)\right] + r_{i-1/3}\Big] - P'_i\left(\left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i+1}\right) + q_i y_i + r_i\Big] \quad (45)$$

Rearranging the terms of (45) we get we get the following 3 term recurrence relation

$$A_i y_{i-1} - B_i y_i + C_i y_{i=1} = D_i;\ i = 1,2...N-1 \quad (46)$$

where

$$A_i = -1 + P_{i-1}\left(1-\frac{\delta}{h}\right)+\frac{h}{8}P'_{i-1}\left(1-\frac{\delta}{h}\right)-\frac{h}{8}q_{i-1}+\frac{h}{18}\left(P'_{i-2/3}-q_{1-2/3}\right)$$

$$\left[5-\frac{(2P_{i-1}-P_i)}{2}\left(1-\frac{\delta}{h}\right)+hq_{i-1}\right]+\frac{h}{36}\left[P'_{i-1/3}-q_{i-1/3}\right]$$

$$\left[\frac{7}{2}-(P_{i-1}-2P_i)\left(1-\frac{\delta}{h}\right)+hq_{i-1}\right]-\frac{3\delta}{8}\left[P'_{i-2/3}+P'_{i-1/3}\right]$$

$$B_i = -1+P_i\left(1-\frac{\delta}{h}\right)-P_{i-1}\frac{\delta}{h}-\frac{\delta}{8}P^1_{i-1}-\frac{h}{36}\left(P'_{i-2/3}-q_{i-2/3}\right)$$

$$\left[\frac{7}{2}+(2P_{i-1}-P_i)\left(1-\frac{2\delta}{h}\right)-hq_i\right]-\frac{h}{18}\left[P'_{i-1/3}-q_{i-1/3}\right]$$

$$\left[5+\frac{(P_{i-1}-2P_i)}{2}\left(1-\frac{2\delta}{h}\right)-hq_i\right]-\frac{3\delta}{8}\left[P'_{i-2/3}+P'_{i-1/3}\right]$$

$$-\frac{h}{8}P'_i\left(1-\frac{\delta}{h}\right)+\frac{h}{8}q_i\ C_i=-P_i\frac{\delta}{h}+\frac{\delta}{36}(2P_{i-1}-P_i)\left(P'_{i-2/3}-q_{i-2/3}\right)$$

$$+\frac{\delta}{36}(P_{i-1}-2P_i)\left(P^1_{i-1/3}-q_{i-1/3}\right)+\frac{\delta}{8}P'_i\ D_i=\frac{h}{8}(r_{i-1}+r_i)+\frac{3h}{8}\left(r_{i-2/3}+r_{i-1/3}\right)$$

$$\frac{-h^2}{36}\left[\left(P'_{i-2/3}-q_{i-2/3}\right)(2r_{i-1}-r_i)+\left(P'_{i-1/3}-q_{i-1/3}\right)(r_{i-1}-2r_i)\right] \quad (47)$$

From equation (46) we get a system of equations with (N+1) unknowns $y_0$to y_N . The boundary conditions (27) with (N-1) equations give (N+1) equations. Solving the system of equations we get the solution ($y_{1,}\, y_2 \ldots. y_{N-1}$).

Numerical Results

Three linear singular perturbation problems with left-end boundary layer and one linear problem with right-end boundary layer, of which two are with constant coefficients and two are with variable coefficients are considered.

Example1: From fluid dynamics for fluid of small viscosity (Reinhardt[20], example 2) we consider the non-homogeneous singular perturbation problem

$$\in\ y''(x)+y'(x)=1+2x;\quad x\in[0,1]$$

With $y(0)=0$ and $y(1)=1$

The exact solution of non-homogeneous singular perturbation problem is

$$y(x)=x(x+1-2\in)+\frac{(2\in-1)\left(1-e^{-x/\in}\right)}{\left(1-e^{-1/\in}\right)}$$

For $\in=10^{-3}$ and $\in=10^{-4}$ the numerical results are specified in the tables 1and 2 for different values of δ satisfying the condition $0<\delta<<1$ respectively. The comparison is shown in figures- 1 and 2.

