



A Non-Uniform Haar Wavelet Method for Coupled Problems

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Available online at: www.isca.in, www.isca.me

Received 26th July 2026, revised 9th August 2026, accepted 10th September 2026

Abstract

In this paper, an approach for the solution of systems of first and second order linear partial singularly perturbed (LPSP) initial and boundary value problems or coupled problems, respectively, has been proposed using non-uniform Haar wavelets in wavelet collocation method. Three different types of meshes have been considered. Maximum absolute residual errors (MAREs) of numerical solutions obtained by proposed method have been presented in tabular form. The numerical results demonstrate that the proposed method yields highly accurate approximations.

Keywords: Singularly Perturbed Differential Equations; Non-Uniform Haar Wavelet; Meshes; Wavelet Collocation Method; Residual Error.

Introduction

Singularly perturbed differential equations constitute an important class of mathematical models that arise in a wide variety of scientific and engineering applications, including fluid mechanics, heat and mass transfer, chemical reaction processes, control systems, elasticity, biological modeling, and transport phenomena. The presence of one or more small perturbation parameters multiplying the highest-order derivatives often produces sharp boundary or interior layers in the solution. These layers introduce stiffness into the governing equations and significantly reduce the effectiveness of conventional numerical methods unless suitably refined meshes or specially designed numerical techniques are employed¹⁻³.

Over the past several decades, considerable effort has been devoted to the development of uniformly convergent numerical methods for singularly perturbed problems. Classical finite difference methods constructed on layer-adapted meshes, such as Shishkin and Bakhvalov meshes have proved successful for many classes of boundary value problems, while subsequent investigations extended uniformly convergent difference methods to parabolic problems with delays, integral boundary conditions, and nonlocal constraints⁴⁻⁷.

Among the various wavelet families, Haar wavelets are particularly attractive due to their simple mathematical structure, sparse operational matrices, and ease of implementation. These properties enable Haar wavelet methods to approximate solutions with relatively low computational cost while maintaining high accuracy. Consequently, Haar wavelet techniques have been successfully applied to regular as well as singularly perturbed differential equations, including elliptic, parabolic, integro-differential, and nonlinear problems^{8,9}.

Kumar and Deswal developed a wavelet-based approximation for two-parameter singularly perturbed boundary value problems with Robin boundary conditions¹⁰. Deswal et al. proposed three-dimensional Haar wavelet methods on non-uniform meshes for singularly perturbed elliptic boundary value problems, demonstrating improved numerical accuracy near boundary layers^{11,12}. Kumar et al. combined Haar wavelets with a nonstandard finite difference scheme for singularly perturbed partial integro-differential equations¹³, while Podila and Sundrani employed Haar wavelets for two-parameter singularly perturbed problems involving discontinuous convection coefficients and source terms¹⁴.

Modern studies have explored differential-difference equations, time-fractional delay partial differential equations, fuzzy singular perturbation models, adaptive control strategies, and machine-learning-based approximations of slow invariant manifolds^{15,16}. These developments highlight both the diversity of applications and the continuing demand for efficient numerical methods capable of accurately resolving layer phenomena in increasingly sophisticated mathematical models.

Motivated by these observations, the present work proposes a non-uniform Haar wavelet collocation method for solving systems of first-order partially singularly perturbed initial value problems and second-order partially singularly perturbed boundary value problems. Three different non-uniform mesh distributions are employed to improve the approximation of boundary layer solutions. The numerical performance of the proposed approach is assessed through several benchmark examples, and the MAREs are compared.

Non-uniform Haar wavelet: For non-uniform Haar wavelets, the closed interval $[a, b]$ is divided into $2M$, where

$M = 2^J$, subintervals of uneven sizes. The i^{th} wavelet for their grid points is defined as follows:

$$\psi_i(\zeta) = \begin{cases} 1, & \eta_1(i) \leq \zeta \leq \eta_2(i) \\ -c_i, & \eta_2(i) \leq \zeta \leq \eta_3(i) \\ 0 & \text{elsewhere} \end{cases} \quad (1)$$

where,

$$\eta_1(i) = \zeta(2k\mu), \quad \eta_2(i) = \zeta[(2k+1)\mu], \quad \eta_3(i) = \zeta[2(k+1)\mu], \quad \mu = M/m \quad (2)$$

$$\text{and } c_i = \frac{\eta_2(i) - \eta_1(i)}{\eta_3(i) - \eta_2(i)}, \quad (3)$$

which, can be calculated from $\int_a^b \psi_i(\zeta) = 0$.

