Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. **2(7),** 7-16, July **(2014)** *Res. J. Mathematical and Statistical Sci.*

An Optimal Control Model of a Closed Economy with Government Participation

Gabriel A. Okyere, Francis T. Oduro and Opuni S. Basoa
Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, GHANA

Available online at: www.isca.in, www.isca.me
Received 27th May 2014, revised 11th June 2014, accepted 10th July 2014

Abstract

This paper seeks to build a deterministic optimal control model to fit the Ghanaian economy. The model looks at a simple closed economy with government participation. The Hamilton-Pontryagin's theory was employed to analyze the systems obtained. Numerical simulations were also used to obtain numerical solutions to the model. Results obtained from the analysis performed suggest that the system is controllable. The short to long term predictions of the system are generally very good. The results obtained also suggest that the 2020 goal of Ghana becoming a middle income economy is attainable if the control measures prescribed are applied.

Keywords: Hamiltons-Pontryagin's, control variables, state variable, Ghanaian economy.

Introduction

It is generally agreed that the mechanism of an economy of any entity is like a complex device, consisting of diverse parts working in unison to attain some set targets. Just as a machine fails any time any part is defective, an economy is in mess any time something is wrong with any of its fundamentals.

For instance, the fallout of the world food shortage in the later part of 2007 and into 2008 and the allied additional effects of the galloping oil prices, the financial and economic melt down are all too fresh in our memories. The repercussions continued into 2009. On the other hand, good economic basics and expectations hold good for economic stability and growth.

Important to governments and policymakers is the economic welfare of the citizenry, mindful of the effects of inflation and unemployment dynamics on the health of the economy, and hence, welfare. The higher the inflationary levels, the more difficult economic and business decisions now, and into the future, are to make. Cost of living becomes increasingly high, and the economy unstable. Interestingly, some minimal inflation is required in any economy to kick-start to make it worthwhile to undertake the risk to start-up any business. So the question is: what is this minimal level of inflation required in the economy to kick-start it? How high can it be in order not to plunge the economy into instability, and hence, chaos?

Unemployment, on the other hand, can be debilitating and self-devaluing as one loses one's stream of income and rather becomes dependent on others. Thus one's welfare plummets and so does one's dignity. Generally, as unemployment rate rises aggregate demand falls and so does investment, production and national income, all things being equal. The economy thus shrinks, and economic well-being worsens. Empirical evidence and general economic theory suggest that there is a trade off between these variables. However, there are also evidences of instances of high rates of unemployment associated with high inflationary rates, the converse is also true. How much trade off between these variables is prudently attainable?

The prime aim of any government is to stabilize its economy, mindful of the need to grow it, over and above the population growth rate. A government may embark on expansionary fiscal or monetary policies (the mechanisms may differ) to induce growth in its economy. But if the economy happens to be at or around its full capacity then the expansionary policy rather leads to over heating it, giving rise to inflationary pressures or unemployment or both. The converse is also true. So to what degree can a government influence its economy?

An economy, in general, consists of lots of interwoven variables forming a complex system. Whereas certain pairs or sets of economic variables may have positive (negative) correlation with one another they may impact on the larger economy along the same or opposite direction. Thus, whereas interest rate and investment are negatively related, interest rates and savings are positively correlated. But, all things being equal, as interest rate rises, savings surge up but investments shrink with a resulting dip in economic fortunes and well being. In this paper we develop an optimal control model for the Ghanaian economy and make predictions based on the developed model.

Model Development

The involvement of government in an economy entails a number of issues, and thus some new variables like money supply, with attendant interest rate and inflation management.

We take it from the expositions from these researches[1-4]. The aggregate demand, $Z(t)$, is given by

$$Z(t) = C(t) + I(t) + G(t) \quad (1)$$

where $G(t)$ is the government expenditure, $C(t)$ is total consumption and $I(t)$ is investment. The dynamics in the market can be introduced in various ways[1,5,6]. Whereas Samuelson[7] and Hicks[8] did so by introducing lags into consumption and investment behaviour, Philips[9-11], as captured by Turnovsky[1,12,13], assumed gradual clearance in the market. He specified this by

$$\dot{Y}(t) = \alpha\{Z(t) - Y(t)\}, \quad \alpha > 0 \quad (2)$$

where $Y(t)$ and $\dot{Y}(t)$ respectively denotes aggregate supply (or national product) and the rate of change in aggregate national product at time t. If aggregate demand exceeds output, supply is increased at a rate proportional to excess demand and vice versa. To complete the sub-model here, behavioural hypotheses are brought to bear on consumption and investment. The simplest of these is to specify that consumption is proportional to current output, that is[1, 12, 14].

$$C(t) = b \cdot Y(t), \qquad 0 < b < 1. \quad (3)$$

Thus by putting equation (1) and equation (3) into equation (2), we obtain

$$\dot{Y}(t) = \alpha\{(b-1)Y(t) + I(t) + G(t)\} \quad (4)$$

$$= b_1 Y(t) + b_2 I(t) + b_3 G(t) \quad (5)$$

where $b_1 = \alpha(b-1)$, $b_2 = b_3 = \alpha$.