Table-1
Numerical outcome of example 1 for ε=0.001, h=0.1

x	y(x)		Exact solution
	δ=0.001	δ=0.005	
0.1	-0.8684178	-0.8684165	-0.8881992
0.2	-0.7582041	-0.7582028	-0.7583992
0.3	-0.6085981	-0.6085968	-60859930
0.4	-0.4387999	-0.4387989	-0.4387993
0.5	-0.249	-0.2489990	-0.2489994
0.6	-0.0392	-0.0391992	-0.0391995
0.7	0.1906	0.19060062	0.1906004
0.8	0.4404	0.44040043	0.4404003
0.9	0.7102	0.71020022	0.7102002

Table-2
Numerical outcome of example 1 for ε=0.0001, h=0.1

x	y(x)		Exact solution
	δ=0.0001	δ=0.0005	
0.1	-0.8878222	-0.88782	-0.8898192
0.2	-0.759838	-0.75984	-0.7598393
0.3	-0.60986	-0.60986	-0.6098593
0.4	-0.43988	-0.43988	-0.4398794
0.5	-0.2499	-0.24990	-0.2498994
0.6	-0.03992	-0.03992	-0.0399195
0.7	0.19006	0.19006	0.1900604
0.8	0.44004	0.44004	0.4400403
0.9	0.71002	0.71002	0.7100202

