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Fixed Point Results on Fuzzy Mappings for Rational Expressions

Garg Surendra Kumar and Shukla Manoj Kumar
Department of Mathematics, Govt. Model Science College, Jabalpur, MP, INDIA

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Abstract

In this paper we prove some fixed point and common fixed point theorems for fuzzy mappings in complete metric space which also include rational expression as a contraction. AMS Subject Classification: 54H25, 47 H10

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Introduction

In 1965 Zadeh[1] introduce the concept of fuzzy sets. After that so many works have been done in fuzzy sets. In 1981 Heilpern[2] define fuzzy mappings are use of fuzzy mappings he proved fixed point theorem which is a fuzzy analogue of the fixed point theorem for multi valued mappings of Nadler[3] , Vijayraju and Marudai[4] generalized the results of Bose and Mukherjee's[5] for fuzzy mappings. So many authors Marudai and Srinivasan[6], Bose and Sahani[7], Butnariu[8,9,10], Chang and Huang[11], Chang[12], Chitra[13], Som and Mukharjee[14] studied fixed point theorems for fuzzy mappings. Lee, Cho, Lee and Kim[15] obtained a common fixed point theorem for a sequence of fuzzy mappings satisfying certain conditions, which is generalization of the second theorem of Bose and Sahini[7].

More recently Vijayraju and Mohanraj[16] obtained some fixed point theorems for contractive type fuzzy mappings which are generalization of Beg and Azam[17]. Fuzzy extension of Kirk and Downing[18] and its prove analogue to the proof of Park and Jeong[19]. In the present paper we are proving some fixed point and common fixed point theorems in fuzzy mappings containing the rational expressions.

Preliminaries

We need following definitions and assumptions:

Definition 2.1 Let X be any metric linear space and d be any metric on X. A fuzzy set in X is a function with domain X and values in [0,1]. If A is a fuzzy set and x ε X, the function value A(x) is called the grade of membership of x in A. The collection of all fuzzy sets in X is denoted by Ғ(x).

Let $A \in$ Ғ(x).and $\alpha \in [0,1]$.The set α –level set of A, denoted by A_α
$A_\alpha = \{ x: A(x) \geq \alpha \}$ if $\alpha \in [0,1]$,
$A_0 = \overline{\{x : A(x) > 0\}}$, whenever $\bar{B}$ is clouser of B
Now we distinguish from the collection Ғ(x) a sub collection of approximate quantities, denoted W(x).

Definition 2.2 A fuzzy subset A of X is an approximate quantity i ff its α-level set is a compact subset (non fuzzy) of X for each α ϵ [0, 1], and $\sup\limits_{x \varepsilon X} A(x) = 1$

When A ϵ W(x) and $A(x_0) = 1$ for some x_0 ϵ W(x), we will identify A with an approximation of x_0. Then we shall define a distance between two approximate quantities.

Definition 2.3 Let A, B ϵ W(x), α ϵ [0,1], define
$$p_\alpha(A,B) = \inf_{x \varepsilon A_\alpha, y \varepsilon B_\alpha} d(x,y), D_\alpha(A,B) = dist(A_\alpha, B_\alpha), d(A,B) = \sup_\alpha D_\alpha(A,B)$$
Where dist. Is Hausdorff distance. The function p_α is called α-spaces, and a distance between A and B. It is easy to see that p_α is non decreasing function of α. We shall also define an order of the family W(x), which characterizes accuracy of a given quantity.

Definition 2.4 Let A, B ϵ W(x). An approximate quantity A is more accurate then B, denoted by $A \subset B$, iff $A(x) \le B(x)$, for each x ϵ X.

Definition 2.5 Let X be an arbitrary set and Y be any metric linear space. F is called a fuzzy mapping iff F is mapping from the set X into W(Y), ie , F(x)ϵW(Y) for each x ϵ X

A fuzzy mapping F is a fuzzy subset on X x Y with membership function F(x,y).The function value F(x,y) is grade of membership of y in F(x).
Let A ϵ F(X), B ϵ F(Y) the fuzzy set $F^{-1}(B)$ in F(X), is defined as $F^{-1}(B)(x) = \sup_{y \varepsilon Y} (F(x, y) \cap B(y))$ where $x \in X$

First of all we shall give here the basic properties of α-space and α-distance between some approximate quantities.