q Mesh: The q mesh is defined as

$$\hat{\zeta}(j) = \frac{1 - q^j}{1 - q^N}, \quad j = 0, 1, \dots, N \quad (4)$$

$$\zeta(j) = \frac{\hat{\zeta}(j-1) + \hat{\zeta}(j)}{2}, \quad j = 1, 2, \dots, N \quad (5)$$

ρ Mesh: The ρ mesh is given as follows

$$M = \frac{N}{2}, \quad \rho = \frac{5 - \varepsilon}{5M}, \quad \delta = \frac{N\rho - 1}{M(N-1)}, \quad (6)$$

$$\zeta_0 = 0, \quad \zeta_j = 1 - (N-j)\rho + \delta(N-j+1)\frac{N-j}{2}, \quad j = 1, 2, \dots, N, \quad (7)$$

$$X_j = \frac{\zeta_{j+1} + \zeta_j}{2}, \quad j = 1, 2, \dots, N. \quad (8)$$

Shishkin Mesh: The Shishkin mesh is given as follows

$$\sigma = \min\left(\frac{1}{\alpha}, \varepsilon \log(N)\right), \quad (9)$$

$$\zeta_j = \frac{2\sigma(j - \frac{1}{2})}{N}, \quad j = 1, 2, \dots, \frac{N}{2} - 1, \quad (10)$$

$$\zeta_j = \sigma + \frac{2(1-\sigma)(j - M - \frac{1}{2})}{N}, \quad j = \frac{N}{2}, \dots, N. \quad (11)$$

Consider the linear system of first order partial singularly perturbed initial value problem as follows:

$$\varepsilon u'(\zeta) + p(\zeta)u(\zeta) + q(\zeta)v(\zeta) = F(\zeta), \quad (12)$$

$$v'(\zeta) + r(\zeta)u(\zeta) + s(\zeta)v(\zeta) = G(\zeta), \quad \forall \zeta \in (0, 1] \quad (13)$$

$$\text{with initial conditions } u(0) = \lambda_1, \quad v(0) = \lambda_2, \quad (14)$$

where, $p(\zeta), q(\zeta), r(\zeta), s(\zeta), F(\zeta)$ and $G(\zeta)$ are continuous functions.

The matrix form of the system of equations (12) and (13) can be written as

$$\begin{pmatrix} \varepsilon \frac{d}{d\zeta} & 0 \\ 0 & \frac{d}{d\zeta} \end{pmatrix} \begin{pmatrix} u(\zeta) \\ v(\zeta) \end{pmatrix} + \begin{pmatrix} p(\zeta) & q(\zeta) \\ r(\zeta) & s(\zeta) \end{pmatrix} \begin{pmatrix} u(\zeta) \\ v(\zeta) \end{pmatrix} = \begin{pmatrix} F(\zeta) \\ G(\zeta) \end{pmatrix}, \quad \forall \zeta \in (0, 1] \quad (15)$$

$$\text{with initial conditions } w(0) = (u(0), v(0))^T = (\lambda_1, \lambda_2)^T. \quad (16)$$

Further, consider the linear system of second order singularly perturbed boundary value problem as follows:

$$\varepsilon u''(\zeta) + p(\zeta)u(\zeta) + q(\zeta)v(\zeta) = F(\zeta), \quad (17)$$

$$v''(\zeta) + r(\zeta)u(\zeta) + s(\zeta)v(\zeta) = G(\zeta), \quad \forall \zeta \in (0, 1] \quad (18)$$

with boundary conditions

$$u(0) = \lambda_1, \quad v(0) = \lambda_2, \quad u(1) = \mu_1, \quad v(1) = \mu_2, \quad (19)$$

The matrix form of the system of equations (17) and (18) can be written as

$$\begin{pmatrix} \varepsilon \frac{d^2}{d\zeta^2} & 0 \\ 0 & \frac{d^2}{d\zeta^2} \end{pmatrix} \begin{pmatrix} u(\zeta) \\ v(\zeta) \end{pmatrix} + \begin{pmatrix} p(\zeta) & q(\zeta) \\ r(\zeta) & s(\zeta) \end{pmatrix} \begin{pmatrix} u(\zeta) \\ v(\zeta) \end{pmatrix} = \begin{pmatrix} F(\zeta) \\ G(\zeta) \end{pmatrix} \quad \forall \zeta \in (0, 1] \quad (20)$$

With initial conditions

$$w(0) = (u(0), v(0))^T = (\lambda_1, \lambda_2)^T \quad (21)$$

$$w(1) = (u(1), v(1))^T = (\mu_1, \mu_2)^T \quad (22)$$

Method of Solution

Wavelet Collocation Method: In this section, a method using integration approach of collocation method for non-uniform Haar wavelet collocation is proposed.

