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A Multi-Variate Reducible Product-Type Estimator in Two-Phase Sampling

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Abstract

To estimate population mean using multiple supplementary variables in a two-phase sampling set-up, a reducible or generalized product estimator has been constructed. It is assumed that the mean of the primary supplementary variable is unavailable and the means of p other (additional) supplementary variables are easily available. After studying reducibility property of the proposed estimator, some of its desirable statistical properties have been analyzed both theoretically and empirically.

Keywords: Supplementary variable, bias, efficiency, product estimator, two-phase sampling.

Introduction

Let y_i and x_i, $i = 1, 2, \dots., N$, be the measurements corresponding to survey variable y and a high negatively correlated supplementary (auxiliary) variable x for the ith unit of a population U whose means are $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ and $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$. As is already known, to estimate $\overline{Y}$ in the absence of information on $\overline{X}$, one needs a two-phase sampling. In the present context permitting simple random sampling without replacement (SRSWOR) at each phase, this sampling scheme consists of the selection of $n_1$units for the first-phase sample $s_1 (s_1 \subset U)$ to observe x for an estimation of $\overline{X}$, and then selection of n_2 units for the second-phase sample s_2 $(s_2 \subset s_1)$ to determine $y -$ values.

Define $\overline{y}_2 = \frac{1}{n_1}\sum_{i\in s_1} y_i$ and $\overline{x}_2 = \frac{1}{n_2}\sum_{i\in s_2} x_i$ as the mean values of y and x respectively for s_2, and $\overline{x}_1 = \frac{1}{n_1}\sum_{i\in s_1} x_i$ as the mean value of x for s_1. Then the traditional product estimator of $\overline{Y}$ is given by $t_P = \bar{y}_2 \frac{\bar{x}_2}{\bar{x}_1}$.

Even though t_P is biased, effect of bias is insignificant for large samples, and the asymptotic mean square error (MSE) expression is given by

$$M(t_P) = \theta_2 S_y^2 + (\theta_2 - \theta_1) R\left(R S_x^2 + 2\rho_{yx} S_y S_x\right), \tag{1}$$

where $\theta_1 = \frac{1}{n_1} - \frac{1}{N}$, $\theta_2 = \frac{1}{n_2} - \frac{1}{N}$, $R = \frac{\overline{Y}}{\overline{X}}$, $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \overline{Y}\right)^2$, $S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \overline{X}\right)^2$and ρ_{yx} is the coefficient of correlation among$y$ and x. t_P is more error-free (précised) than the direct estimator $\overline{y}_2$ when $\frac{\beta_{yx}}{R} < -\frac{1}{2}$, such that β_{yx} is regression coefficient of y on x.

Following Chand and Kiregyera, improvement over t_P in the sense of reduction of its MSE is possible by involving an additional supplementary variable[1-3]. This technique may be called *Chand-Kiregyera(C-K) Technique* that encompasses estimating $\overline{X}$ from s_1 exploiting prior details on an additional supplementary variable to be used in place of $\overline{x}_1$in any standard estimator. Although many researchers followed the idea of wielding an additional supplementary variable to compose varieties of estimators, only a handful efforts have been given to generate product-type estimators. Considering availability of multiple additional auxiliary variables, the present paper not only considers a generalized product-type estimator under C-K approach but also develops another new generalized product-type estimator under a modified approach called *Redesigned Approach*[4].

Association of Multiple Additional Supplementary Variables

Consider a situation where prior details on p cheaply and easily accessible additional supplementary variables $z_1, z_2, \dots, z_p$ (may be called as $z -$variables) with $\boldsymbol{z}' = (z_1, z_2, \dots, z_p)$, are obtainable. Assume that these $z -$variables acquire high correlation with y and x. Define $\overline{\boldsymbol{Z}}' = \left(\overline{Z}_1, \overline{Z}_2, \dots, \overline{Z}_p\right)$ as vector of their population means, where $\overline{Z}_j = \frac{1}{N}\sum_{i=1}^{N} z_{ji}$ and z_{ji} is the observation for z_j on the $i -$th unit of U, $j = 1, 2, \dots, p$; $i = 1, 2, \dots, N$. It is also accepted that $\overline{Z}_j$'s are known precisely.