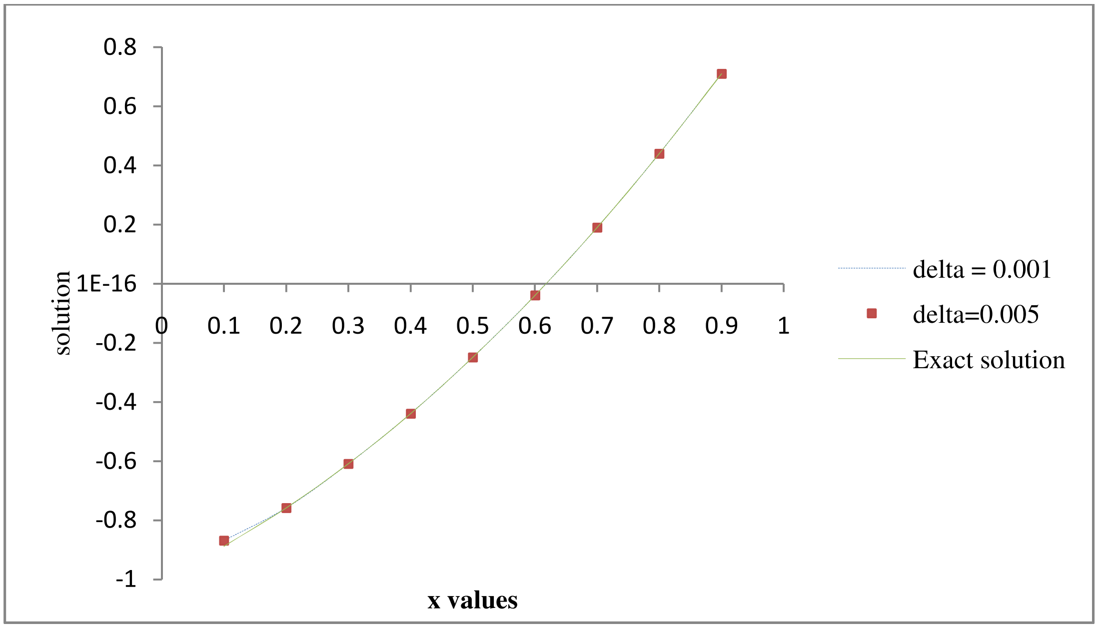

Figure-1
Comparison of numerical results with ε=0.001

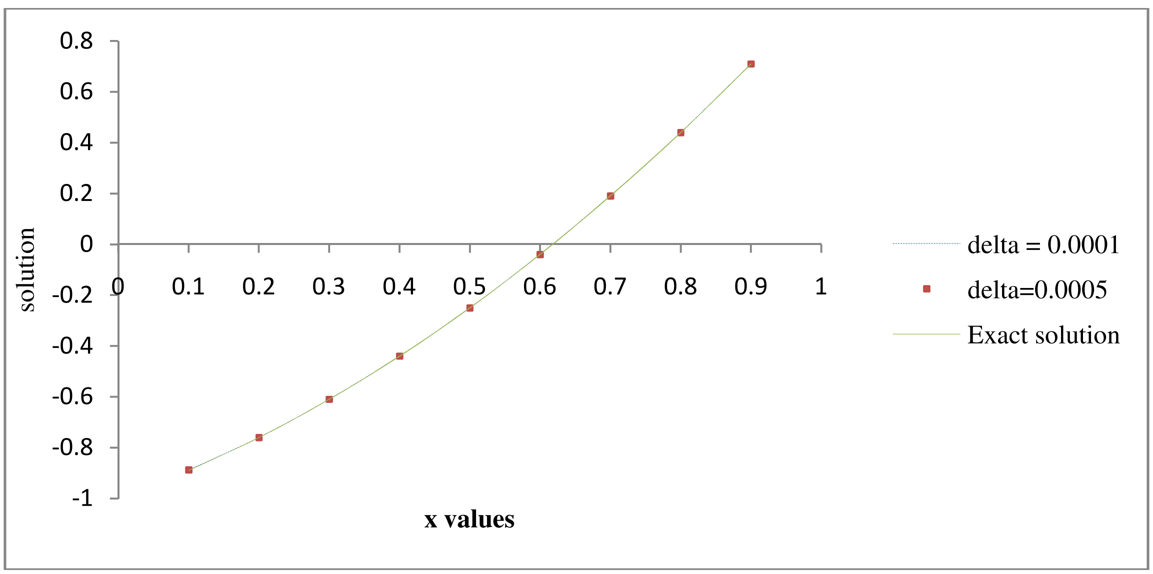

Figure-2
Comparison of numerical results with ε=0.0001

Example2: From Bender and Orszag [2] (page 480, problem 9.17 with $\alpha = 0$)
we consider the homogeneous singular perturbation problem

$$\in y''(x) + y'(x) - y(x) = 0; \quad x \in [0,1]$$

With $y(0) = 1$ and $y(1) = 1$

The exact solution of homogeneous singular perturbation problem is

$$y(x) = \frac{\left[\left(e^{m2} - 1\right)e^{m_1 x} + \left(1 - e^{m1}\right)e^{m_2 x}\right]}{\left(e^{m2} - e^{m1}\right)}$$

where $m_1 = \dfrac{-1 + \sqrt{1 + 4\in}}{2\in}, m_2 = \dfrac{-1 - \sqrt{1 + 4\in}}{2\in}$

For $\in = 10^{-3}$ and $\in = 10^{-4}$, the results are specified in the table 3 and table 4 respectively. The comparison is shown in figures 3 and 4.

Table-3
Numerical outcome of example 2 for ε=0.001, h=0.01

x	y(x)		Exact solution
	δ=0.001	δ=0.005	
0	1	1	1
0.02	0.38084	0.380839	0.375678
0.04	0.383299	0.383299	0.383260
0.06	0.390992	0.390991	0.390994
0.08	0.398882	0.398882	0.398885
0.1	0.406932	0.406932	0.406935
0.2	0.449685	0.449685	0.449688
0.3	0.496929	0.496929	0.496932
0.4	0.549138	0.549137	0.549140
0.5	0.606831	0.606831	0.606833
0.6	0.670585	0.670585	0.670588
0.7	0.741038	0.741038	0.741040
0.8	0.818893	0.818893	0.818894
0.9	0.904927	0.904927	0.904928
1	1	1	1

Table-4
Numerical outcome of example 2 for ε=0.0001, h=0.01

x	y(x)		exact solution
	δ=0.0001	δ=0.0005	
0	1	1	1
0.02	0.375423	0.375423	0.37534787
0.04	0.38294382	0.38294382	0.38292964
0.06	0.39067871	0.39067871	0.39066455
0.08	0.39856984	0.39856984	0.3985557
0.10	0.40662035	0.40662036	0.40660625
0.20	0.44937876	0.44937877	0.4493649
0.30	0.49663346	0.49663346	0.49662006
0.40	0.54885726	0.54885726	0.54884456
0.50	0.60657267	0.60657268	0.60656098
0.60	0.67035719	0.67035719	0.67034685
0.70	0.74084901	0.74084901	0.74084044
0.80	0.81875344	0.81875344	0.81874712
0.90	0.90484995	0.90484995	0.90484646
1.00	1	1	1