Method for System of First Order LPSP Initial Value Problems: Consider the system as follows

$$\varepsilon u'(\zeta) + p(\zeta)u(\zeta) + q(\zeta)v(\zeta) = F(\zeta) \quad (23)$$

$$v'(\zeta) + r(\zeta)u(\zeta) + s(\zeta)v(\zeta) = G(\zeta) \quad (24)$$

$$\text{subject to initial conditions } u(0) = \lambda_1, \quad v(0) = \lambda_2. \quad (25)$$

$$\text{To solve this system, consider } u'(\zeta) = \sum_{i=1}^N p_i H_i(\zeta) \quad (26)$$

$$\text{and } v'(\zeta) = \sum_{i=1}^N q_i H_i(\zeta) \quad (27)$$

Now, integration of equation (26) from 0 to ζ , gives

$$u(\zeta) = \sum_{i=1}^N p_i P_i(\zeta) + u(0). \quad (28)$$

Also, integration of equation (27) from 0 to ζ , yields

$$v(\zeta) = \sum_{i=1}^N q_i P_i(\zeta) + v(0). \quad (29)$$

Now, putting the values of $u'(\zeta)$, $v'(\zeta)$, $u(\zeta)$ and $v(\zeta)$ from equations (26), (27), (28) and (29) in equations (23) and (24) respectively, results into

$$\varepsilon \left[\sum_{i=1}^N p_i H_i(\zeta) \right] + p(\zeta) \left[\sum_{i=1}^N p_i P_i(\zeta) + u(0) \right] + q(\zeta) \left[\sum_{i=1}^N q_i P_i(\zeta) + v(0) \right] = F(\zeta) \quad (30)$$

and

$$\sum_{i=1}^N q_i H_i(\zeta) + r(\zeta) \left[\sum_{i=1}^N p_i P_i(\zeta) + u(0) \right] + s(\zeta) \left[\sum_{i=1}^N q_i P_i(\zeta) + v(0) \right] = G(\zeta) \quad (31)$$

On simplifying the system of equations (30) and (31), gives

$$\sum_{i=1}^N [\varepsilon p_i H_i(\zeta) + p(\zeta) p_i P_i(\zeta) + q(\zeta) q_i P_i(\zeta)] = F(\zeta) - p(\zeta) u(0) - q(\zeta) v(0) \quad (32)$$

and

$$\sum_{i=1}^N [q_i H_i(\zeta) + r(\zeta) p_i P_i(\zeta) + s(\zeta) q_i P_i(\zeta)] = G(\zeta) - r(\zeta) u(0) - s(\zeta) v(0) \quad (33)$$

The equations (32) and (33) are system of linear equations with unknown Haar wavelet coefficients p_i 's and q_i 's. Solving these equations and consequently substituting the values of Haar wavelet coefficients p_i 's in equation (28) and q_i 's in equation (29), the non-uniform Haar wavelet approximate solutions of the system of first order LPSP initial value problems are obtained.

Method for System of Second Order LPSP Boundary Value Problems: Consider the system as follows

$$\varepsilon u''(\zeta) + p(\zeta)u(\zeta) + q(\zeta)v(\zeta) = F(\zeta), \quad (34)$$

$$v''(\zeta) + r(\zeta)u(\zeta) + s(\zeta)v(\zeta) = G(\zeta), \quad \forall \zeta \in (0,1] \quad (35)$$

with boundary conditions

$$u(0) = \lambda_1, \quad v(0) = \lambda_2, \quad u(1) = \mu_1, \quad v(1) = \mu_2. \quad (36)$$

