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Characterization of Uniform Distribution u (0,θ) through Expectation

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Abstract

For characterization of uniform distribution one needs any arbitrary non constant function only in place of approaches such as independence of sample mean and variance, correlation of minimum and maximum in a random sample of size two, moment conditions, inequality of Chernoff, available in the literature. Path breaking different approach based on identity of distribution and equality of expectation of function of random variable was used in characterizing uniform distribution through expectation of non constant function of random variable with examples for illustrative purpose.

Keywords: Characterization, distribution, correlation, expectation, variable.

Introduction

Characterizations was independently develop in different branches of applied probability and pure mathematics. Characterizations theorem are located on borderline between probability theory and mathematical statistics. It is of general interest to mathematical community, to probabilists and statistician as well as to researchers and practitioner industrial engineering and operation research and various scientist specializing in natural and behavior science, in particular those who are interested in foundation and application of probabilistic model building[1,2].

Geary[3] stated that given sample of size $n \geq 2$ independent observations come from some distribution on the line then sample mean and variance are independent is the necessary and sufficient condition for to be from normally distribution. The need of some regularity condition for Geary's characterization of normal distribution have been removed by successive refinement[2]. Similar characterization for uniform distribution by Kent[4] asserted that if $n \geq 2$ i.i.d random angels from distribution defined by density on circle, sample mean direction and resultant length are independent if and only if angels come from uniform distribution.

Various approaches for characterization of uniform distribution are available in the literature. It is well known that minimum and maximum in a random sample of size two are positively correlate and coefficient of correlation is less or equal to one half. Bartoszyn'ski[5] proposed that a result of this type might exist in connection with a problem in cell division. Since the two daughter cells cannot always be distinguished later, the times till their further division can only be recorded as the earlier event and the later event. The correlation between these ordered pairs thus may provide the only information on the independence of the two events. Terreel[6] showed that the coefficient of correlation is one half if and only if random sample comes from rectangular distribution. Terreel's proof is computational nature and use properties of Legendre polynomial. Lopez-Bldzquez[7] gave ease proof for Terreel's characterization and obtained shaper bound on the coefficient of correlation.

Uniform distribution $U(0,1)$ is neatly characterized by two moment conditions: $E[Max(X_1, X_2)] = \frac{2}{3}$ and $E(X_1^2) = \frac{1}{3}$ by Lin[8]. Using two suitable moments of order statistics Too (1989) characterize uniform and exponential distribution[9-13]. Huang[16] studied density estimation by wavelet-based reproducing kernels and further doing error analysis for bias reduction in a spline-ased multi resolution, Chow[17] (1999) studied *n*-fold convolution modulo one and characterize uniform distribution on interval zero to one.

Inequality of Chernoff[18,19,] assert that" if X is normally distributed with mean 0 and variance 1 and if g is absolutely continuous and g(X) has finite variance, then" $E[\{g'(X)\}^2] \geq V[g(X)]$ and equality holds if and only $g(X)$ is linear. Chernoff proved this result using Hermite polynomials where as Chris[20] proved inequality of Chernoff by using Cauchy- Schwarz inequality and Fubini's theorem. Sumrita[21] (1990) studied Chernoff-type inequalities for distributions on [-1,1] having symmetric unimodal densities and gave characterization of uniform distributions by inequalities of chernoff-type.

This research note provides characterization of uniform distribution with probability density function (pdf)

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} \; ; \; a < x < \theta < b; \\ 0 \; ; \; otherwise. \end{cases} \qquad (1.1)$$

where $-\infty \le a < b \le \infty$ are known constants and $(1/\theta)$ is everywhere differentiable function. Since range is truncated by θ from right $a = 0$.

The characterization of uniform distribution through expectation of function, $\phi(X)$ in section 2 and section 3 is for illustrative examples.

Characterization Theorem: Let X be a random variable with distribution function F. Assume that F is continuous on the interval $[a, b]$, where $-\infty \le a < b \le \infty$. Let $\phi(X)$ and $g(X)$ be two distinct differentiable and intregrable functions of X on the interval $[a, b]$ where $-\infty \le a < b \le \infty$. Then

$$E\left[g(X) + X\frac{d}{dX}g(X)\right] = g(\theta). \tag{2.1}$$

is the necessary and sufficient condition for pdf $f(x,\theta)$ of F to be $f(x,\theta)$ defined in (1.1).