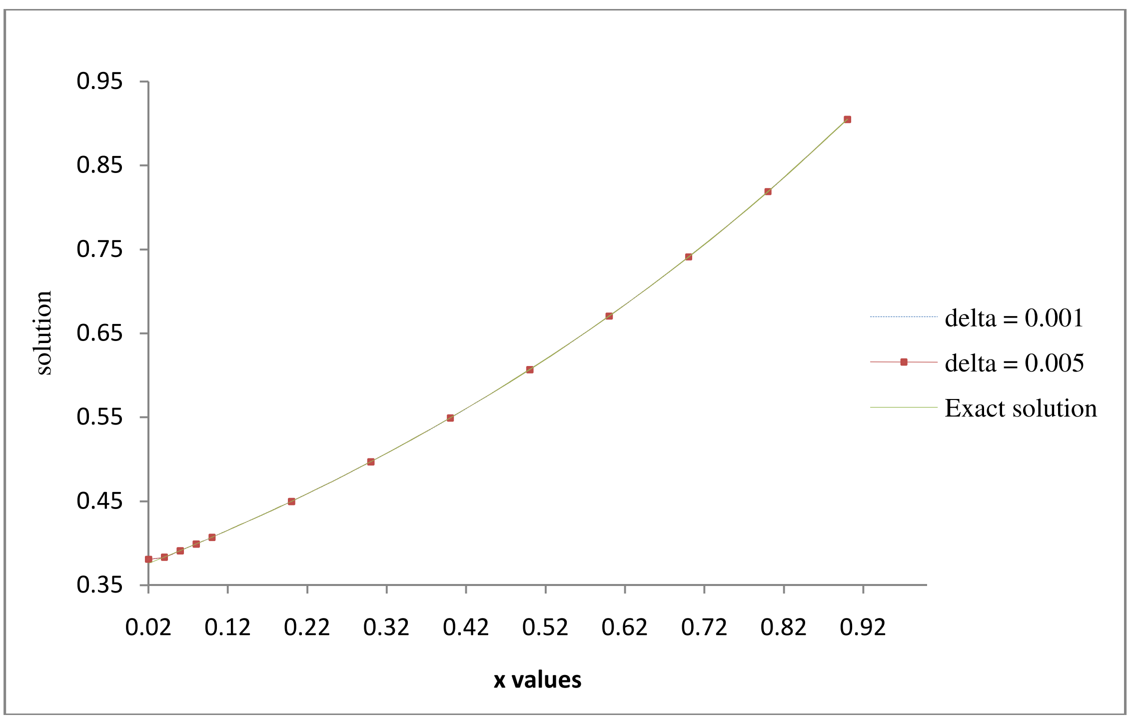

Figure-3
Comparison of numerical results with ε=0.001

Example 3: Consider the singular perturbation problem with variable coefficients from Kevorkian[5] (page 33, with $\alpha = -\frac{1}{2}$)

$$\in y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \left(\frac{1}{2}\right) y(x) = 0, 0 \le x \le 1,$$
$$y(0) = 0,\ y(1) = 1$$

We have taken consistently valid approximation (Nayten[6], page 148) as per the exact solution that is

$$y(x) = \frac{1}{(2 - x)} + \left(\frac{1}{2}\right) e^{\left(-\left(x - \frac{x^2}{4}\right)/\in\right)}$$

The results obtained by our method and by Simpson's 1/3 rule for $\in = 10^{-3}$ and for $\in = 10^{-4}$ are specified in the tables 5 and 6 respectively, and the comparison is shown in figures 5 and 6. The comparison of errors in both cases is given in the tables 7 and 8 respectively.

