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Results on Hankel Determinants for the Class of Univalent Functions with a Generalized Derivative Operator

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Abstract

In this article, we introduce a new subclass of analytic functions defined by a generalized generalized derivative operator. The Hankel determinant for the nonlinear functional $|a_2a_4 - a_3^2|$ is obtained. *Our result extends corresponding previously known result*.

Keywords: Univalent functions, Hankel determinant, Complex plane C, Analytic function, generalized derivative operator.

Introduction

The Hankel determinant play an important role in study of univalent functions in complex analysis in the unit disk $\mathrm{U} = \{z : |z| < 1\}$. It is essential in various majors of mathematics and applied sciences. It has been of interest to many researchers. One of the most important results in the study involving Hankel determinants is the second Hankel determinant, it is estimated and given by $|a_2a_4 - a_3^2|$ which has been considered by many authors, the pioneering work of Duren introduced the concept of univalent analytic functions and showed that second coefficient $|a_3 - a_2^2| \le 1$ is proved[1].

Noor, determined the rate of growth of $H_q(n)$ for the class of functions with bounded boundary rotation[2]. While Ehrenborg studied the Hankel determinant of exponential polynomials and Layman obtained estimates the Hankel transform of an integer sequence and some of its properties[3,4]. As we know, the various subclasses of the univalent functions were investigated by many authors and improved the results of the second Hankel determinant, for instant, Janteng, et al, found a sharp bound for the function f in the subclass R of S in the unit disk, they have shown $|a_2a_4 - a_3^2| < \frac{4}{9}$, then they found the value of the second Hankel determinant for the class of star like and convex functions $|a_2a_4 - a_3^2| < 1$, and $|a_2a_4 - a_3^2| < \frac{1}{8}$ respectively[5,6].

The sharp bound for the functions in the classes of strike functions of order β and convex functions of order β were found by Lee et al.[7]. Moreover, Kund & Mishra[8] examined a class of analytic functions associated with the Carlson-Shaffer operator in the unit disk to estimates the second Hankel determinant. Recently, other scholars have focused on deriving sharp upper bounds for different classes and operators[9-11].

In this paper, we introduce a new class $\mathrm{R}^*(\lambda_1, \lambda_2, l, n, \alpha)$ by using a generalized derivative operator and fined the sharp of the second Hankel for this class. In addition, various other known results are also pointed out and find some interesting corollaries on the class of normalized analytic functions in the open unit disk. Our results certainly generalized several results obtained earlier.

The Hankel determinants $H_q(n)$ ($n = 1, 2, 3, \cdots, q = 1, 2, 3, \cdots$) of the function f are defined by Noonan and Thomas [12]

$$H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k+q-1} & a_{k+q} & \cdots & a_{k+2q-2} \end{vmatrix}.$$

Let $f \in \mathrm{A}$ the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in \mathrm{U}), \tag{1}$$

Let P be the family of all functions p analytic in U for which the real of the function and p great than zero and

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad (z \in \mathrm{U}). \tag{2}$$

In this current work, authors will show sharp estimate for Hankel determinant of the univalent functions class which define in the formula (1).

Definition 1: Amer and Darus[13] had recently introduced a new generalized derivative operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$, also this operator has been used in many of research papers[14-17].

Let $f \in \mathrm{A}$.then the generalized derivative operator is defined by

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^k, \qquad (3)$$

Where $n,m \in N_0 = \{0,1,2,...\},\ \lambda_2 \ge \lambda_1 \ge 0, l \ge 0,$

and $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}.$

We also a class of univalent functions denoted by $\mathrm{R}^*(\lambda_1,\lambda_2,l,n,\alpha)$ of A is introduced.