Phillips' input was to introduce various policy rules for $G(t)$. Much of this was developed in terms of this simple model. However, most of the ensuing literature, as well as much of Phillips' own contributions endogenized investment by using some form of the accelerator theory. The effect of this is to raise the order of the equilibrium dynamics, thereby generating a richer array of time paths for output and other relevant variables[1].

To model the relationship between the time rate of change of investment and other variables in the national product identity, we employ the accelerator model of investment streams[15,16]. The accelerator model is a model of business investment, which in its simplest form, relates the level of investment to the rate of change in output, here, GDP. This is invariable referred to as the business cycle, and it may explain the cyclical fluctuations in national product. Here, consumption is assumed to be determined by the last period's income. Also, the capital stock that entrepreneurs desire is proportional to the last period's output[14,16,17]. By relaxing these assumptions and casting the original idea in a continuous dynamic format, we have:

$$K^d(t) = \nu Y(t), \quad 0 < \nu < 1 \quad (6)$$

$$\dot{K}(t) = \varepsilon\{K^d(t) - K(t)\} = I(t) \quad (7)$$

where $K^d(t)$ is the desired capital stock, ν and η are constants to be determined. From equation (6) and (7), we obtain

$$\dot{I}(t) = \eta\{\nu\dot{Y}(t) - \dot{K}(t)\} = \eta\{\nu\dot{Y}(t) - I(t)\} \quad (8)$$

which is the same as espoused[1]. Now, using the aggregate demand and aggregate national income identity, then equation (8) becomes

$$\dot{I}(t) = \eta\{\nu\dot{Y}(t) - I(t)\} = \eta\{\nu\alpha\{Z(t) - Y(t)\} - I(t)\}$$

$$= \eta_1 \cdot Y(t) + \eta_2 \cdot I(t) + \eta_3 \cdot G(t), \quad (9)$$

where $\eta_1 = \eta\nu\alpha(b-1)$, $\eta_2 = \eta(\nu\alpha - 1)$ and $\eta_3 = \eta\nu\alpha$.

With the involvement of government in the economy, monetary policies, in addition to fiscal policies, are executed at various points in time. This requires that we add up the dynamics in money supply and demand, inflation and unemployment, as well as their linkages with aggregate demand and supply (product). Now, suppose $M^d(t)$, $M^s(t)$ and $r(t)$ respectively denote the aggregate demand for money, aggregate supply of money and the equilibrium nominal interest rate charged, each at any time t in an economy. Then, from economic theory, demand for money is an increasing function of aggregate income (product) and nominal price level, and a decreasing function of the nominal interest rate. Thus

$$M^d(t) = L(Y(t), r(t), P(t))$$

and since the price index, $P(t)$, is related to the inflationary rate, $p(t)$, by extension,

$$\mathrm{M}^{\delta}(\tau) = \Lambda(\Psi(\tau), \rho(\tau), \pi(\tau))$$

$$= \mu_1 Y(t) + \mu_2 p(t) - \mu_3 r(t), \quad \mu_1, \mu_2, \mu_3 > 0$$

assuming, for simplicity, a linear relationship, we obtain[1]:

$$M^d(t) = (\mu_1 + \kappa\mu_2)Y(t) + \mu_2\pi(t) - \mu_3 r(t).$$

Also

$$\dot{r}(t) = \gamma_0\{M^d(t) - M^s(t)\}, \qquad \gamma_0 \in R \quad (10a)$$

$\Rightarrow$

$$\dot{r}(t) = \gamma_0(\mu_1 + \kappa\mu_2)Y(t) - \gamma_0\mu_3 r(t) + \gamma_0\mu_2\pi(t) - \gamma_0 \cdot M^s(t).$$

The data values on GDP, that is $Y(t)$, and money supply, that is $M^s(t)$, are all in real terms, which suggests that interest rate should also be in real terms. Preliminary analysis suggests the use of real interest rates. Now let $r_r(t)$ denote the real interest rate at any time t, then $r_r(t) = r(t) - p(t)$. We obtain[1] $r_r(t) = r(t) - \{\kappa Y(t) + \pi(t)\}$. Thus

$$\dot{r}_r(t) = \dot{r}(t) - \kappa\dot{Y}(t) - \dot{\pi}(t)$$
$$= \gamma_1 \cdot Y(t) + \gamma_2 \cdot I(t) + \gamma_3 \cdot r_r(t) + \gamma_4 \cdot \pi(t) + \gamma_5 \cdot G(t) + \gamma_6 \cdot M^s(t) \quad (10b)$$

where $\gamma_1 = \gamma_0(\mu_1 + \kappa\mu_2 - \kappa\mu_3) - \kappa\alpha(b-1)$, $\gamma_2 = \gamma_2 = -\kappa\alpha$, $\gamma_3 = -\gamma_0\mu_3$, $\gamma_6 = -\gamma_0$ and $\gamma_4 = \gamma_0(\mu_2 - \mu_3) - \rho\kappa$.