Lemma 2.1: Let x ε X,A ε W(X),and {x} be a fuzzy set with membership function equal a characteristic function of set {x}. If {x} is subset of a then $p_\alpha(x,A) = 0$ for each αϵ[0,1].

Lemma 2.2 $p_\alpha(x, A) \le d(x,y) + p_\alpha(y,A)$ for any x,y ϵX .

Lemma 2.3 If $\{x_0\}$ is subset of A, then $p_\alpha(x_0,B) \le D_\alpha(A,B)$ for each B ϵ W(X).

Lemma 2.4[20] : Let (X,d) be a complete metric space , T be a fuzzy mapping from X into W(X) and $x_0 \epsilon X$, then there exists $x_1 \epsilon X$ such that $\{x_1\} \subset T\{x_0\}$

Lemma 2.5[15] Let A, B, ϵ W(X).then for each $\{x\} \subset A$, there exists $\{y\} \subset B$ such that $D(\{x\}, \{y\}) \le D(A, B)$
Let X be a non empty set and I = [0,1].A fuzzy set of X is an element of I^x .For A,B $\in I^x$ we denote $A \subseteq B$ if and only if $A(x) \le B(x)$ for each x ϵX.

Definition (2.6)[3]. An intuitionist fuzzy set (i-fuzzy set) a of X is an object having the form $A = \langle A^1, A^2 \rangle$, where $A^1, A^2 \varepsilon I^x$ and $A^1(x) + A^2(x) \le 1$ for each x ε X. We denote by IFS(X) the family of all i-fuzzy sets of X.

Definition (2.7) [6] Let x_α be a fuzzy point of X. We will say that $\langle x_\alpha, 1 - x_\alpha \rangle$ is an i-fuzzy point of x and it will be denoted by $[x_\alpha]$. In particular $[x] = \langle \{x\}, 1 - \{x\} \rangle$ will be called an i- point of X.

Definition (2.8) [3] Let A, B ε IFS(X). Then $A \subset B$ if and only if $A^1 \subset B^1$ and $B^2 \subset A^2$

Remark 2.1 Notice $[x_\alpha] \subset A$ if and only if $x_0 \subset A^1$
Let (X,d) be a metric space. The α- level set of A is denoted by $A_\alpha = \{x : A(x) \ge \alpha\}$ *if* $\alpha \in]0,1]$ *and* $A_0 = \overline{\{x : A(x) \succ o\}}$, *Where* $\bar{B}$ *denotes the clouser of (nonfuzzy set)* B Heilpern[10] called a fuzzy mapping from the set of X into a family $W(X) \subset I^x$ defined as $A \in W(X)$ if and only if A_α is compact and convex in X for each $\alpha \in (0, 1]$ and

Sup {Ax: x $\in$ X} = 1. In this context we give the following definitions.

Definition (2.9)[10] Let X be a metric space and α ϵ [0,1]. Consider the following family $W_\alpha(X) = \{A \in I^X : A_\alpha \text{ is nonempty, compact and convex}\}$

Now we define the family if i-fuzzy sets of X as follows: $\Phi_\alpha(X) = \{A \in IFS(X) : A^1 \in W_\alpha(X)\}$, it is clear that $\alpha \in I$, $W(X) \subset \Phi_\alpha(X)$.

Definition (2.10)[6] **:** Let x_α be a fuzzy point of X. we will say that x_α is a fixed fuzzy point of the fuzzy mapping F over X if $x_\alpha \subset F(x)$ (i.e. the fixed degree of x is at least α). In particular and according to[2], if $\{x\} \subset F(x)$,we say that x is a fixed point of F.