Definition 2: A function $f \in \mathrm{A}$.is said to be in the class $\mathrm{R}^*(\lambda_1,\lambda_2,l,n,\alpha)$

$$\mathrm{R}^*(\lambda_1,\lambda_2,l,n,\alpha) =$$

$$\left\{ f \in \mathrm{A} : \Re\left\{ (1-\alpha)\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} + \alpha(I^m(\lambda_1,\lambda_2,l,n)f(z))' \right\} > 0 \right\}$$

Where
$n,m \in N_0 = \{0,1,2,...\}, \lambda_2 \ge \lambda_1 \ge 0, l \ge 0, \alpha > 0, z \in \mathrm{U}.$

Remark 1: The subclass $\mathrm{R}^*(0,0,0,1,1) = \mathrm{R}$ was studied systematically by MacGregor[18].

Lemma 1: If $p \in \mathrm{P}$ then $|c_k| \le 2$, for all k[19].

Lemma 2: Let the function $p \in \mathrm{P}$ be given by the power series (2). Then, $2c_2 = c_1^2 + x(4-c_2),$
for some x, $|x| \le 1$, and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1 x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)2(1-|x|^2)z,$$

for some z, $|z| \le 1$ see [20].

Results

Theorem 1 . If $f \in \mathrm{R}^*$, then

$$\left|a_2a_4 - a_3^2\right| \le \frac{16(1+l)^{2(m-1)}(1+2\lambda_2)^{2m}}{(1+2\alpha)^2(1+2\lambda_1+l)^{2(m-1)}(n+1)^2(n+2)^2},$$

where $\alpha > 0.$

Proof: Since $f \in \mathrm{R}^*$

$$(1-\alpha)\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} + \alpha(I^m(\lambda_1,\lambda_2,l,n)f(z))' = p(z), \qquad (4)$$

Equating coefficients in (4), we get,

$$\left.\begin{aligned} a_2 &= \frac{c_1(1+l)^{m-1}(1+\lambda_2)^m}{(1+\alpha)(1+\lambda_1+l)^{m-1}(n+1)}, \\ a_3 &= \frac{2c_2(1+l)^{m-1}(1+2\lambda_2)^m}{(1+2\alpha)(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}, \\ a_4 &= \frac{6c_3(1+l)^{m-1}(1+3\lambda_2)^m}{(1+3\alpha)(1+3\lambda_1+l)^{m-1}(n+1)(n+2)(n+3)}. \end{aligned}\right\} \qquad (5)$$

From (5), it can be established that

$$\left|a_2a_4 - a_3^2\right| =$$

$$\left|\frac{6c_1c_3(1+l)^{2(m-1)}(1+\lambda_2)^m(1+3\lambda_2)^m}{(1+\alpha)(1+3\alpha)(1+\lambda_1+l)^{m-1}(1+3\lambda_1+l)^{m-1}(n+1)^2(n+2)(n+3)}\right.$$

$$\left. - \frac{4c_2^2(1+l)^{2(m-1)}(1+2\lambda_2)^{2m}}{(1+2\alpha)^2(1+2\lambda_1+l)^{2(m-1)}(n+1)^2(n+2)^2}\right|.$$

Now assuming,

$$X_1 = \frac{(1+\lambda_1+l)^{m-1}(n+1)}{(1+l)^{m-1}(1+\lambda_2)^m}, \quad X_2 = \frac{(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}{2(1+l)^{m-1}(1+2\lambda_2)^m},$$

and

$$X_3 = \frac{(1+3\lambda_1+l)^{m-1}(n+1)(n+2)(n+3)}{(1+l)^{m-1}(1+3\lambda_2)^m}.$$