The traditional Philips relation seeks to explain the empirically based inverse (negative) relation between the rate of growth of nominal (money) wage and the rate of unemployment[18]. By extension, however, this idea is used to analyse problems of inflation and unemployment. In its original state

$$w(t) = f(U(t)), \qquad f'(U(t)) < 0 \quad (11)$$

where $w(t) = \dot{W}(t)/W(t)$ is the rate of growth of money wage, $W(t)$, and $U(t)$ is the rate of unemployment, at any time, t. The justification for the adaptation of equation (14) to explain the linkage between inflation and unemployment is based on the fact that mark-up pricing is a common usage. Hence, a positive $w(t)$, which reflects a growing nominal wage cost, might invariably carry inflationary implications. Thus, inflation, like $w(t)$, is a function of $U(t)$. The inflationary pressure of a positive $w(t)$ can, nonetheless, be offset by a rise in labour productivity, supposed to be exogenous, and here denoted by γ. Hence, inflationary effect can persist only to the extent by which nominal wage grows faster than productivity[18]. Let $p(t)$ denote the rate of inflation, that is, the rate of growth of nominal price level $P(t)$, at any time t (i.e., $p(t) = \dot{P}(t)/P(t)$). Hence,

$$p(t) = w(t) - \gamma. \quad (12)$$

From equation (11) and (12), and assuming a linear form for $f(U(t))$, that is, $f(U(t)) = \alpha_0 - \alpha_1 U(t)$, say, where α_0, $\alpha_1 > 0$, then

$$p(t) = \alpha_0 - \gamma - \alpha_1 U(t). \quad (13)$$

However, economists prefer to use the expectation-augmented form[19] of equation (11), which states that

$$w(t) = f(U(t)) + \alpha_2\pi(t), \qquad 0 < \alpha_2 \le 1$$

where $\pi(t)$ denotes the expected rate of inflation at any time t. Using equation (19), then equation (13) becomes

$$p(t) = \alpha_0 - \gamma - \alpha_1 U(t) + \alpha_2\pi(t), \qquad 0 < \alpha_2 \le 1. \quad (14)$$

The rate of change in expected inflation is proportional to the difference between actual inflation and expected inflation[1,18]. Thus

$$\dot{\pi}(t) = \rho\{p(t) - \pi(t)\}, \; 0 < \rho \le 1$$

i.e. $\dot{\pi}(t) = \rho\{\alpha_0 - \gamma - \alpha_1 U(t) + \alpha_2\pi(t) - \pi(t)\}$,

$$= \rho(\alpha_2 - 1)\pi(t) - \rho\alpha_1 U(t)$$

absorbing the constant term, in conformity with optimal control problems.

From equation (14), $U(t)$ impacts on $p(t)$, largely through the supply side[18]. They argue that $p(t)$ also affects $U(t)$. For instance, the rate of inflation may affect the consumption-saving decisions of the public, and thus aggregate demand for domestic goods and services, and the latter, in turn, influence the rate of unemployment. The rate of inflation can also affect the success of government policies of demand management. Based on the inflation rate, a given quantum of money expenditure (fiscal policy) could turn out varying degree of real expenditure. Similarly, a given rate of nominal-money expansion (monetary policy) could result in varying rates of real-money expansion.

Considering the required feedback through monetary policy only, for simplicity, put μ to represent the rate of growth in nominal money balance, $M(t)$ [18]. That is, $\mu = \dot{M}(t)/M(t)$. Assuming that the rate of change in unemployment is inversely proportional to the rate of growth in real money, then

$$\dot{U}(t) = -\phi\{\mu - p(t)\}, \quad \phi > 0$$
$$= \phi\alpha_2\pi(t) - \phi\alpha_1 U(t)$$

absorbing the constant term. Now, by combining equations (5), (9), (10b) and $\dot{\pi}(t) = \rho_\kappa \cdot Y(t)$; one of Turnovsky's expositions[1], together, we obtain the system defining the product and money markets (constraint) dynamics in the closed economy with government input. Thus we have:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$

where,

$$A = \begin{pmatrix} b_1 & b_2 & 0 & 0 \\ \eta_1 & \eta_2 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \rho_\kappa & 0 & 0 & 0 \end{pmatrix} \text{ and } \quad B = \begin{pmatrix} b_3 & 0 \\ \eta_3 & 0 \\ \gamma_5 & \gamma_6 \\ 0 & 0 \end{pmatrix}$$

$x(t) = (Y(t) \quad I(t) \quad r(t) \quad \pi(t))^T$ and $u(t) = (G(t) \quad M^s(t))^T$.

Suppose the objective functional, $V_{u(t)}(t)$, is defined in terms of $u(t)$, that is, in terms of a combination of fiscal and monetary policy options. We could have made a combination of the control and state functions, but for simplicity, we use:

$$V_{u(t)}(t) = \int_0^{T_f} F(x(t), u(t), t)\,dt = \frac{1}{2}\int_0^{T_f} e^{-\delta t} \cdot u^T(t) \cdot N \cdot u(t)\,dt$$

where $e^{-\delta t}$ is the time discounting factor (i.e., δ is the force of interest), N is positive semi-definite square matrix, T_f is the planning horizon, and the T on $u(t)$ denotes a vector transpose. Thus we maximize the objective functional

$$V_{u(t)}(t) = \int_0^{T_f} F(x(t), u(t), t)dt = \frac{1}{2}\int_0^{T_f} e^{-\delta t} \cdot u^T(t) \cdot N \cdot u(t)dt$$

subject to $\dot{x}(t) = A \cdot x(t) + B \cdot u(t) = g(x(t), u(t), t)$

where $x(0) = x^0 \geq 0$, $\lambda(T_f) = 0$, $x(T_f) = x^{(T_f)} \geq 0$ and $T_f \geq 0$.

