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Random Common Fixed Point Result in Polish Space

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Abstract

In this paper we find necessary conditions for convergence to common fixed point under random iteration scheme. And find, random common fixed point of pair of JSR random operators satisfying generalized contractive condition in the framework of symmetric spaces. AMS (2000): 54H25, 47H10

Keywords: Symmetric space, Polish space JSR random operator, random common fixed point

Introduction

Random fixed point theorems are stochastic generalization of classical fixed point theorems. To find the solution of nonlinear random system the random fixed point theory is used by so many authors. Beg[1,2] and Beg and Shahzad[3,4] find common random fixed point results and random coincidence points of a pair of compatible random operator. Mathematician Beg and shahzad[4,5] used different iteration process to obtain common random fixed points. Recently, Mehta et.al[6] find some results in such direction.

In this paper we find necessary conditions for convergence to a common fixed point of two pair of JSR random operators satisfying generalized contractive conditions in Polish and symmetric spaces.

Preliminaries

In this paper we denote (Ω, Σ) as measurable space. A symmetric set X and a non-negative real valued function d is defined on XxX such that for all x, y $\in$X we have (i) d (x,y) = 0 iff x=y, and (ii) d (x,y) = d (y,x). For $\varepsilon > 0$ and $x \in X$, $B(x, \varepsilon)$ denote the spherical open ball centered at x and radius ε. A topology t(d) on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$ where d is defined as above.

Now $x_n \to x$ as $n\to\infty$ if and only if $x_n \to x$ in topology t(d). Let F be a subset of X. The mapping $T:\Omega\times F\to F$ is a random map if and only if for each fixed $x\in F$, the mapping $T(.,x):\Omega\to F$ is measurable .A mapping $\xi:\Omega\to X$ is *measurable* if $\xi^{-1}(U) \in \Sigma$ for each open subset U of X. The mapping T is continuous if for each $\omega \in \Omega$, the mapping $t(\omega,.):F\to X$ is continuous. A measurable mapping $\xi:\Omega\to X$ is a random fixed point of random map $T:\Omega\times F\to F$ if and only if $T(\omega,\xi(\omega)=\xi(\omega)$ for each $\omega\in\Omega$.We denote the set of random fixed point of a random map T by RF(T) and the set of all measurable mapping from Ω into a symmetric a space X by $M(\Omega,X)$. We denote the nth iterate $T(\omega, T(\omega,\ldots\ldots\ldots,T(\omega,x)$ of T by $T_n(\omega,x)$.

Definition 1. Let X be a separable complete space i.e. X is Polish space. Random operators $S,T:X\to X$ is said to be **T-JSR mapping** if
$\alpha d(T(\omega,\xi_n(\omega),S(\omega,\xi_n(\omega))) \leq \alpha\, d(T(\omega,T(\xi_n(\omega)),T(\omega,\xi_n(\omega))) = 0 \ \forall\omega\in\Omega$
where α= lim sup or lim inf and $\{(\xi_n,(\omega)\}$ is a sequence in X such that $\lim T(\omega,\xi_n(\omega) = \lim S(\omega, \xi_n(\omega) = \xi(\omega)$.

Definition 2. X be a polish space and random operators $S,T:\Omega\times X\to X$ are said to be weakly T-JSR if $T(\omega,\xi(\omega))=S(\omega,\xi(\omega))$ for some ξ in $M(\Omega,X)$, then $S(\omega,T(\omega,\xi(\omega)))\leq T(\omega,T(\omega,\xi(\omega)))$.

Definition 3. $\{x_n\}$,$\{y_n\}$ are two sequences in symmetric space (X,d) and for $x,y \in X$, the space satisfy the following conditions
3(a) $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x_n ,y) = 0$ then x = y

3(b) $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x_n ,y_n) = 0$ then $\lim_{n\to\infty} d(y_n,x) = 0$

Definition 4. Let $\{x_n\}$, $\{y_n\}$ be two sequences in Polish space (X,d) and x in X. The space X is said to satisfy condition (H_E) if $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,y_n) = 0$ then $\lim_{n\to\infty} d(x_n,y_n) = 0$

Definition 5. The two random mappings $S,T : \Omega x X \to X$ are said to satisfy property (I) if there exists a sequence $\{\xi_n\}$ in $M(\Omega,X)$ such that for some ξ in $M(\Omega,X)$, $\lim_{n\to\infty} d(T(\omega,\xi_n(\omega),\xi(\omega))) = \lim_{n\to\infty} d(S(\omega,\xi_n(\omega),\xi(\omega))) = 0\ \forall \omega\in\Omega$.
Where d is a symmetric function on X

Results

Theorem 3.1: Let (X, d) be a Polish space that satisfy 3(a) and (H_E). Let S and T are S-JSR random operators from Ωx X to X which satisfy the property (I) and following condition
$d(T(\omega,x),T(\omega,y)) \le [\max\{d(S(\omega,x),S(\omega,y)),d(S(\omega,x),T(\omega,x))+d(S(\omega,y),T(\omega,y)), d(S(\omega,x),T(\omega,y))+d(S(\omega,y),T(\omega,x))\}]\quad \forall\ x,y \in X$ and $\omega\in\Omega$ ……………………………………(3.1).