Main Results

Theorem 3.1: (X, d) be a complete metric space. Let F be continuous fuzzy mapping from X into $W_\alpha(X)$ satisfying the following condition: There exists K ϵ (0,1] such that

$$D_\alpha\big(F(x), F(y)\big) \le K(M(x,y)),\ For\ all\ x, y\ \varepsilon\ X\ with\ x \ne y,\ and$$

$$M(x,y) = \phi\left\{\begin{array}{l} d(x,y),\ p_\alpha(x,Fx),\ p_\alpha(y,Fy),\ p_\alpha(x,Fy),\ \dfrac{d(x,y)+p_\alpha(x,Fy)}{1+d(x,y)\,p_\alpha(x,Fy)}, \\ \dfrac{p_\alpha(x,Fx)+p_\alpha(x,Fy)}{1+p_\alpha(x,Fx)\,p_\alpha(x,Fy)},\ \dfrac{p_\alpha(y,Fy)+p_\alpha(x,Fy)}{1+p_\alpha(y,Fy)\,p_\alpha(x,Fy)} \end{array}\right\}$$

Then there exists x ϵ X such that x_α is a fixed fuzzy point of F iff x_0, x_1 ϵX such that $x_1 \epsilon F(x_0)_\alpha$ with $\sum_{n=1}^{\infty} k^n d(x_0, x_1) \prec \infty$ for $\alpha \epsilon (0,1]$. In particular if α = 1 then x is a fixed point of F.

Proof: If there exists x ϵX such that x_α is fixed fuzzy point of F, i.e. $x_\alpha \subset F(x)$ then $\sum_{n=1}^{\infty} k^n d(x,x) = 0$. Let x_0 ϵ K and suppose that there exists $x_1 \epsilon (F(x_0))_\alpha$ such that $\sum_{n=1}^{\infty} k^n d(x_0, x_1) \prec \infty$. Since $(F(x_1))_\alpha$ is a nonempty compact subset of X, then there exists $x_2 \subset (F(x_1))_\alpha$, such that : $d(x_1, x_2) = p_\alpha(x_1, F(x_1)) \le D_\alpha(F(x_0), F(x_1))$

By induction we construct a sequence $\{x_n\}$ in X such that $x_n \subset (F(x_{n-1}))_\alpha$, and $d(x_n, x_{n+1}) \le D_\alpha(F(x_n), F(x_{n-1}))$. Since K is given to be the non-decreasing, so $d(x_n, x_{n+1}) \le K\{M(x,y)\}$

$$= K\ \phi\left\{\begin{array}{l} d(x_n, x_{n-1}),\ p_\alpha(x_n, F(x_n)),\ p_\alpha(x_{n-1}, F\big(x_{n-1}\big)),\ p_\alpha(x_n, F\big(x_{n-1}\big)), \\ \dfrac{d(x_n, x_{n-1}) + p_\alpha(x_n, F(x_{n-1}))}{1 + d(x_n, x_{n-1})\,p_\alpha(x_n, F(x_{n-1})))},\ \dfrac{p_\alpha(x_n, F(x_n)) + p_\alpha(x_n, F(x_{n-1}))}{1 + p_\alpha(x_n, F(x_n))\,p_\alpha(x_n, F(x_{n-1}))}, \\ \dfrac{p_\alpha(x_{n-1}, F(x_{n-1})) + p_\alpha(x_n, F(x_{n-1}))}{1 + p_\alpha(x_{n-1}, F(x_{n-1}))\,p_\alpha(x_n, F(x_{n-1}))} \end{array}\right\}$$

$$= K\ \phi\left\{\begin{array}{l} d(x_n, x_{n-1}), d(x_n, x_{n+1}), (x_{n-1}, x_n), d(x_n, x_n), \dfrac{d(x_n, x_{n-1}) + d(x_n, (x_n)}{1 + d(x_n, x_{n-1})\,d(x_n, (x_n))}, \\ \dfrac{d(x_n, x_{n+1}) + (x_n, x_n)}{1 + d(x_n, x_{n+1})\,d(x_n, x_n)}, \dfrac{d(x_{n-1}, x_n) + d(x_n, x_n)}{1 + d(x_{n-1}, x_n)\ d(x_n, x_n)} \end{array}\right\}$$