Hamilton-Pontryagin Equations for the System: Let $H(x(t), \lambda(t), u(t), t)$ denote the Hamiltonian function for the system. Hence, the requisite Hamiltonian equation becomes

$$H(x(t), \lambda(t), u(t), t) = F(x(t), u(t), t) + g(x(t), u(t), t) \cdot \lambda(t)$$

$$H(x(t), \lambda(t), u(t), t) = \frac{1}{2} e^{-\delta t} \cdot u^T(t) \cdot N \cdot u(t) + \{A \cdot x(t) + B \cdot u(t)\} \cdot \lambda(t).$$

Here, $\lambda(t)$ is a 4×1 column vector representing the costate functional. The Hamilton-Pontryagin equations is given as:

$$H_x = F_x + g_x \cdot \lambda(t) = A^T \cdot \lambda(t) = -\dot{\lambda}(t) \quad (15)$$

$$H_\lambda = g = A \cdot x(t) + B \cdot u(t) = \dot{x}(t) \quad (16)$$

$$H_u = F_u + g_u \cdot \lambda(t) = e^{-\delta t} N \cdot u(t) + B^T \cdot \lambda(t) = 0 \quad (17)$$

$\lambda(T_f) = 0$, $x(0) = x^0 \geq 0$ and $x(T_f) = x^{(T_f)} \geq 0$.

From Equation (15) and Equation (17), we respectively obtain

$$\lambda(t) = e^{-tA^T} \lambda^0$$

$$u(t) = -e^{\delta t} \cdot N^{-1} \cdot B^T \cdot \lambda(t) = e^{\delta t} \cdot C \cdot e^{-tA^T} \cdot \lambda^0$$

where λ^0 is the initial value for $\lambda(t)$, and $C = -N^{-1} \cdot B^T$. Hence, we have

$$\dot{x}(t) = A \cdot x(t) - e^{\delta t} \cdot B \cdot N^{-1} \cdot B^T \cdot e^{-tA^T} \cdot \lambda^0 = A \cdot x(t) + D \cdot f(t)$$

For $D = -B \cdot N^{-1} \cdot B^T$ and $f(t) = e^{\delta t} \cdot \lambda(t)$.

Results and Analysis

The data sets from the Ghana Statistical Services, Bank of Ghana annual reports and World Bank published database on the Ghanaian economy were used for the estimation of the model parameters. The data items used here are real GDP, GDP growth rates and capital stock as well as the nominal money supply (M2+), interest rates and government expenditures. The rest are the CPI, inflation rates and the GDP deflating factors for the various years within the specified data period. The data sets used spans the period 1992 to 2009, except capital data set or calculations directly involving capital stock which spans the period 1992 to 2006, for lack of data. By multiplying by 100 and dividing through the nominal money supply and government expenditure values by the GDP deflator, 1993 base year, we obtain real money supply, M^s, and government expenditures, G, in millions of Ghana Cedis, at 1993 constant prices. By subtracting inflation figures from the nominal interest rate values gives us the real interest rate, r_r, data values.

To obtain the expected inflation, π, we run a simple Auto-regressive Moving Average (ARIMA) on the inflation data, then pick the model specification that gives the best prediction for the inflation values, and also, have the best inferential statistics. By regressing real capital, K, against real GDP, Y, we obtain expected values for capital stock which we can christen here as our desired capital values, K^d. We also obtain our investment values, that is, I, which is the time change in capital stock. These are illustrated in table 7.

The original data does not contain demand for money values, M^d, neither does it contain consumption expenditure values, C. However, by regressing M^s against Y, CPI and r_r, we obtain expectations for money supply. By the assumption in economic theory that demand is equal to supply at equilibrium, we can conveniently redefine these regression fits as our demand for money estimates. To obtain real consumption expenditure values, C, over the years, we multiple the Y data values by the assumed MPC value of 0.85.

We use the recurrent relation $\dot{Y}_t = (Y_t - Y_{t-1})/(t-(t-1)) = Y_t - Y_{t-1}$, to obtain the time change in Y, that is $\dot{Y}$. This analogy holds for the other time changes. The right hand side involving any of the equations involving time changes is, however, explicitly given. Hence, we work them out as given in the relevant model equations, where necessary. Preliminary results from regressing analysis suggest that the equation for the time change in real interest rate, $\dot{r}_r(t)$ should neither include $I(t)$ nor $Y(t)$, and also not both $G(t)$ and $M^s(t)$ at same time.

Analysis of Data and Fitting the Model Parameters: It was realized that, all the parameters, except ρ_K, are statistically significant, in spite of the fact that they do not ensure an exact fitting of their respective regressions. Even though ρ_K is not statistically significant, and highly so, it still will not affect our results and analysis so much since it does not run through other parameters, at worst it will only affect expected inflation. But given expected inflation figures and that of GDP, the current value will be our best bet. The same applies to γ_6. Moreover, experimental laws derived under idealised assumptions rarely do hold exactly[20]. Also, even if a model seems to fit the reality only poorly, it may still give valuable qualitative information. Based on this, we still shall use the estimated values of ρ_K and γ_6 in our work.