In the usual definition, let $\boldsymbol{S_{zz}} = \begin{pmatrix} S_{z_1}^2 & \cdots & S_{z_1 z_p} \\ \vdots & \ddots & \vdots \\ S_{z_p z_1} & \cdots & S_{z_p}^2 \end{pmatrix}$, $\boldsymbol{S}_{\boldsymbol{yz}}' = \left(S_{yz_1}, S_{yz_2}, \dots, S_{yz_p}\right)$, $\boldsymbol{S}_{\boldsymbol{xz}}' = \left(S_{xz_1}, S_{xz_2}, \dots, S_{xz_p}\right)$, $\boldsymbol{\beta_{yz}} = \boldsymbol{S_{zz}^{-1} S_{yz}}$ and $\boldsymbol{\beta_{xz}} = \boldsymbol{S_{zz}^{-1} S_{xz}}$, where $S_{z_j}^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(z_{ji} - \overline{Z}_j\right)^2$, $S_{z_j z_k} =$

$\frac{1}{N-1}\sum_{i=1}^{N}\left(z_{ji}-\overline{Z}_j\right)\left(z_{ki}-\overline{Z}_k\right)$,$S_{yz_j}=\frac{1}{N-1}\sum_{i=1}^{N}\left(y_i-\overline{Y}\right)\left(z_{ji}-\overline{Z}_j\right)$ and $S_{xz_j}=\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i-\overline{X}\right)\left(z_{ji}-\overline{Z}_j\right),\ j\neq k=1,2,\dots,p.$

Let us further define $\rho_{yz_j}=S_{yz_j}/S_yS_{z_j}$ and $\rho_{xz_j}=S_{xz_j}/S_xS_{z_j}$, $j=1,2,\dots,p$ respectively as the coefficients of simple correlation among (y,z_j) and (x,z_j); and $\rho_{y.\boldsymbol{z}}^2=\frac{\boldsymbol{S}_{yz}'\boldsymbol{S}_{zz}^{-1}\boldsymbol{S}_{yz}}{S_y^2}$ as the squared of the coefficient of multiple correlation of y with the elements of $\boldsymbol{z}$.

In the considered two-phase sampling set-up, s_1 is used to accumulate information on x and all z −variables where as s_2 on y only. Define $\overline{\boldsymbol{z}}_{\boldsymbol{1}}'=\left(\bar{z}_{11},\bar{z}_{12},\dots,\bar{z}_{1p}\right)$ and $\overline{\boldsymbol{z}}_{\boldsymbol{2}}'=\left(\bar{z}_{21},\bar{z}_{22},\dots,\bar{z}_{2p}\right)$ such that $\bar{z}_{1j}=\frac{1}{n_1}\sum_{i\epsilon s_1}z_{ji}$ and $\bar{z}_{2j}=\frac{1}{n_2}\sum_{i\epsilon s_2}z_{ji}$ for $j=1,2,\dots,p$.

Let us first discuss a generalized product-type estimator that can be considered under the C-K method in the presence of p z −variables. Superseding $\bar{x}_1-\boldsymbol{A}'\left(\overline{\boldsymbol{z}}_{\boldsymbol{1}}-\overline{\boldsymbol{Z}}\right)$ for $\bar{x}_1$ in t_P, the following generalized product-type estimator can be defined:

$$t_{MP}=\bar{y}_2\frac{\bar{x}_2}{\left[\bar{x}_1-\boldsymbol{A}'(\overline{\boldsymbol{z}}_{\boldsymbol{1}}-\overline{\boldsymbol{Z}})\right]},$$

where $\boldsymbol{A}'=\left(A_1,A_2,\dots,A_p\right)$ such that A_j's are known constants (coefficients) normally decided to control MSE of the estimator. Note that, for various choices of the coefficients, t_{MP} clearly defines a class or a system of estimators of $\overline{Y}$. Asymptotic MSE expression for t_{MP} is given as

$$\mathrm{M(t_{MP})=M(t_P)+\theta_1RA'S_{zz}\left(RA+2\beta_{yz}\right)}. \tag{2}$$

From (2), $M(t_{MP})<M(t_P)$ *i.e.*, t_P is less efficient than t_{MP} if $\boldsymbol{A}<-2\frac{\boldsymbol{\beta_{yz}}}{R}$. (3)

Hence, under this reasonable limitation, t_{MP} can give a remarkable increase of precision over t_P and accordingly selection of $\boldsymbol{A}$ can be made. Note that this selection is not only influenced by the correlation of x with y but also by the correlations of all z −variables with y.