Also, let $c_1 = c$ $,(0 \le c \le 2),$ and use lemma 2 we get

$$\left|a_2a_4 - a_3^2\right| =$$

$$\left\{\begin{aligned} &\left|\frac{(c^4+4\alpha c^4)(X_2^2 - X_1X_3) + (4\alpha^2c^4X_2^2 - 3\alpha^2c^4X_1X_3)}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3}\right. \\ &+ \frac{[(2c^2+8\alpha c^2)(X_2^2 - X_1X_3) + (8\alpha^2c^2X_2^2 - 6\alpha^2c^2X_1X_3)]x(4-c^2)}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ &- \frac{x^2(4-c^2)\left[(c^2+4\alpha c^2)(X_2^2 - X_1X_3) + (4\alpha^2c^2X_2^2 - 3\alpha^2c^2X_1X_3) + 4X_1X_3 + 16\alpha X_1X_3 + 12\alpha^2X_1X_3\right]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ &\left. + \frac{c(4-c^2)(1-|x|^2)z}{2(1+\alpha)(1+3\alpha)X_1X_3}\right|. \end{aligned}\right.$$

replacement of $|x|$ by ρ give by

$$\left|a_2a_4 - a_3^2\right| \le \left\{\begin{aligned} &\frac{(c^4+4\alpha c^4)(X_2^2 - X_1X_3) + (4\alpha^2c^4X_2^2 - 3\alpha^2c^4X_1X_3)}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ &+ \frac{[(c^2+8\alpha c^2)(X_2^2 - X_1X_3) + (4\alpha^2c^2X_2^2 - 3\alpha^2c^2X_1X_3)]\rho(4-c^2)}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ &+ \frac{\rho^2(4-c^2)\left[(c^2+4\alpha c^2)(X_2^2 - X_1X_3) + (4\alpha^2c^2X_2^2 - 3\alpha^2c^2X_1X_3)\right]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ &+ \frac{\rho^2(4-c^2)\left[4X_1X_3 + 16\alpha X_1X_3 + 12\alpha^2X_1X_3 - 2cX_2^2(1+2\alpha)^2\right]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ &+ \frac{c(4-c^2)}{2(1+\alpha)(1+3\alpha)X_1X_3} \end{aligned}\right.$$

$$= F(\rho), \qquad (6)$$

with $z = |x| \le 1$.

Let the upper bound for (6) is attained at an interior point of $\{(\rho,c) : \rho \in [0,1], c \in [0,2]\}$, then

$$\frac{\partial F}{\partial \rho}=\frac{[(c^2+4\alpha c^2)(X_2^2-X_1X_3)+(4\alpha^2c^2X_2^2-3\alpha^2c^2X_1X_3)](4-c^2)}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \qquad (7)$$

$$+\frac{\rho(4-c^2)\left[(c^2+4\alpha c^2)(X_2^2-X_1X_3)+(4\alpha^2c^2X_2^2-3\alpha^2c^2X_1X_3)\right]}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3}$$

$$+\frac{\rho(4-c^2)\left[4X_1X_3+16\alpha X_1X_3+12\alpha^2X_1X_3-2cX_2^2(1+2\alpha)^2\right]}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3}.$$

Let $F'(\rho)>0$ and consequently F is increasing and $\max_{(0\le\mu\le1)} F(\rho)=F(1),$ which contradicts our assumption of having the maximum value at the interior of $\rho\in[0,1].$ Now let

$$G(c)=F(1)=\begin{cases} \dfrac{(c^4+4\alpha c^4)(X_2^2-X_1X_3)+(4\alpha^2c^4X_2^2-3\alpha^2c^4X_1X_3)}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ +\dfrac{[(c^2+8\alpha c^2)(X_2^2-X_1X_3)+(4\alpha^2c^2X_2^2-3\alpha^2c^2X_1X_3)](4-c^2)}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ +\dfrac{(4-c^2)\left[(c^2+4\alpha c^2)(X_2^2-X_1X_3)+(4\alpha^2c^2X_2^2-3\alpha^2c^2X_1X_3)\right]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ +\dfrac{(4-c^2)\left[4X_1X_3+16\alpha X_1X_3+12\alpha^2X_1X_3-2cX_2^2(1+2\alpha)^2\right]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ +\dfrac{c(4-c^2)}{2(1+\alpha)(1+3\alpha)X_1X_3}, \end{cases}$$