$$= K\ \phi\left\{\begin{array}{l} d(x_n, x_{n-1}), d(x_n, x_{n+1}), (x_{n-1}, x_n), d(x_n, x_n), d(x_n, x_{n-1}), \\ \dfrac{d(x_n, x_{n-1})}{1}, \dfrac{d(x_n, x_{n+1})}{1}, \dfrac{d(x_{n-1}, x_n)}{1} \end{array}\right\}$$

$$= K\{d(x_n, x_{n-1})\}$$

$$= K(p_\alpha d(x_{n-1}, F(x_{n-1}) \le K\big[D_\alpha\big(F(x_{n-1}, F(x_{n-2})\big)\big] \le K\big(Kd(x_{n-1}, x_{n-2})\big) \ldots\ldots\ldots K^n d(x_0, x_1)$$

$$\Rightarrow d(x_n, x_{n+m}) \le d(x_n, x_{n+1}) + \ldots\ldots.. + d(x_{n+m-1}, x_{n+m}) \le K^n\ d(x_0, x_1) + \ldots\ldots.. + K^{n+m-1} d(x_0, x_1)$$

$$= \sum_{k=n}^{k=n+m-1} K^k d(x_0, x_1)$$

Since $\sum_{n=1}^{\infty} k^n d(x_0, x_1) \prec \infty$ it follows that there exists u such that $d(x_n, x_{n+m}) < u$ ϵX. Therefore the sequence $\{x_n\}$ is a Cauchy sequence in X and X is complete therefore $\{x_n\}$ converges to x ϵ X. By the help of lemma 2.1 and 2.2 we have

$$p_\alpha(x, F(x)) \le d(x, x_n) + p_\alpha(x_n, F(x)) \le d(x, x_n) + D_\alpha(F(x_{n-1}), F(x)) \le d(x, x_n) + K\, d(x_{n-1}, x)$$

Consequently, $p_\alpha\left(x, F(x)\right) = 0,$ *and by lemma* 2.1 $x_\alpha \subset F(x)$ Clearly x_α is a fixed fuzzy point of the fuzzy mapping F over X. In particular if α = 1 then x is a fixed point of F. Now we will generalize this theorem for common fixed point.

Theorem 3.2: Let (X, d) be a complete metric space. Let T and S be continuous fuzzy mappings from X into W_α (X) and F: X→ W_α (X) be a mapping such that

$I\ \ .F(X) \subset S(X) \bigcap T(X)$

$II\ \ \{S, F\}$ *and* $\{T, F\}$ *are R − weakly commuting mappings.*

$III\ \ D_\alpha(Fx, Fy) \le K\left[M(x, y)\right]\ \ \forall\, x, y \in X$ *with* $x \ne y$*, where* $M(x, y)$ *is defined as*

$$M(x, y) = K\,\phi\left\{\begin{array}{l} D_\alpha(Sx, Ty), D_\alpha(Sx, Fx), D_\alpha(Ty, Fy), D_\alpha(Fx, Ty), \\ \dfrac{D_\alpha(Sx, Ty) + D_\alpha(Fx, Ty)}{1 + D_\alpha(Sx, Ty) D_\alpha(Fx, Ty)}, \dfrac{D_\alpha(Sx, Fx) + D_\alpha(Fx, Ty)}{1 + D_\alpha(Sx, Fx) D_\alpha(Fx, Ty)} \end{array}\right\}$$

Where K is non decreasing function such that $K : [0, \infty) \to [0, \infty)$,

$K(0) = 0$ and $K(t) \prec t\ \ \forall\, t \in (0, \infty)$, $\alpha \in (0, 1]$ and then $\exists\, x \in X$ such that x_α is common fixed fuzzy point of S, T and F if and only if x_0, x_1

$\in X$ such that $\sum_{n=1}^{\infty} K^n d(x_0, x_1) \prec \infty$.

In perticuler if $\alpha = 1$, then x is common fixed point of S, T and F.