If $T(\omega,X) \subset S(\omega,X)$ and one of $T(\omega,X)$ or $S(\omega,X)$is a complete subspace of X for every $\omega\in\Omega$, then T and S have unique and common random fixed point.

Proof: Since random operators S and T satisfy the property (I) therefore there exists a sequence $\{\xi_n\}$ in $M(\Omega,X)$ such that for some $\xi \in M(\Omega,X)$ and for every $\omega\in\Omega$
$\lim_{n\to\infty} d(T(\omega,\xi_n(\omega),\xi(\omega))) = \lim_{n\to\infty} d(S(\omega,\xi_n(\omega),\xi(\omega))) = 0$………(3.2)

Then by property (H_E), we have $\lim_{n\to\infty} d(T(\omega,\xi_n(\omega),S(\omega,\xi_n(\omega))=0$ for every $\omega\in\Omega$.

(A) Suppose $S(\omega,X)$ is a complete subspace of X for every $\omega\in\Omega$. Let $\xi_1:\Omega\to X$ be the limit of the sequence of measurable mapping $\{S(\omega,\xi_n(\omega))\}$ and $S(\omega,\xi_n(\omega))$ in $S(\omega,X)$ for every $\omega\in\Omega$ and $n\in N$. Since X is separable, therefore $\xi_1 \in M(\Omega,X)$.
Moreover $\xi_1(\omega)\in S(\omega,X)$ for every $\omega\in\Omega$. Then this allows obtaining the measurable mapping $\bar{\xi}:\Omega\to X$ such that $\xi(\omega) = S(\omega, \bar{\xi}(\omega))$.Now for every $\omega\in\Omega$ we show that that $T(\omega, \bar{\xi}(\omega)) = S(\omega, \bar{\xi}(\omega))$. Let us consider

$d(T(\omega, \bar{\xi}(\omega)),T(\omega,\xi_n(\omega))) \le \max[d(S(\omega, \bar{\xi}(\omega)),S(\omega,\xi_n(\omega))), \{d(S(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega)))+ d(S(\omega,\xi_n(\omega)),T(\omega,\xi_n(\omega)))\}/2, \{d(S(\omega, \bar{\xi}(\omega)),T(\omega,\xi_n(\omega)))+d(S(\omega,\xi_n(\omega)),T(\omega, \bar{\xi}(\omega)))\}/2]$

$d(T(\omega, \bar{\xi}(\omega)),T(\omega,\xi_n(\omega))) \le \max[d(\xi(\omega)),S(\omega,\xi_n(\omega))), \{d(\xi(\omega)),T(\omega, \bar{\xi}(\omega)))+d(S(\omega,\xi_n(\omega)),T(\omega,\xi_n(\omega)))\}/2, \{d(\xi(\omega)),T(\omega,\xi_n(\omega)))+d(S(\omega,\xi_n(\omega)),T(\omega, \bar{\xi}(\omega)))\}/2]$

Therefore by (3.2) and limit we obtain $d(T(\omega, \bar{\xi}(\omega)),\xi(\omega))\le d(\xi(\omega),T(\omega, \bar{\xi}(\omega)))$

which gives contradiction therefore $\xi(\omega) = T(\omega, \bar{\xi}(\omega)) \Rightarrow T(\omega, \bar{\xi}(\omega)) = S(\omega, \bar{\xi}(\omega))$.

The weak T-JSR of random mapping T and S implies that for some $\bar{\xi}$ in $M(\Omega,X)$.

Now we show that $T(\omega,T(\omega, \bar{\xi}(\omega))) = T(\omega, \bar{\xi}(\omega))$ for every $\omega\in\Omega$. Consider

$d(T(\omega, \bar{\xi}(\omega)),T(\omega,\xi(\omega))) \le \max[d(S(\omega, \bar{\xi}(\omega)),S(\omega,\xi(\omega))), \{d(S(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega)))+d(S(\omega,\xi(\omega)),T(\omega,\xi(\omega)))\}/2, \{d(S(\omega, \bar{\xi}(\omega)), T(\omega,\xi(\omega))),d(S(\omega,\xi(\omega)),T(\omega, \bar{\xi}(\omega)))\}/2], \le \max[d(T(\omega, \bar{\xi}(\omega)),S(\omega,T(\omega, \bar{\xi}(\omega))), \{d(T(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega)))+d(S(\omega,T(\omega, \bar{\xi}(\omega))), T(\omega,T(\omega, \bar{\xi}(\omega)))\}/2, d(T(\omega, \bar{\xi}(\omega)),T(\omega,T(\omega, \bar{\xi}(\omega)))+d(S(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega, \bar{\xi}(\omega)))\}/2]$