then,

(8)

$$G'(c)=\left.\begin{array}{l} \dfrac{(4c^3+16\alpha c^3)(X_2^2-X_1X_3)+(16\alpha^2c^3X_2^2-12\alpha^2c^3X_1X_3)}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ +\dfrac{[(2c+16\alpha c)(X_2^2-X_1X_3)+(8\alpha^2cX_2^2-6\alpha^2cX_1X_3)](4-c^2)}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ -\dfrac{2c[(c^2+8\alpha c^2)(X_2^2-X_1X_3)+(4\alpha^2c^2X_2^2-3\alpha^2c^2X_1X_3)]}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ +\dfrac{(4-c^2)\left[(2c+8\alpha c)(X_2^2-X_1X_3)+(8\alpha^2cX_2^2-6\alpha^2cX_1X_3)-2X_2^2(1+2\alpha)^2\right]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ -\dfrac{2c\left[(c^2+4\alpha c^2)(X_2^2-X_1X_3)+(4\alpha^2c^2X_2^2-3\alpha^2c^2X_1X_3)\right]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ -\dfrac{2c\left[4X_1X_3+16\alpha X_1X_3+12\alpha^2X_1X_3-2cX_2^2(1+2\alpha)^2\right]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)X_1X_2^2X_3} \\ +\dfrac{4-3c^2}{2(1+\alpha)(1+3\alpha)X_1X_3}, \end{array}\right\}$$

therefore (8) implies $c=0,$ which is a contradiction. Thus any maximum points of G must be on the boundary of $c\in[0,2].$ However, $G(c)\ge G(2)$ and thus G has maximum value at $c=0.$

The upper bound for (6) corresponds to $\rho=1$ and $c=0,$ in which case

$$\left|\frac{6c_1c_3(1+l)^{2(m-1)}(1+\lambda_2)^m(1+3\lambda_2)^m}{(1+\alpha)(1+3\alpha)(1+\lambda_1+l)^{m-1}(1+3\lambda_1+l)^{m-1}(n+1)^2(n+2)(n+3)}\right.$$

$$\left.-\frac{4c_2^2(1+l)^{2(m-1)}(1+2\lambda_2)^{2m}}{(1+2\alpha)^2(1+2\lambda_1+l)^{2(m-1)}(n+1)^2(n+2)^2}\right|\le\frac{16(1+l)^{2(m-1)}(1+2\lambda_2)^{2m}}{(1+2\alpha)^2(1+2\lambda_1+l)^{2(m-1)}(n+1)^2(n+2)^2}.$$

The proof of Theorem 1 is evidently complete.

Remark 2. If $\alpha>0,$ then we get the corresponding functional $|a_2a_4-a_3^2|,$ for the class $f\in\mathrm{R}^*(0,0,0,0,\alpha)=R(\alpha),$ studied in G. Murugusundaramoorthy and N. Magesh [21] as in the following corollary.

Corollary 1. *If* $f\in R(\alpha),$ then $|a_2a_4-a_3^2|\le\dfrac{4}{(1+2\alpha)^2},$ the result is sharp.

Remark 3. If $\alpha=1,$ then we get the corresponding functional $|a_2a_4-a_3^2|,$ for the class $f\in\mathrm{R}^*(0,0,0,0,1)=\mathrm{R},$ studied in A. Janteng et al.[5] as in the following corollary.

Corollary 2. *If* $f\in\mathrm{R},$ then $|a_2a_4-a_3^2|\le\dfrac{4}{9},$ the result is sharp.

Conclusion

In this study, we define a new class of univalent functions using the generalized derivative operator. Then we investigated the bounds of the second Hankel determinant for this class. Our results extend previously known results in the literature. Recently, researchers have explored various problems related to analytic and univalent functions, with a particular focus on derivative and integral operators. These topics offer vast opportunities for further exploration, as detailed in the existing literature[22-27].

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