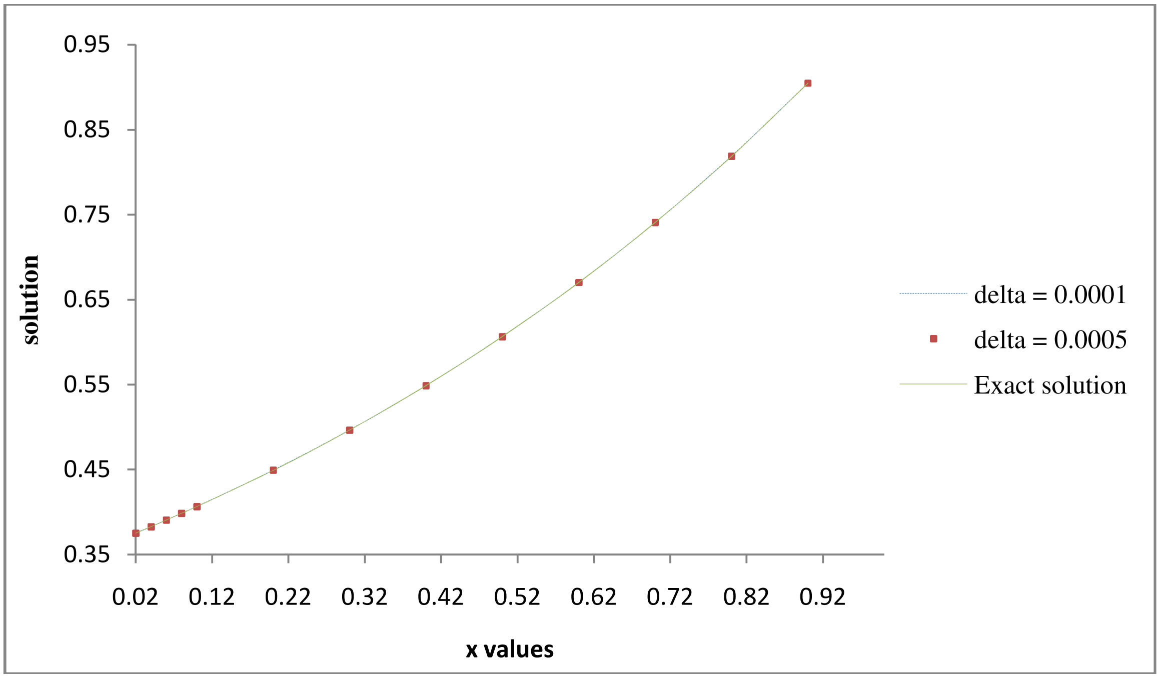

Figure-4
Comparison of numerical results with ε=0.0001

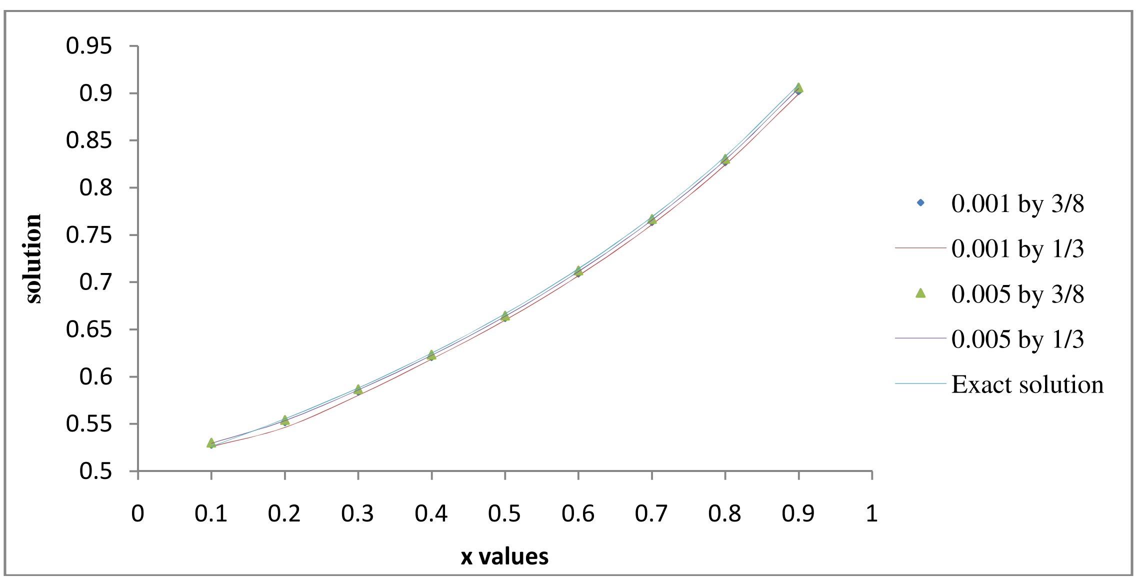

Figure-5
Comparison of numerical results with ε=0.001

Table-5
Numerical outcome of example 3 for ε=0.001, h=0.1

x	y(x)				exact solution
	by 3/8 rule δ=0.001	by 1/3 rule δ=0.001	by 3/8 rule δ=0.005	by 1/3 rule δ=0.005	
0	1	1	1	1	1
0.1	0.527732	0.526295	0.530153	0.529536	0.526316
0.2	0.551594	0.546451	0.554211	0.553544	0.555556
0.3	0.583946	0.580445	0.586717	0.58601	0.588235
0.4	0.620398	0.618668	0.62334	0.622589	0.625
0.5	0.661701	0.659856	0.664837	0.664036	0.666667
0.6	0.708892	0.706915	0.712249	0.711391	0.714286
0.7	0.763324	0.761195	0.766937	0.766012	0.769231
0.8	0.826802	0.824496	0.830714	0.829712	0.833333
0.9	0.901782	0.899267	0.906047	0.904954	0.909091
1	1	1	1	1	1

Table-6
Numerical outcome of example 3 for ε=0.0001, h=0.1