$\le \max[d(T(\omega, \bar{\xi}(\omega)),T(\omega,T(\omega, \bar{\xi}(\omega))), \{d(T(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega)))+d(T(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega,T(\omega, \bar{\xi}(\omega)))\}/2, \{d(T(\omega, \bar{\xi}(\omega)),T(\omega,T(\omega, \bar{\xi}(\omega)))+d(T(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega, \bar{\xi}(\omega)))\}/2]$
$\le d(T(\omega, \bar{\xi}(\omega)),T(\omega,T(\omega, \bar{\xi}(\omega)))] < d(T(\omega, \bar{\xi}(\omega)),T(\omega,\xi(\omega)))$

Which is contradiction, therefore $T(\omega,T(\omega, \bar{\xi}(\omega))) = T(\omega, \bar{\xi}(\omega))$ i.e. $T(\omega, \bar{\xi}(\omega))$ is a random fixed point of T. Again,
$d(S(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega, \bar{\xi}(\omega))) \le d(T(\omega,T(\omega, \bar{\xi}(\omega))),T(\omega, \bar{\xi}(\omega))), = d(T(\omega, \bar{\xi}(\omega)),T(\omega, \bar{\xi}(\omega))) = 0.$

It implies that $T((\omega, \bar{\xi}(\omega))$ is also fixed point of S. Thus $T((\omega, \bar{\xi}(\omega))$ is common fixed point of S and T.

(B) Suppose T (ω,X) is complete subspace of X for every $\omega\in \Omega$. As T $(\omega,X) \subset S(\omega,X)$ the proof is same as (A).

Uniqueness:- Let ν and $\bar{\nu}$ from Ω to X are two common fixed point of S and T .
Let $\nu(\omega) \neq \bar{\nu}(\omega)$ then by 3.1 we have
$d(\nu(\omega), \bar{\nu}(\omega)) = d(T(\omega,\nu(\omega)),T(\omega, \bar{\xi}(\omega))), \leq \max [d(S(\omega,\nu(\omega)),S(\omega, \bar{\nu}(\omega))), \{d(S(\omega,\nu(\omega)),T(\omega,\nu(\omega)))+d(S(\omega, \bar{\nu}(\omega)), T(\omega, \bar{\nu}(\omega)))\}\backslash 2, \{d(S(\omega,\nu(\omega)),T(\omega, \bar{\nu}(\omega))),d(S(\omega, \bar{\nu}(\omega)),T(\omega,\nu(\omega)))\}\backslash 2], \leq d(\nu(\omega), \bar{\nu}(\omega))$

Which gives contradiction, therefore $\nu(\omega) = \bar{\nu}(\omega)$ for every $\omega\in \Omega$.

Example

4.1:- Let $\Omega = [0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of [0,1]. Let X = R with $d(x,y) = a^{|x-y|} - 1$, where $a > 1$ and clearly d is symmetric on R. Define random operators S and T from Ωx X to X as $S(\omega,x) = (1-\omega^2+2x)/3$ and $T(\omega,x) = (1-\omega^2+3x)/4$.

Also sequence of mapping $\xi_n : \Omega \rightarrow X$ is defined by $\xi_n(\omega) = 1+(1/n) -\omega^2$ for every $\omega \in \Omega$ and $n \in N$.
Define measurable mapping $\xi : \Omega \rightarrow X$ as $\xi(\omega) = 1-\omega^2$ for every $\omega \in \Omega$.
$\lim_{n\to\infty} d(T(\omega,\xi_n(\omega)),\xi(\omega)) = \lim_{n\to\infty} a^{|T(\omega,\xi n(\omega)) - \xi(\omega)|} - 1 = \lim_{n\to\infty} a^{3/4n} - 1 = 0$

and $\lim_{n\to\infty} d(S(\omega,\xi n(\omega)),\xi(\omega))=\lim_{n\to\infty} a^{|T(\omega,\xi n(\omega))-\xi(\omega)|}-1= \lim_{n\to\infty} a^{2/3n} -1=0.$

Clearly S and T satisfy property I.

4.2:- Let S and T from ΩxX to X as $S(\omega,x) = (1-\omega^2+2x)/3$,
$T(\omega,x) = (1-\omega^2+3x)/4$ and let $\bar{\xi}(\omega) = 1-\omega^2$ for every $\omega \in \Omega$.
Then $T(\omega, \bar{\xi}(\omega)) = 1-\omega^2 = S(\omega, \bar{\xi}(\omega))$
$\Rightarrow S(\omega,T(\omega, \bar{\xi}(\omega))) < T(\omega,T(\omega, \bar{\xi}(\omega)))$.

Conclusion

Thus S and T are weakly T-JSR operators. Also as in 4.1 S and T are weakly T-JSR operators and $T(\omega, \bar{\xi}(\omega)) = 1-\omega^2$ is a unique random fixed point of S and T.

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