Proof: Let for $x_0 \in X$ there exists x_1 and x_2 such that $x_1 \in (S(x_1))_\alpha \subset (F(x_0))_\alpha$ and $x_2 \in (T(x_2))_\alpha \subset (F(x_1))_\alpha$ .By induction one can construct a sequence $\{ x_n \}$ in X such that

$$x_{2n+1} \in \left(Sx_{2n+1}\right)_\alpha \subset \left(F(x_{2n}\right)_\alpha .\ And\ \ x_{2n+2} \in \left(Tx_{2n+2}\right)_\alpha \subset \left(F(x_{2n+1}\right)_\alpha .$$

Since K is given to be non-decreasing. So $d(x_n, x_{n+1}) \le D_\alpha(F(x_{n-1}), F(x_n)) \le K .M(x_{n-1}, x_n)$

$$= K\,\phi\left\{\begin{array}{l} D_\alpha(Sx_{n-1}, Tx_n), D_\alpha(Sx_{n-1}, Fx_{n-1}), D_\alpha(Tx_n, Fx_n), D_\alpha(Fx_{n-1}, Tx_n), \\ \dfrac{D_\alpha(Sx_{n-1}, Tx_n) + D_\alpha(Fx_{n-1}, Tx_n)}{1 + D_\alpha(Sx_{n-1}, Tx_n) D_\alpha(Fx_{n-1}, Tx_n)}, \dfrac{D_\alpha(Sx_{n-1}, Fx_{n-1}) + D_\alpha(Fx_{n-1}, Tx_n)}{1 + D_\alpha(Sx_{n-1}, Fx_{n-1}) D_\alpha(Fx_{n-1}, Tx_n)} \end{array}\right\}$$

$$= K\,\phi\left\{\begin{array}{l} d(x_{n-1}, x_n), d(x_n, x_n), d(x_n, x_{n+1}), d(x_n, x_n), \\ \dfrac{d(x_{n-1}, x_n) + (x_n, x_n)}{1 + d(x_{n-1}, x_n) d(x_n, x_n)}, \dfrac{d(x_{n-1}, x_n) + d(x_n, x_n)}{1 + (x_{n-1}, x_n)(x_n, x_n)} \end{array}\right\}$$

$$= K\left\{d(x_n, x_{n-1})\right\}$$

Therefore $d(x_n, x_{n+1}) \le Kd(x_{n-1}, x_n) = KD_\alpha\left(F(x_{n-2}), F(x_{n-1})\right) \le K^2 d(x_{n-1}, x_{n-2}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \prec K^n\, d(x_0, x_1)$

$$\Rightarrow d(x_n, x_{n+m}) \le d(x_n, x_{n+1}) + \ldots\ldots.. + d(x_{n+m-1}, x_{n+m}) \le K^n\, d(x_0, x_1) + \ldots\ldots\ldots + K^{n+m-1} d(x_0, x_1)\ = \sum_{j=n}^{k=n+m-1} K^j\, d(x_0, x_1)$$

Since $\sum_{n=1}^{\infty} k^n d(x_0, x_1) \prec \infty$ it follows that there exists u such that $d(x_n, x_{n+m}) < u \in X$. Therefore the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to $x \in X$ and $\left(Sx_{2n+1}\right)_\alpha$, $\left(Tx_{2n+2}\right)$ also converges on X.

Since $\{S,F\}$ and $\{T,F\}$ are R – weakly commuting mappings. So $p_\alpha(x, F(x)) \le d(x, x_n) + p_\alpha(x_n, F(x)) \le d(x, x_n) + D_\alpha(F(x_{n-1}), F(x)) \le d(x, x_n) + Kd(x_{n-1}, x)$ Consequently, $p_\alpha(x, F(x)) = 0$, *and by lemma* 2.1 $x_\alpha \subset F(x)$

Conclusion

Clearly x_α is a common fixed fuzzy point of the fuzzy mapping F, S and T over X. In particular if $\alpha = 1$ then x is a common fixed point of F, S and T.

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