The optimum value of A that computed in the usual manner to minimize $M(t_{MP})$ is

$$\mathrm{\widehat{A}=-R^{-1}S_{zz}^{-1}S_{yz}=-R^{-1}\beta_{yz}}. \tag{4}$$

Utilization of this optimum value gives minimum $M(t_{MP})$*i.e.*, the minimum MSE bound of t_{MP} as

$$\mathrm{M_{min}(t_{MP})=M(t_P)-\theta_1S_{yz}'S_{zz}^{-1}S_{yz}=M(t_P)-\theta_1S_y^2\rho_{y.z}^2}\ ,(5)$$

and the optimum *i.e.*, minimum MSE bound estimator of t_{MP} as

$$\hat{t}_{MP}=\bar{y}_2\frac{\bar{x}_2}{\left[\bar{x}_1+R^{-1}\boldsymbol{\beta}_{\boldsymbol{yz}}'(\overline{\boldsymbol{z}}_{\boldsymbol{1}}-\overline{\boldsymbol{Z}})\right]}.$$

For the case of one z −variable $z_j, j=1,2,\dots,p$, say, $\hat{t}_{MP}=t=\bar{y}_2\frac{\bar{x}_2}{\left[\bar{x}_1-d(\bar{z}_{1j}-\overline{Z}_j)\right]}$, a composite estimator considered earlier[5]. See that t is generalizable for various selections of d. For example, t_P, $t_{RP}=\bar{y}_2\frac{\bar{x}_2\bar{z}_{1j}}{\bar{x}_1\overline{Z}_j}$, $t_{PP}=\bar{y}_2\frac{\bar{x}_2\overline{Z}_j}{\bar{x}_1\bar{z}_{1j}}$ and $t_{RGP}=\bar{y}_2\frac{\bar{x}_2}{\left[\bar{x}_1-\beta_{xz_j}(\bar{z}_{1j}-\overline{Z}_j)\right]}$ appear as specified cases of t if $d=0$, $\frac{\bar{x}_1}{\bar{z}_{1j}}$, $-\frac{\bar{x}_1}{\overline{Z}_j}$ and $\beta_{xz_j}=\frac{S_{xz_j}}{S_{z_j}^2}$ respectively. For this case, also note that

$$\hat{t}_{MP}\longrightarrow\hat{t}=\bar{y}_2\frac{\bar{x}_2}{\bar{x}_1-\frac{\beta_{yz_j}}{R}\left(\bar{z}_{1j}-\overline{Z}_j\right)}$$

and

$$M_{min}(t_{MP})\longrightarrow M_{min}(t)=M(\hat{t})=M(t_P)-\theta_1S_y^2\rho_{yz_j}^2. \tag{6}$$

The Proposed Reducible Product-Type Estimator

In many times the C-K approach has been denounced on the ground that it encourages substitution of $\bar{x}_1$ only by another estimator of $\overline{X}$ taking into account of one z −variable using data on s_1 but without considering $\bar{x}_2$ which happens to be less efficient estimate than $\bar{x}_1$ for estimating $\overline{X}$[4]. This approach therefore fails to exploit information contents on the additional supplementary variable z_j at different phases of sample selection. Keeping this in mind, the authors developed a more refined system contemplating certain modification over the C-K approach for the adequate use of available information on single z −variable z_j and to bring increased precision over t_P. They also called this technique a *Redesigned Technique* that involves making use of two difference estimators *viz*., $\bar{x}_1-\delta\left(\bar{z}_{1j}-\overline{Z}_j\right)$ established on s_1 and $\bar{x}_2-\eta\left(\bar{z}_{2j}-\bar{z}_{1j}\right)$ established on s_2 in lieu of $\bar{x}_1$ and $\bar{x}_2$ respectively in t_P. This methodology prompted them to define a new more generalized or a reducible (as named by the authors) product-type estimator:

$$\ell^{(G)}=\bar{y}_2\frac{\bar{x}_2-\eta\left(\bar{z}_{2j}-\bar{z}_{1j}\right)}{\bar{x}_1-\delta\left(\bar{z}_{1j}-\overline{Z}_j\right)}.$$

The reducibility characteristic of $\ell^{(G)}$ brings a system of estimators of product-type for $\overline{Y}$. Taking $\eta=0$ and $\delta=d$, $\ell^{(G)}=t$. This implies that the generalized estimator t forms a sub-class of the class generated by $\ell^{(G)}$. On the other hand, if $\eta=0$ and $\delta=0$, $\ell^{(G)}=t_P$ *i.e*., our base estimator; and if $\eta=0$, $\ell^{(G)}\longrightarrow t_{RP}$, t_{PP} and t_{RGP} for $\delta=\frac{\bar{x}_1}{\bar{z}_{1j}}$, $-\frac{\bar{x}_1}{\overline{Z}_j}$ and β_{xz_j} respectively.