To solve this system, assume that

$$u''(\zeta) = \sum_{i=1}^N p_i H_i(\zeta), \quad (37)$$

and

$$v''(\zeta) = \sum_{i=1}^N q_i H_i(\zeta). \quad (38)$$

Now, integrating equation (37) from 0 to ζ , gives

$$u'(\zeta) = \sum_{i=1}^N p_i P_i(\zeta) + u'(0). \quad (39)$$

Also, integration of equation (38) from 0 to ζ , leads to

$$v'(\zeta) = \sum_{i=1}^N q_i P_i(\zeta) + v'(0). \quad (40)$$

$$\text{where, } P_i(\zeta) = \int_0^\zeta H_i(\zeta) d\zeta.$$

Again integrating equations (39) and (40) from 0 to ζ , gives

$$u(\zeta) = \sum_{i=1}^N p_i \Omega_i(\zeta) + \zeta u'(0) + u(0), \quad (41)$$

and

$$v(\zeta) = \sum_{i=1}^N q_i \Omega_i(\zeta) + \zeta v'(0) + v(0), \quad (42)$$

respectively.

Further, integrating the equations (37) and (38) from 0 to 1, respectively, results into

$$u'(0) = u(1) - u(0) - \sum_{i=1}^N p_i \Xi_i(\zeta), \quad (43)$$

and

$$v'(0) = v(1) - v(0) - \sum_{i=1}^N q_i \Xi_i(\zeta), \quad (44)$$

$$\text{where, } \Xi_i(\zeta) = \int_0^1 P_i(\zeta) d\zeta.$$

Now, putting the values of $u'(0)$ and $v'(0)$ from equations (43) and (44) in equations (41) and (42), respectively, yields

$$u(\zeta) = \sum_{i=1}^N p_i \Omega_i(\zeta) + \zeta \left(u(1) - u(0) - \sum_{i=1}^N p_i \Xi_i(\zeta) \right) + u(0), \quad (45)$$

$$v(\zeta) = \sum_{i=1}^N q_i \Omega_i(\zeta) + \zeta \left(v(1) - v(0) - \sum_{i=1}^N q_i \Xi_i(\zeta) \right) + v(0). \quad (46)$$

Now, substituting the values of $u''(\zeta)$, $v''(\zeta)$, $u(\zeta)$ and $v(\zeta)$ from equations (37), (38), (45) and (46) in equations (34) and (35) respectively, the following system of linear equations is obtained

$$\begin{aligned} \varepsilon \left[\sum_{i=1}^N p_i H_i(\zeta) \right] + p(\zeta) \left[\sum_{i=1}^N p_i \Omega_i(\zeta) + \zeta \left(u(1) - u(0) - \sum_{i=1}^N p_i \Xi_i(\zeta) \right) + u(0) \right] \\ + q(\zeta) \left[\sum_{i=1}^N q_i \Omega_i(\zeta) + \zeta \left(v(1) - v(0) - \sum_{i=1}^N q_i \Xi_i(\zeta) \right) + v(0) \right] = F(\zeta), \end{aligned} \quad (47)$$

and

$$\sum_{i=1}^N q_i H_i(\zeta) + r(\zeta) \left[\sum_{i=1}^N p_i \Omega_i(\zeta) + \zeta \left(u(1) - u(0) - \sum_{i=1}^N p_i \Xi_i(\zeta) \right) + u(0) \right] + s(\zeta) \left[\sum_{i=1}^N q_i \Omega_i(\zeta) + \zeta \left(v(1) - v(0) - \sum_{i=1}^N q_i \Xi_i(\zeta) \right) + v(0) \right] = G(\zeta) \quad (48)$$

On simplifying the system (47) and (48), leads to

$$\sum_{i=1}^N p_i \left[\delta H_i(\zeta) + p(\zeta) (\Omega_i(\zeta) - \zeta \Xi_i(\zeta)) \right] + \sum_{i=1}^N q_i \left[q(\zeta) (\Omega_i(\zeta) - \zeta \Xi_i(\zeta)) \right] = F(\zeta) - p(\zeta) [\zeta (u(1) - u(0)) + u(0)] - q(\zeta) [\zeta (v(1) - v(0)) + v(0)], \quad (49)$$

and

$$\sum_{i=1}^N q_i \left[H_i(\zeta) + s(\zeta) (\Omega_i(\zeta) - \zeta \Xi_i(\zeta)) \right] + \sum_{i=1}^N p_i \left[r(\zeta) (\Omega_i(\zeta) - \zeta \Xi_i(\zeta)) \right] = G(\zeta) - r(\zeta) [\zeta (u(1) - u(0)) + u(0)] - s(\zeta) [\zeta (v(1) - v(0)) + v(0)]. \quad (50)$$

The equations (49) and (50) comprise a system of linear equations with unknown Haar wavelet coefficients p_i 's and q_i 's, which can be solved by using any method such as Gauss elimination method. Substituting the values of Haar wavelet coefficients p_i 's in equation (45) and q_i 's in equation (46), the non-uniform Haar wavelet approximate solutions of the system of second order LPSP boundary value problems are obtained.