Table-1
Parameter Estimates and Inferential Analysis

Parameter	Value	Standard Error	T-Value	P-Value	ANOVA	
					F-Value	P-Value
α	0.13853	0.02924	4.74	0.000	22.45	0.000
ν	0.13956	0.05272	2.65	0.021	7.01	0.021
η	-1.1981	0.4357	-2.75	0.017	7.56	0.017
γ_3	0.7986	0.2635	3.03	0.009	4.61	0.29
γ_6	-0.01255	0.3141	-0.40	0.695		
ρ_κ	0.000201	0.005916	0.03	0.973	0.00	0.973
(1) Coef of G^2	0.5509	0.1452	3.79	0.002	25.72	0.000
(1) Coef of $(M^s)^2$	0.5459	0.1539	3.55	0.003		
(1) Coef of $G \cdot M^s$	-1.0885	0.2987	-3.64	0.003		
(2) Coef of G^2	0.04096	0.01058	3.87	0.002	9.55	0.001
(2) Coef of $(M^s)^2$	0.03868	0.01121	3.45	0.004		
(2) Coef of $G \cdot M^s$	-0.07946	0.02176	-3.65	0.003		
(3) Coef of G^2	0.005012	0.001416	3.54	0.003	21.11	0.000
(3) Coef of $(M^s)^2$	0.004939	0.001501	3.29	0.005		
(3) Coef of $G \cdot M^s$	-0.009880	0.002914	-3.39	0.004		

The parameters in $\dot{Y}(t)$ and $\dot{r}_r(t)$ were fitted by assuming a constant term, but discarded in the model, in conformity to control models. The coefficients of $G(t)$, $G(t)\cdot M^s(t)$ and $M^s(t)$, which happen to be the coefficient matrix of the utility function, the ones prefixed (1) are obtained by regressing GDP against $G(t)^2$, $M^s(t)^2$ and $G(t)\cdot M^s(t)$. The response variables in the case of those marked (2) and (3) are accordingly inflation and GDP growth rate. Assuming that the policy objective is to ensure growth in the economy, then:

$$A = \begin{pmatrix} -0.0207795 & 0.13853 & 0 & 0 \\ 0.003474474 & 1.174936837 & 0 & 0 \\ 0 & 0 & 0.7986 & 0 \\ 0.000201 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.13853 & 0 \\ -0.02805834 & 0 \\ 0 & -0.01255 \\ 0 & 0 \end{pmatrix} \quad N = \frac{1}{10^3}\begin{pmatrix} 5.012 & -4.94 \\ -4.94 & 4.939 \end{pmatrix}$$

$$\Rightarrow C = -N^{-1}\cdot B^T \approx \begin{pmatrix} -1951.132325 & 326.242663 & 176.796856 & 0 \\ -1951.527371 & 326.308717 & 179.373653 & 0 \end{pmatrix}$$

and

$$D = -B\cdot N^{-1}\cdot B^T \approx 10^2 \begin{pmatrix} -2.70290361 & 0.45194396 & 0.24491668 & 0 \\ 0.45194396 & -0.07556812 & -0.04095174 & 0 \\ 0.24491668 & -0.04095174 & -0.02251139 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The controllability matrix is given as

$$C = \begin{pmatrix} 0.1385 & 0.0000 & -0.0068 & 0.0000 & -0.0044 & 0.0000 & -0.0052 & 0.0000 \\ -0.0281 & 0.0000 & -0.0325 & 0.0000 & -0.0382 & 0.0000 & -0.0449 & 0.0000 \\ 0.0000 & -0.0126 & 0.0000 & -0.0100 & 0.0000 & -0.0080 & 0.0000 & -0.0064 \\ 0.0000 & 0.0000 & 0.00003 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix}$$

The rank C=4, since the row rank of a given matrix is always equal to the column rank, the system under consideration is controllable. The analytical solution becomes:

$$\lambda(t) \approx \begin{pmatrix} -0.105115e^{0.02118t} - 0.000035e^{-1.17534t} \\ 0.012174e^{0.02118t} - 0.012174e^{-1.17534t} \\ 0.884222e^{-0.7986t} \\ 0 \end{pmatrix}$$

$$u(t) = \begin{pmatrix} G(t) \\ M^s(t) \end{pmatrix} \approx \begin{pmatrix} 209.0649641e^{0.076182t} - 3.9026889e^{-1.120339t} + 156.3277248e^{-0.7436t} \\ 209.1072934e^{0.076182t} - 3.9034791e^{-1.120339t} + 158.6061856e^{-0.7436t} \end{pmatrix}$$

which are respectively the costate and control functions of the system, using the initial vector value set $u^0 = (361.49 \quad 363.81)^T$. This is a vector of 2008 real government expenditure and money supply values, in millions of Ghana cedis and at 1993 constant prices, for the control vector. The analytical solution to the state vector for the constructed dynamic system can similarly as $x^0 = (795.13 \quad 35.0 \quad 9.12 \quad 10.66)^T$, which is the initial value set for the state vector function. The first two are monetary values, in millions of Ghana Cedis, at 1993 constant prices. The investment component, in the second position, is a projection chosen to conform to the data values given.

Evaluation and Analysis of Model Results: These solutions which give the best solution sets use 2008 data values as the initial values. However, given that we do not have any data values for investment beyond 2006, we use an approximately nice estimate for it taking into consideration its 2006 value. The real interest rate value is, however, altered to yield somewhat better solution. Table 2 shows the model solution for the state variables, whereas table 3 gives us the numeric solution of the control and the associated costate functions.