x	y(x)				exact solution
	by 3/8 rule δ=0.0001	by 1/3 rule δ=0.0001	by 3/8 rule δ=0.0005	by 1/3 rule δ=0.0005	
0	1	1	1	1	1
0.1	0.522382	0.520901	0.524902	0.524256	0.526316
0.2	0.55085	0.549284	0.553515	0.552831	0.555556
0.3	0.583253	0.581595	0.586072	0.585349	0.588235
0.4	0.619707	0.617945	0.622701	0.621932	0.625
0.5	0.661022	0.659142	0.664212	0.663392	0.666667
0.6	0.708238	0.706225	0.711654	0.710776	0.714286
0.7	0.76272	0.760551	0.766395	0.765449	0.769231
0.8	0.826281	0.823932	0.83026	0.829235	0.833333
0.9	0.9014	0.898837	0.905736	0.904618	0.909091
1	1	1	1	1	1

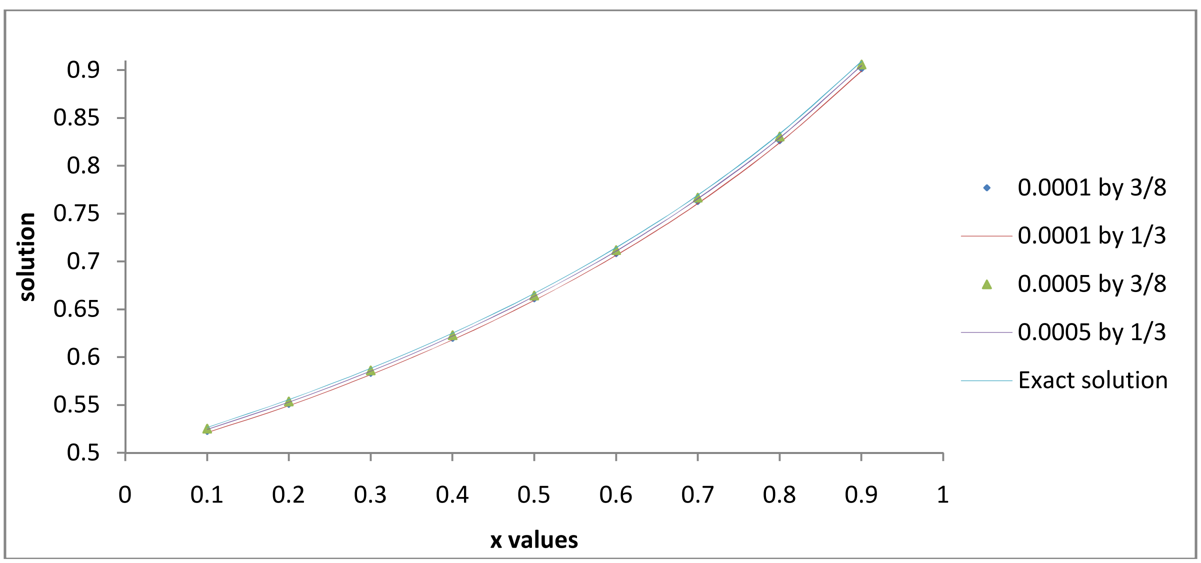

Figure-6
Comparison of numerical results with ε=0.0001
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Table-7
Comparison of absolute errors by Simpson's 3/8 rule and by Simpson's 1/3 rule for example 3 with ε=0.001 and h=0.1