Under their designed technique using p z −variables, the following reducible estimator, a direct multi-variate extension of $\ell^{(G)}$, is proposed:

$$\ell_{MP}^{(G)}=\bar{y}_2\frac{\bar{x}_2-\boldsymbol{C}_{\boldsymbol{2}}'(\overline{\boldsymbol{z}}_{\boldsymbol{2}}-\overline{\boldsymbol{z}}_{\boldsymbol{1}})}{\bar{x}_1-\boldsymbol{C}_{\boldsymbol{1}}'(\overline{\boldsymbol{z}}_{\boldsymbol{1}}-\overline{\boldsymbol{Z}})},$$

where: $C_1' = (C_1, C_2, \ldots, C_p)$ and $C_2' = (C_{21}, C_{22}, \ldots, C_{2p})$ are vectors of known coefficients decided to reduce $M(\ell_{MP}^{(G)})$ as per requirement. Notice that $\ell_{MP}^{(G)}$ is compiled when $\bar{x}_1$ and $\bar{x}_2$ in t_P are replaced by $\bar{x}_1 - C_1'(\bar{z}_1 - \bar{Z})$ and $\bar{x}_2 - C_2'(\bar{z}_2 - \bar{z}_1)$ respectively. Furthermore, note that $\ell_{MP}^{(G)} = t_{MP}$ if $C_1' = A$ and $C_2' = 0$.
An asymptotic MSE of $\ell_{MP}^{(G)}$ is obtained as

$$M(\ell_{MP}^{(G)}) = M(t_P) + \theta_1 R C_1' S_{zz}(RC_1 + 2\beta_{yz})$$

$$+(\theta_2 - \theta_1) R C_2' S_{zz}(RC_2 - 2\beta_{yz} - 2R\beta_{xz}). \quad (7)$$

As from (7) it is difficult to get both necessary and sufficient conditions, the following sufficient conditions are presented for warranting an appreciable gain in precision of $\ell_{MP}^{(G)}$ over t_P *i.e.*, $M(\ell_{MP}^{(G)}) < M(t_P)$:

$$\boldsymbol{C_1} < -2\frac{\boldsymbol{\beta_{yz}}}{R} \text{ and } \boldsymbol{C_2} < \frac{2(\boldsymbol{\beta_{yz}} + R\boldsymbol{\beta_{xz}})}{R}. \quad (8)$$

Hence, to meet (8) selections of the coefficient vectors $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ don't depend on the impact of x on y but on the impacts of all z −variables on both x and z.
To explain situations where the gain in efficiency of $\ell_{MP}^{(G)}$ over t_{MP} is remarkable, from (2) and (7) it is deduced that

$$M(\ell_{MP}^{(G)}) = M(t_{MP}) + \theta_1 R(C_1 - A)'[R(C_1 + A) + 2\beta_{yz}]$$

$$+(\theta_2 - \theta_1) R C_2' S_{zz}(RC_2 - 2\beta_{yz} - 2R\beta_{xz}). \quad (9)$$

This implies that $\ell_{MP}^{(G)}$ would be more efficient than t_{MP} when the following conditions are met: either

$$A < C_1 < -\frac{RA + 2\beta_{yz}}{R} \text{ or } -\frac{RA + 2\beta_{yz}}{R} < C_1 < A, \quad (10)$$

and

$$C_2 < \frac{2(\beta_{yz} + R\beta_{xz})}{R}. \quad (11)$$

But when $C_1 = A$, (11) is sufficient for $M(\ell_{MP}^{(G)}) < M(t_{MP})$.

It is very important to remark that the comparisons of $\ell_{MP}^{(G)}$ with t_P and t_{MP} would of course be meaningful only when t_P out performs over the direct estimator $\bar{y}_2$ *i.e.*, if $2\beta_{yx} < -R$.