The results suggest unreliable model prediction. Predictions for $I(t)$ and $r_r(t)$ beyond the first year are not good. In the case of GDP, it is highly unrealistic beyond the fourth year. Predictions for $G(t)$ and $M^s(t)$ go down sharply initially before picking up gradually, which is not in conformity with the originally data. This erratic behaviour arises mainly due to the erratic nature of investment, giving unrealistic parameter estimates in the time change in investment. This can be seen from analytic solutions. Smoothing the investment data by running trend analysis, we obtain an improved parameter estimates that result in better and more reliable solution. Again, the time change in real interest is refitted to include only expected inflation. Thus, the matrix *B* does not have any element in the second column, posing problems like, rank deficiency, near singularity, bad scaling and ill-conditioning in some of the intermediary matrices used. However, resulting solutions look good.

Inferential analysis of the latter regression is good. The numerical solutions for the smoothened model are provided in table 4 and table 5. The problems here are that the first costate gives negative values and that real interest rate and expected are always increasing due to the lack of randomness in the model. The initial value vector used is $x^0 = (79513 \quad 4262 \quad 9.12 \quad 10.66)^T$, where 42.62 is the projected value for investment arising from the trend analysis for 2008.

Table-2
Numerical System State Solution

Time	GDP @ 1993 Prices (GH¢' M)	Investment @ 1993 Prices (GH¢' M)	Real Interest Rate (%)	Expected Inflation (%)
1	831.96	103.95	13.91	10.82
2	880.85	329.09	25.37	11.00
3	986.51	1058.12	51.05	11.18
4	1281.63	3416.70	107.99	11.40
5	2189.63	11047.10	234.26	11.74
6	5077.34	35733.65	514.50	12.41
7	14365.30	115603.23	1136.85	14.19
8	44354.62	374011.04	2519.47	19.51
9	141314.10	1210058.40	5591.68	36.31
10	454940.44	3914991.30	12418.86	90.20
11	1469559.30	12666486.01	27591.12	264.10
12	4752149.45	40980928.34	61309.66	826.26
13	15372487.81	132589013.23	136245.91	2644.57
14	49733247.52	428976319.91	302785.45	8526.96
15	160903369.83	1387902990.70	672906.61	27558.20

Table-3
Numerical Solution of the Control andCostate Functions of the Model

Time	Government Expenditure @ 1993 Prices (GH¢' M)	Money Supply @ 1993 Prices (GH¢' M)	Costate Function 1	Costate Function 2	Costate Function 3	Costate Function 4
1	298.66	299.79	-0.11	0.01	0.40	0
2	278.39	278.96	-0.11	0.01	0.18	0
3	279.41	279.71	-0.11	0.01	0.08	0
4	291.49	291.67	-0.11	0.01	0.04	0
5	309.78	309.90	-0.12	0.01	0.02	0
6	332.02	332.11	-0.12	0.01	0.01	0
7	357.22	357.30	-0.12	0.01	0.00	0
8	384.98	385.06	-0.12	0.01	0.00	0
9	415.20	415.29	-0.13	0.01	0.00	0
10	447.95	448.05	-0.13	0.02	0.00	0
11	483.36	483.46	-0.13	0.02	0.00	0
12	521.59	521.70	-0.14	0.02	0.00	0
13	562.87	562.99	-0.14	0.02	0.00	0
14	607.42	607.54	-0.14	0.02	0.00	0
15	655.50	655.63	-0.14	0.02	0.00	0

Table-4
Numerical Solution for State Vector for Modified Model

Time	GDP @ 1993 Prices (GH¢' M)	Investment @ 1993 Prices (GH¢' M)	Real Interest Rate (%)	Expected Inflation (%)
1	837.19	55.69	11.48	10.82
2	884.64	72.93	13.88	11.00
3	938.32	95.68	16.32	11.18
4	999.33	125.73	18.80	11.37
5	1069.03	165.45	21.33	11.58
6	1149.18	217.95	23.90	11.81
7	1242.02	287.40	26.52	12.05
8	1350.44	379.27	29.20	12.31
9	1478.18	500.83	31.94	12.59
10	1630.09	661.72	34.74	12.90
11	1812.51	874.67	37.62	13.25
12	2033.68	1156.60	40.57	13.63
13	2304.42	1529.85	43.62	14.07
14	2638.93	2024.06	46.77	14.56
15	3055.86	2678.48	50.03	15.13

The fitted model of the closed economic with government participation for the Ghanaian economy is described by:

$$\dot{x}(t)=\begin{pmatrix}\dot{Y}(t)\\ \dot{I}(t)\\ \dot{r}_r(t)\\ \dot{\pi}(t)\end{pmatrix}=\begin{pmatrix}-0.0207795 & 0.13853 & 0 & 0\\ 0.000384673 & 0.280885516 & 0 & 0\\ 0 & 0 & 0 & 0.22\\ 0.000201 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}Y(t)\\ I(t)\\ r_r(t)\\ \pi(t)\end{pmatrix}+\begin{pmatrix}0.13853 & 0\\ -0.00256448 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}G(t)\\ M^S(t)\end{pmatrix}$$

$$\dot{\lambda}(t)=\begin{pmatrix}=0.0207795 & 0.000384673 & 0 & 0.000201\\ 0.13853 & 0.280885516 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0.22 & 0\end{pmatrix}\begin{pmatrix}\lambda_1(t)\\ \lambda_2(t)\\ \lambda_3(t)\\ \lambda_4(t)\end{pmatrix}$$

and $u(t)=e^{\delta t}\cdot C\cdot\lambda(t)$, where C, and δ are as defined earlier.