Error Analysis: Theorem. Let $w(\zeta)$ be a square integrable function with bounded first derivative on $(0,1)$ and $w(\zeta_j)$ be Haar wavelet approximation of $w(\zeta)$. Then the error norm at J^{th} level satisfies the inequality

$$\|E\| = |w(\zeta) - w(\zeta_j)| \leq 2D\sqrt{K} \left(\frac{2^{-2(J+1)}}{3} \right)^2. \quad (51)$$

MAREs: In case, if the exact solution of the system of first order LPSP initial value problem is not known, then MARE is calculated by the formula

$$E = \max |\varepsilon u'(\zeta_j) + p(\zeta_j)u(\zeta_j) + q(\zeta_j)v(\zeta_j) - F(\zeta_j)|, \quad (52)$$

where, $u'(\zeta_j)$, $u(\zeta_j)$ and $v(\zeta_j)$ are given in equations (39), (45) and (46), respectively, and $j = 1, 2, \dots, N$; $N = 2J + 1$.

In case, if the exact solution of the system of second order LPSP boundary value problem is not known, then MARE is calculated by the formula

$$E = \max |\varepsilon u''(\zeta_j) + p(\zeta_j)u(\zeta_j) + q(\zeta_j)v(\zeta_j) - F(\zeta_j)|, \quad (53)$$

where, $u''(\zeta_j)$, $u(\zeta_j)$ and $v(\zeta_j)$ are given in equations (37), (45) and (46), respectively, and $j = 1, 2, \dots, N$; $N = 2J + 1$.

Numerical Examples

Example-1: Consider the following system of first order partial singularly perturbed initial value problem:

$$\varepsilon u'(\zeta) + (2 + \zeta)u(\zeta) - \left(1 + \frac{\zeta}{2}\right)v(\zeta) = 5\zeta + \frac{1}{2}, \quad (54)$$

$$v'(\zeta) - (1 + \zeta)u(\zeta) + (2 + \zeta)v(\zeta) = \zeta e^\zeta, \quad \forall \zeta \in (0,1) \quad (55)$$

$$\text{with initial conditions } u(0) = 2, \quad v(0) = 2. \quad (56)$$

The exact solution of the Example-1 is not known. MAREs are obtained by non-uniform Haar wavelet on ρ mesh and Shishkin mesh, respectively and have been given in the Tables 1 and 2 for different values of perturbation parameter ε .

Table-1: MAREs obtained by non-uniform Haar wavelet on ρ mesh for various values of ε .

$\varepsilon \setminus N$	64	128	256	512	1024
2^{-2}	4.8701E-14	2.6868E-13	9.1446E-13	5.9259E-14	2.4202E-13
2^{-6}	7.5723E-14	1.0251E-12	7.0238E-13	7.3614E-13	1.5762E-12
2^{-10}	1.2123E-14	4.6704E-13	6.1608E-13	6.6497E-13	3.6561E-12
2^{-14}	2.0324E-13	1.5995E-12	7.5747E-12	2.3983E-11	9.7295E-12
2^{-18}	2.1731E-13	1.5863E-12	1.1096E-11	1.3411E-10	1.9492E-10
2^{-22}	1.4677E-13	2.0127E-12	1.4508E-11	1.7891E-10	2.6338E-10
2^{-26}	2.0933E-13	2.0608E-12	1.3163E-11	1.3062E-10	2.7274E-10
2^{-30}	4.4147E-13	1.8439E-12	1.0245E-11	1.3652E-10	2.7150E-10
2^{-34}	1.5742E-13	1.6831E-12	1.1305E-11	1.2828E-10	2.7483E-10

Table-2: MAREs obtained by non-uniform Haar wavelet on Shishkin mesh for various values of ε .