Proof. Given $f(x,\theta)$ defined in (1.1), for necessity of (2.1) if $\phi(X)$ is such that $g(\theta) = E[\phi(X)]$ where $g(\theta)$ is differentiable function then

$$g(\theta) = \int_a^\theta \phi(x) f(x,\theta)\, dx \tag{2.2}$$

Differentiating with respect to θ on both sides of (2.1) and replacing X for θ and after simplification

$$\phi(X) = g(X) + X\frac{d}{dX}g(X) \tag{2.3}$$

which establishes necessity of (2.1). Conversely given (2.1), let $k(x,\theta)$ be pdf of r.v. X such that

$$g(\theta) = \int_a^\theta \left[g(x) + x\frac{d}{dx}g(x)\right] k(x,\theta)\, dx. \tag{2.4}$$

Since $(1/\theta)$ is decreasing on the interval (a, θ) where $-\infty \le a < b \le \infty$ following identity holds :

$$g(\theta) = \frac{1}{\theta}\int_a^\theta \left[\frac{d}{dx} x g(x)\right] dx. \tag{2.5}$$

Differentiating integrand of (2.5) one gets

$$g(\theta) = \int_a^\theta \left[g(x) + x\frac{d}{dx}g(x)\right]\frac{1}{\theta} dx. \tag{2.6}$$

And (2.6) will be

$$g(\theta) = \int_a^\theta \phi(x)\frac{1}{\theta} dx \tag{2.7}$$

where $\phi(x)$ is function of X derived in (2.3). From (2.4) and (2.7) by uniqueness theorem

$$k(x,\theta) = \frac{1}{\theta}. \tag{2.8}$$

Since $(1/\theta)$ is decreasing on the interval (a, θ) where $-\infty \le a < b \le \infty$ and since $a = 0$ integrating (2.8) on both sides one gets

$$1 = \int_a^\theta k(x,\theta) dx. \tag{2.9}$$

Hence $k(x,\theta)$ derived in (2.8) reduces to $f(x,\theta)$ defined in (1.1), which establishes sufficiency of (2.1).

Remark 2.1. Using $\phi(X)$ derived in (2.3), the $f(x,\theta)$ given in (1.1) can be determined by

$$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X) - g(X)} \tag{2.10}$$

and pdf is given by

$$f(x,\theta) = \frac{\frac{d}{dx}(T(x))}{T(\theta)} \tag{2.11}$$

where $T(X)$ is increasing function for $-\infty \le a < b \le \infty$ with $T(a) = 0$ such that it satisfies

$$M(X) = \frac{d}{dx}[\log(T(X))]. \tag{2.12}$$

Applications

Example-1: Using method described in the remark characterization of uniform distribution through p^{th} quantile $Q_p(\theta) = \theta p$ is illustrated.

$$g(\theta) = \theta p$$
$$g(X) = Xp$$
$$\phi(X) = g(X) + X\frac{d}{dX}g(X) = 2Xp$$
$$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X) - g(X)} = \frac{1}{X}$$
$$\frac{d}{dx}[\log(X)] = \frac{1}{X} = M(X)$$
$$T(X) = X$$
$$f(x,\theta) = \frac{\frac{d}{dx}(T(x))}{T(\theta)} = \frac{1}{\theta}$$

Example-2: Using method described in the remark characterization of uniform distribution through p^{th} quantile $e^{-\theta}$ is illustrated.

$$g(\theta) = e^{-\theta}$$
$$g(X) = e^{-X}$$
$$\phi(X) = g(X) + X\frac{d}{dX}g(X) = (1 - X)e^{-X}$$
$$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X) - g(X)} = \frac{1}{X}$$
$$\frac{d}{dx}[\log(X)] = \frac{1}{X} = M(X)$$
$$T(X) = X$$
$$f(x,\theta) = \frac{\frac{d}{dx}(T(x))}{T(\theta)} = \frac{1}{\theta}$$

Example-3: The pdf $f(x,\theta)$ defined in (1.1) can be characterized through non constant function such as

$$g_i(\theta) = \begin{cases} \frac{\theta}{2}; for\ i = 1, Mean, \\ \frac{\theta^r}{r+1}; for\ i = 2,\ r^{th}\ \text{raw moment}, \\ e^{\theta}\ ; for\ i = 3, \\ e^{-\theta}; for\ i = 4, \\ \theta p; for\ i = 5,\ p^{th}\ \text{quantile}, \\ \frac{t}{\theta}; for\ i = 6,\ \text{distribution function at t}, \\ 1 - \frac{t}{\theta}; for\ i = 7,\ \text{Relibility at t}, \\ 1 - \frac{t}{\theta}\ ; for\ i = 8,\ \text{Hazard function}, \end{cases}$$

and by using

$$[\varphi_i(X) - g_i(X)] = \begin{cases} \frac{x}{2}; for\ i = 1, Mean, \\ \frac{r}{r+1}x^r; for\ i = 2,\ r^{th}\ \text{raw moment}, \\ xe^{x}; for\ i = 3, \\ -xe^{-x}; for\ i - 4, \\ xp; for\ i = 5,\ \text{pth quantile}, \\ -\frac{t}{x}; for\ i = 6,\ \text{distribution Function at t}, \\ \frac{t}{x}; for\ i = 7,\ \text{Relibility Function at t}, \\ \frac{x}{(x-t)^2}; for\ i = 8,\ \text{Hazard Function}, \end{cases}$$

and defining $M(X)$ given in (2.10) and substituting $T(X)$ as appeared in (2.12) for (2.11).

Note that to characterize pdf given in (1.1) one needs any arbitrary non constant function only.

Conclusion

To characterize pdf given in (1.1) one needs any arbitrary non constant function only.

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