The real GDP predictions (at 1993 constant prices) suggests that in twelve years time, which according to our model is 2020, since our reference point is 2008, the real GDP would be GH¢2,033.68 million, at 1993 constant prices, see Table 3.3.8. (Note that our monetary values were quoted in million Ghana Cedis.) This translates to more than 2.5 times of that of 2008 real GDP. But real GDP of GH¢795.13 million at 2008 translates to GH¢17,617.60 million, at current prices (2008 prices), according Bank of Ghana (BOG) figures. This in turn translates to US$14,758.82 million, using the inter-bank average (of buying and selling the US dollar) exchange rate of GH¢1.1937 to US$1 by close of 2008, from BOG data.

This suggests that the vision 2020 agenda of hitting the middle income bracket is seriously dicey, if no pragmatic and well-integrated control measures. We may need to work much harder to get there. We need to be doing an average of between 8 to 12 percentage growth rate in GDP, or more, depending on where in the middle income bracket we wish to be by 2020.

Table-5
Numerical Solution of the Control and Costate Vectors of the Modified Model

Time	Government Expenditure @ 1993 Prices (GH¢' M)	Money Supply @ 1993 Prices (GH¢' M)	Costate Function 1	Costate Function 2	Costate Function 3	Costate Function 4
1	392.04	392.11	-0.19	0.02	0	0
2	423.62	423.70	-0.19	0.04	0	0
3	457.56	457.65	-0.20	0.05	0	0
4	494.08	494.18	-0.20	0.07	0	0
5	533.39	533.50	-0.21	0.07	0	0
6	575.75	575.86	-0.21	0.08	0	0
7	621.39	621.51	-0.22	0.09	0	0
8	670.59	670.73	-0.22	0.09	0	0
9	723.64	723.79	-0.22	0.10	0	0
10	780.86	781.01	-0.23	0.10	0	0
11	842.56	842.73	-0.23	0.10	0	0
12	909.12	909.30	-0.24	0.11	0	0
13	980.92	981.11	-0.24	0.11	0	0
14	1058.37	1058.58	-0.25	0.11	0	0
15	1141.92	1142.15	-0.25	0.12	0	0

Table-6
Projected GDP and Per Capita Income

Year	Projected GDP (GH¢'M) @ '93 Const Prices	Projected Population (Million)	Projected GDP Growth Rate (%)	Projected Per Capita Income (US$)	Projected Per Capita Income Growth Rate (%)
2008	795.13	22.901000	7.3000	644.4600	-
2009	837.19	23.395662	5.2897	664.6297	3.1297
2010	884.64	23.901008	5.6678	687.9433	3.5078
2011	938.32	24.417270	6.0680	714.8282	3.9080
2012	999.33	24.944683	6.5020	745.8664	4.3420
2013	1069.03	25.483488	6.9747	781.7774	4.8147
2014	1149.18	26.033931	7.4975	823.5044	5.3375
2015	1242.02	26.596264	8.0788	872.2460	5.9188
2016	1350.44	27.170743	8.7293	929.5467	6.5693
2017	1478.18	27.757631	9.4591	997.3956	7.2991
2018	1630.09	28.357196	10.2768	1078.3525	8.1168
2019	1812.51	28.969712	11.1908	1175.7363	9.0308
2020	2033.68	29.595457	12.2024	1293.8086	10.0424
2021	2304.42	30.234719	13.3128	1438.1046	11.1528
2022	2638.93	30.887789	14.5160	1615.7971	12.3560
2023	3055.86	31.554966	15.7992	1836.1790	13.6392