Following conventional optimization procedure, the optimum values of $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ to minimize $M(\ell_{MP}^{(G)})$ in (9) are determined as

$$\hat{C}_1 = -R^{-1} S_{zz}^{-1} S_{yz} = -R^{-1}\beta_{yz} \quad (12)$$

$$\hat{C}_2 = R^{-1} S_{zz}^{-1} S_{yz} + S_{zz}^{-1} S_{xz} = R^{-1}\beta_{yz} + \beta_{xz}. \quad (13)$$

Evaluating (9) for $C_1 = \hat{C}_1$ and $C_2 = \hat{C}_2$, after simplification, the minimum MSE bound of $\ell_{MP}^{(G)}$ is derived as

$$M_{min}(\ell_{MP}^{(G)}) = M(t_P) - \theta_1 S_y^2 \rho_{y.z}^2 - (\theta_2 - \theta_1)(S_{yz} + RS_{xz})' S_{zz}^{-1}(S_{yz} + RS_{xz}). \quad (14)$$

See that the minimum MSE bound expression for $\ell_{MP}^{(G)}$ relies upon the partial correlation of y and x for fixed $\boldsymbol{z}$, and multiple correlations of y and x with $\boldsymbol{z}$.

Conclusively, a minimum MSE bound estimator of $\ell_{MP}^{(G)}$ corresponding to equation (14) is

$$\hat{\ell}_{MP}^{(G)} = \bar{y}_2 \frac{\bar{x}_2 - (R^{-1}\boldsymbol{\beta_{yz}} + \boldsymbol{\beta_{xz}})'(\bar{\boldsymbol{z}}_2 - \bar{\boldsymbol{z}}_1)}{\bar{x}_1 + R^{-1}\boldsymbol{\beta}_{yz}'(\bar{\boldsymbol{z}}_1 - \bar{\boldsymbol{Z}})}.$$

Here we also straight forwardly derive that when one additional supplementary variable z_j has been used,

$$\hat{\ell}_{MP}^{(G)} \longrightarrow \hat{\ell}_P^{(G)} = \bar{y}_2 \frac{\bar{x}_2 - \left(\frac{\beta_{yz_j}}{R} + \beta_{xz_j}\right)(\bar{z}_{2j} - \bar{z}_{1j})}{\bar{x}_1 + \frac{\beta_{yz_j}}{R}(\bar{z}_{1j} - \bar{Z}_j)},$$

and

$$M_{min}(\ell_{MP}^{(G)}) \longrightarrow M_{min}(\ell_P^{(G)}) = M(\hat{\ell}_P^{(G)})$$

$$= M(t_P) - S_y^2\left[\theta_1 \rho_{yz_j}^2 + (\theta_2 - \theta_1)\left(\rho_{yz_j} + \frac{C_x}{C_y}\rho_{xz_j}\right)^2\right]. \quad (15)$$

Efficiency Comparison

Various conditions procured above to show $\ell_{MP}^{(G)}$ is more efficient than t_P and t_{MP} are hard to verify unless they are tried to a definite surveyed situation. However, from the said conditions it may be inferred that the composed redesigned methodology has scope to bring improvements over that of Chand-Kiregyera. But for more clarification, precision of $\ell_{MP}^{(G)}$ compared t_P and t_{MP} has been evaluated in term of minimum MSE bound. For this let us modify (14) as

$$M_{min}(\ell_{MP}^{(G)}) = M(t_P) - \theta_1 S_y^2 \rho_{y.\boldsymbol{z}}^2 - (\theta_2 - \theta_1)\boldsymbol{U}'\boldsymbol{S}_{\boldsymbol{zz}}^{-1}\boldsymbol{U}, \quad (16)$$

such that $\boldsymbol{U}' = (U_1, U_2, \ldots, U_p)$ and $U_j = \frac{1}{N-1}\sum_{i=1}^{N}(g_i - \overline{G})(z_{ji} - \overline{Z}_j), j = 1, 2, \ldots, p$, where $g_i = y_i + Rx_i$ and $\overline{G} = \frac{1}{N}\sum_{i=1}^{N} g_i = \overline{Y} + R\overline{X}$. $\boldsymbol{S_{zz}}$ being a variance–covariance matrix, is necessarily positive definite and so also $\boldsymbol{S}_{\boldsymbol{zz}}^{-1}$. Hence, the quadratic form $U'\boldsymbol{S}_{\boldsymbol{zz}}^{-1}U$ is positive definite, *i.e.*, $U'\boldsymbol{S}_{\boldsymbol{zz}}^{-1}U \geq 0$[6]. Then, from (5), (14) and (16)

$$M_{min}(\ell_{MP}^{(G)}) < M_{min}(t_{MP}) < M(t_P), \quad (17)$$

which establishes that $\hat{\ell}_{MP}^{(G)}$ is more efficient than both $\hat{t}_{MP}$ and t_P. This outcome simply confirms that the methodology used to formulate $\ell_{MP}^{(G)}$ is superior to that used for t_{MP} under the minimum MSE bound criterion.