$\varepsilon \setminus N$	64	128	256	512	1024
2^{-2}	2.3581E-13	6.3616E-14	1.4088E-13	1.2069E-13	1.4088E-13
2^{-6}	2.2891E-13	3.4759E-13	6.5280E-13	2.3513E-13	9.0102E-13
2^{-10}	4.6474E-12	5.7835E-12	3.1263E-12	8.3663E-12	1.0842E-11
2^{-14}	5.4188E-11	8.3047E-11	7.6559E-11	2.4077E-10	2.0673E-10
2^{-18}	1.3551E-09	6.6944E-10	3.1046E-09	4.5758E-09	2.3658E-09
2^{-22}	1.1750E-08	2.5296E-08	4.0195E-08	2.2049E-08	5.6705E-08
2^{-26}	4.5770E-07	1.8034E-07	6.1171E-07	9.6331E-07	9.7524E-07
2^{-30}	1.6048E-06	4.7842E-06	3.2528E-06	1.5017E-05	1.8888E-05
2^{-34}	3.2147E-05	3.4636E-05	1.6767E-04	1.7457E-04	3.6062E-04

Table-3: MAREs obtained by non-uniform Haar wavelet on Shishkin mesh for various values of ε .

$\varepsilon \setminus N$	64	128	256	512
2^{-2}	8.1877E-14	9.2358E-14	9.6601E-14	3.3459E-13
2^{-6}	1.0568E-13	1.0112E-13	9.6501E-14	3.0337E-13
2^{-10}	3.2368E-13	1.4556E-13	1.8085E-13	2.2248E-13
2^{-14}	1.4556E-13	2.0789E-13	1.3100E-13	2.4689E-13
2^{-18}	3.4778E-13	1.4677E-13	4.2066E-13	2.1389E-13
2^{-22}	1.2344E-13	2.0139E-13	4.0633E-13	3.0336E-13
2^{-26}	1.0188E-13	3.0337E-13	3.3084E-13	3.2458E-13
2^{-30}	1.0112E-13	4.0124E-13	1.2656E-13	2.2248E-13
2^{-34}	1.2344E-13	3.1586E-13	1.1323E-13	3.0863E-13

Table-4: MAREs obtained by non-uniform Haar wavelet on q mesh for various values of ε .

$\varepsilon \setminus N$	64	128	256	512
2^{-2}	6.1227E-14	8.3466E-14	3.1356E-13	2.2445E-13
2^{-6}	5.2377E-14	8.3466E-14	3.3469E-13	2.5577E-13
2^{-10}	5.2377E-14	8.1888E-14	3.0358E-13	2.5577E-13
2^{-14}	5.1239E-14	8.1888E-14	3.0357E-13	2.5577E-13
2^{-18}	5.2377E-14	8.1888E-14	3.1486E-13	2.5577E-13
2^{-22}	5.2377E-14	8.1888E-14	3.0358E-13	2.5577E-13
2^{-26}	5.2377E-14	8.1888E-14	3.1456E-13	2.5577E-13
2^{-30}	6.1227E-14	8.1888E-14	3.1579E-13	2.5577E-13
2^{-34}	5.2377E-14	8.1888E-14	3.2557E-13	2.5577E-13

Example-2: Consider the following system of second order partial singularly perturbed boundary value problem:

$$-\varepsilon v''(\zeta) + 2(1 + \zeta)^2 u(\zeta) - (1 + \zeta^3)v(\zeta) = 2e^\zeta, \quad (57)$$

$$-v''(\zeta) - 2\cos\left(\frac{\pi\zeta}{4}\right)u(\zeta) + 2.2e^{1-\zeta}v(\zeta) = 10\zeta + 1, \quad \forall \zeta \in (0,1) \quad (58)$$

with boundary conditions $u(0) = v(0) = 0$ and $y_1(1) = y_2(1) = 0$. (59)

The exact solution of the Example 2 is not known. MAREs are obtained by non-uniform Haar wavelet on Shishkin mesh and q mesh, respectively and have been given in the Table-3 and 4 for different values of perturbation parameter ε .

Conclusion

The systems of first and second order LPSP initial and boundary value problems have been solved using non-uniform Haar wavelets on Shishkin mesh, ρ mesh and q mesh. It is observed that the use of non-uniform Haar wavelet on different meshes such as Shishkin mesh, ρ mesh and q mesh can yield more accurate results.

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