Table-7
A Table Showing the Main Data Sets used in the Model Development

Year	GDP '93 Prices (GH¢ M)	GDP Growth Rate	GDP Deflator	Capital '93 Price (GH¢ M)	CPI 2001 Base Y	CPI 1993 Base Y	Inflatn	Govt Exp Current P (GH¢ M)	Money S Current P (GH¢ M)	Nominal Interest Rate
1992	368.91	3.88	75.98	1.66	9.63	78.33	13.33	50.07	51.93	29.00
1993	387.25	4.85	100.00	14.77	12.29	100.00	27.66	82.18	66.16	39.00
1994	399.91	3.30	130.16	6.15	16.49	134.18	34.18	114.96	96.74	37.50
1995	416.00	4.11	186.36	25.46	28.18	229.21	70.82	171.37	132.96	41.79
1996	435.12	4.60	260.60	42.53	37.38	304.07	32.66	260.40	178.49	45.32
1997	453.39	4.20	311.29	23.54	45.16	367.41	20.83	384.82	331.88	46.63
1998	474.67	4.70	364.38	28.09	52.28	425.28	15.75	448.68	390.38	40.50
1999	495.69	4.41	415.16	10.94	59.49	483.97	13.80	509.49	453.32	30.00
2000	514.21	3.69	528.05	7.20	83.61	680.17	40.54	804.85	633.83	47.00
2001	535.71	4.00	710.66	30.46	101.41	824.98	21.29	982.73	1,024.80	42.50
2002	560.08	4.50	872.42	9.91	116.79	950.13	15.17	1,332.17	1,536.81	38.50
2003	589.47	5.20	1,122.33	69.99	144.31	1,173.98	23.56	1,898.13	2,117.37	32.75
2004	622.35	5.60	1,282.29	69.22	161.31	1,312.27	11.78	2,622.95	2,666.72	28.75
2005	658.87	5.90	1,476.17	13.50	183.74	1,494.81	13.91	3,224.82	3,046.79	26.00
2006	701.20	6.20	1,664.58	87.15	203.81	1,658.04	10.92	3,873.47	4,230.26	26.00
2007	741.21	6.06	1,895.01	-	229.79	1,869.44	12.75	5,624.53	5,750.72	24.25
2008	795.13	7.27	2,215.77	-	271.46	2,208.37	18.13	8,009.82	8,061.20	27.25
2009	828.53	4.20	2,568.66	-	314.83	2,561.27	15.98	8,248.24	10,233.28	32.75

Conclusion

In this study, we developed an optimal control models for a simple closed economy with government participation. The results obtained suggest that, the prediction reliability of the model over time is very good, except for the long term predictions of real interest rates and the medium to long term ones of expected inflation which are a bit suspect. Even though the predictions of the models look fairly good, especially in relation to past performance records on GDP and investment values, the prime objective of the model here is not to offer predictions for the economy. Rather the objective is to offer these predictions in the face of undertaking certain control measures. Thus, the predictions offered by the models need not necessarily conform to empirical performance of the economy.

References

1. Turnovsky S.J., Stabilization Theory and Policy: 50 Years after Philips Curve A Plenary Address presented at 2008 ESAM08 Conference, Market Models: Policy Frontiers in the A. W. Philips Traditions, July 9–11, **(2008),** Wellington, N. Zealand(http://www.econ.washington.edu /…/detail.asp).
2. Somashekar N.T., Modern Macroeconomics Theory, Anmol Publishing PVT Ltd, **(2005)**
3. Mc Eachern W.A, Macroeconomics, A Contemporary Introduction, 5th Ed., South Western College Publishing, **(2000)**
4. Tinbergen J., On the Theory of Economic Policy, North-Holland, Amsterdam, **(1952)**
5. Turnovsky S.J., Optimal Stabilization Policies for Deterministic and Stochastic Linear Economic Systems, *Review of Economic Studies,* **40,** 79-96 **(1973)**
6. Turnovsky S.J., Optimal Stabilization Policies for Stochastic Linear Systems: The Case of Correlated Multiplicative and Additive Disturbances, Review of Economic Studies, **43,** 191-194 **(1976)**
7. Samuelson P.A., Interaction Between the Multiplier Analysis and the Principle of Acceleration, Review of Economic Statistics, **21,** 75-78 **(1939)**
8. Hicks J.R., A Contribution to the Theory of the Trade Cycle, Oxford University Press, Oxford, International Financial Statistics, Year Books **(2007)**
9. Phillips A.W., Stabilisation Policy in a Closed Economy, Economic Journal, **64,** 290-323 **(1954)**
10. Phillips A.W., Stabilisation Policy and the Time Form of Lagged Responses, Economic Journal, 67, 265-277 **(1957)**
11. Phillips A.W., The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861-1957,Economica, **25,** 283-299 **(1958)**
12. Turnovsky S.J., Macroeconomic Analysis and Stabilization Policy, Cambridge University Press, Cambridge, U.K., **(1977a)**

13. Turnovsky S.J., Optimal Control of Linear Systems with Stochastic Coefficients and Additive Disturbances, in J. D. Pitchford and S. J. Turnovsky, (eds.), Applications of Control Theory to Economic Analysis, (North-Holland, Amsterdam), 293-335 **(1977b)**

14. Cass D., Optimum Growth in an Aggregate Model of Capital Accumulation, *Review of Economic Studies,* 32, 233-240 **(1965)**

15. Lovell M.C., Economics with Calculus, World Scientific Publishing Co. Pte Ltd, **(2004)**

16. Froyen R.T., Macroeconomics, Theories and Policies, 8th Ed., Pearson Education Inc., **(2005)**

17. Lovell M.C. and E. Prescott, Money Multiplier Accelerator Interaction, *Southern Economic Journal,* **35,** 60-72 **(1968)**

18. Chiang A.C. and Wainwright K., Fundamental Methods of Mathematical Economics, 4th Ed., (Int. Ed), McGraw-Hill / Irwin of McGraw-Hill Companies Inc., **(2005)**

19. Friedman M., A Monetary and Fiscal Framework for Economic Stability, *American Economic Review,* **38,** 245-264 **(1948)**

20. Kreyszig E., Advanced Engineering Mathematics, 9th Ed., John Wiley and Sons, **(2006)**

21. Ghana Statistical Services, Annual Reports, 1976-2009 **(2009)**

22. Bank of Ghana, Annual Reports, 1979-2008, Bank of Ghana, **(2008)**