Empirical Study

To authenticate previous theoretical outcomes relating to the recommended generalized estimators t_{MP} and $\ell_{MP}^{(G)}$, five populations with two z −variables (z_1 and z_2), as detailed below, have been considered.
Population 1 [7]: N = 32 automobiles, y =miles/gallon, x = displacement, z_1 = horse power, z_2 = weight.
Population 2 [8]: N = 64 countries, y = child mortality, x = female literacy rate, z_1 =per capita GNP, z_2 =total fertility rate.
Population 3 [9]: N = 46 observations, y =evaporation, x = integrated area under daily humidity curve, z_1 =minimum daily relative humidity, z_2 = integrated area under daily air temperature curve.
Population 4 [10]: N = 45 observations, y = pigment creatinine, x = Phosphate (mg/mL), z_1 = volume (mL), z_2 = creatinine (mg/mL).

Population 5[11]: N = 44 married couples of medium and high-class families, y = no. of ever born children, x = education level of mother, z_1 = education level of father, z_2 = duration of marriage

To avoid complicacies, we focused on the minimum MSE so that only minimum MSE bound estimators $\hat{t}$, $\hat{\ell}_P^{(G)}$, $\hat{t}_{MP}$ and $\hat{\ell}_{MP}^{(G)}$ along with the base estimator t_P were taken for comparison. Relative efficiencies (REs) of these equipotential estimators compared to $\bar{y}_2$ whose variance is $V(\bar{y}_2) = \theta_2 S_y^2$, are compiled in Table-1 for specific values of n_1 and n_2.

Table-1 shows that t_P works better than the direct estimator $\bar{y}_2$ but as desired, its performance over all minimum MSE bound estimators is considerably inferior. Among four minimum MSE bound estimators, $\hat{t}$ turns out as the worst performer and appears to be less efficient than $\hat{\ell}_P^{(G)}$ although the efficiency loss in population 4 is marginal when established on z_1. $\hat{\ell}_{MP}^{(G)}$ emerges as the best performer followed by $\hat{\ell}_P^{(G)}$in most cases even though the efficiency gain of $\hat{\ell}_{MP}^{(G)}$ compared to $\hat{\ell}_P^{(G)}$ in population 4 is just marginal. Although this empirical study has a limited scope, its overall findings indicate that $\hat{\ell}_{MP}^{(G)}$ is superior to others on the ground of MSE.

Conclusion

Reviewing foregoing theoretical as well as empirical findings under the two-phase sampling network with numerous additional supplementary variables, we may eventually conclude that the imputed redesigned method with reference to the new reducible estimator $\ell_{MP}^{(G)}$ is not likely inferior to the C-K method and can be applied in many sample surveys for constructing estimators under the considered situations.

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Table–1: REs of Comparable Estimators w.r.t. $\bar{y}_2$(in%).

Estimator	Supplementary variable(s) used	Population				
		1 $n_1 = 10$ $n_2 = 5$	2 $n_1 = 19$ $n_2 = 10$	3 $n_1 = 14$ $n_2 = 7$	4 $n_1 = 14$ $n_2 = 7$	5 $n_1 = 13$ $n_2 = 7$
t_P	x	152.48	157.94	117.81	119.39	116.50
$\hat{t}$	x, z_1	256.19	178.49	152.90	137.06	143.13
	x,z_2	271.29	229.52	150.77	151.59	171.56
$\hat{\ell}_P^{(G)}$	x, z_1	360.15	182.81	201.44	137.72	144.09
	x, z_2	349.02	231.37	208.45	153.02	202.38
$\hat{t}_{MP}$	x, z_1, z_2	280.82	250.42	166.42	151.83	178.74
$\hat{\ell}_{MP}^{(G)}$	x, z_1, z_2	403.40	259.66	227.91	153.27	224.55

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