x	Error			
	by 3/8 rule δ=0.001	by 1/3 rule δ=0.001	by 3/8 rule δ=0.005	by 1/3 rule δ=0.005
0	0	0	0	0
0.1	0.001420	2.08E-05	0.003840	0.003220
0.2	0.003962	0.009105	0.001344	0.002012
0.3	0.004289	0.00779	0.001518	0.002225
0.4	0.004602	0.006332	0.00166	0.002411
0.5	0.004966	0.006811	0.001829	0.002631
0.6	0.005394	0.007371	0.002036	0.002895
0.7	0.005907	0.008036	0.002293	0.003218
0.8	0.006531	0.008837	0.00262	0.003622
0.9	0.007309	0.009824	0.003044	0.004137
1	0	0	0	0

Table-8
Comparison of absolute errors by Simpson's 3/8 rule and by Simpson's 1/3 rule for example 3 with ε=0.0001 and h=0.1

x	Error			
	by 3/8 rule δ=0.0001	by 1/3 rule δ=0.0001	by 3/8 rule δ=0.0005	by 1/3 rule δ=0.0005
0	0	0	0	0
0.1	0.003934	0.005415	0.001414	0.002060
0.2	0.004706	0.006272	0.002041	0.002724
0.3	0.004982	0.006641	0.002163	0.002886
0.4	0.005293	0.007055	0.002299	0.003068
0.5	0.005645	0.007524	0.002454	0.003274
0.6	0.006047	0.008061	0.002631	0.00351
0.7	0.006511	0.008679	0.002836	0.003782
0.8	0.007052	0.009401	0.003074	0.004099
0.9	0.007691	0.010253	0.003355	0.004473
1	0	0	0	0

Example4. Consider the singular perturbation problem

$$\in\ y''(x) - y'(x) = 0, x \in [0,1]$$

With $y(0) = 1$ and $y(1) = 0$.

The exact solution of singular perturbation problem is

$$y(x) \equiv \frac{\left(e^{\frac{x-1}{\in}} - 1\right)}{\left(e^{-1/\in} - 1\right)}$$

At $x = 1$ this problem has a boundary layer. For $\in = 10^{-3}$ and for $\in = 10^{-4}$ the results are specified in the tables 9 and 10 respectively.

Table-9
Numerical outcome of example 4 for ε=0.001, h=0.01

x	y(x)		exact
	δ=0.002	δ=0.005	
0.1	1	1	1
0.2	1	1	1
0.3	1	1	1
0.4	1	1	1
0.5	1	1	1
0.6	1	1	1
0.7	1	1	1
0.8	1	1	1
0.9	1	1	1
0.92	1	1	1
0.94	0.999999	0.999999	1

Table-10
Numerical outcome for example 4 with ε=0.0001, h=0.01

x	y(x)		exact
	δ=0.0002	δ=0.0005	
0.1	1	1	1
0.2	1	1	1
0.3	1	1	1
0.4	1	1	1
0.5	1	1	1
0.6	1	1	1
0.7	1	1	1
0.8	1	1	1
0.9	1	1	1
0.92	1	1	1
0.94	1	1	1

Conclusion

We consider four linear boundary value problems. Their numerical solution and absolute errors are given at different values of δ by fixing the mesh size h. The approximate solution and exact solutions at the grid points are summarized in the tabular form. Further the approximate solution and exact solution are shown graphically. It is observed that the approximate solution is very near to the exact solution. From the tables it is seen that the maximum absolute error is almost same for the both values of ε. It is also observed that the method is more accurate for the problems with constant coefficients when compared to the problems with variable coefficients. It is also observed that the results obtained are more or less same as the numerical solution obtained by employing Simpson's one third rule in the case of singular perturbation problems with constant coefficients. For singular perturbation problems with variable coefficients the results obtained are more close to the exact solution rather than the solution obtained by employing Simpson's 1/3 rule.

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