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A family of modified King's methods with seventh order convergence

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Abstract

Based on King's fourth order methods, we derive a family of seventh order methods for the solution of non linear equations. In terms of computational cost the family requires three evaluations of the function and one evaluation of first derivative. The proposed family increases the order of King's fourth order methods and efficiency index from 1.587 to 1.682. The methods are compared with closest competitors in a series of numerical examples and theoretical order of convergence is verified.

Keywords: Nonlinear equations, newton's method, king's methods, order of convergence, efficiency.

Introduction

Finding the root of nonlinear equation is a classical problem in scientific computation[1,2]. In this paper, we consider iterative methods to find a simple root *r* of the nonlinear equation $f(x)=0$, where $f: R \to R$ be the continuously differentiable real function. Newton's method[3] is an important and basic method for finding simple roots and is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0,1,2,3,.... \tag{1}$$

A number of ways are considered by many researchers[4–10] to improve the local order of convergence of Newton's method at the expanse of additional evaluations of functions and derivatives. In particular, King[4] developed a one-parameter family of fourth order methods defined by

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} &= w_i - \frac{f(x_i)+\beta f(w_i)}{f(x_i)+(\beta-2)f(w_i)} \frac{f(w_i)}{f'(x_i)}. \end{aligned} \tag{2}$$

where $\beta \in R$ is a constant. This family requires two evaluations of the function f, and one evaluation of first derivative f' per iteration. This method is seen to be efficient than Newton's method since it adds only one evaluation of the function but its order of convergence increases from two to four. The famous Ostrowski's method[5, 6] is a member of this family for the case $\beta = 0$.

Recently, based on King's or Ostrowski's methods some higher order methods have been proposed and analyzed for solving non linear equations. Grau and Díaz-Barrero[6], Sharma and Guha[7] and Chun and Ham[8] have developed sixth order methods each requires three f and one f' evaluations per iteration. Kou et al.[9] presented a family of Ostrowski's method with seventh order convergence which is given by

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ H_2(x_i, w_i) &= [f(x_i) - 2f(w_i)]^{-1} f(w_i), \\ z_i &= w_i - H_2(x_i, w_i)(x_i - w_i), \\ H_\alpha(w_i, z_i) &= [f(w_i) - \alpha f(z_i)]^{-1} f(z_i), \\ x_{i+1} &= z_i - [(1 + H_2(x_i, w_i))^2 + H_\alpha(w_i, z_i)] \frac{f(z_i)}{f'(x_i)}. \end{aligned} \tag{3}$$

where $\alpha \in R$ is a constant. Bi et al.[10] developed a family of King's methods with seventh order convergence defined by

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)},$$

$$z_i = w_i - \frac{f(x_i) + \beta f(w_i)}{f(x_i) + (\beta - 2) f(w_i)} \frac{f(w_i)}{f'(x_i)},$$

$$x_{i+1} = z_i - \frac{f(z_i)}{f[z_i, w_i] + f[z_i, x_i, x_i](z_i - w_i)}. \quad (4)$$

Through this work, based on King's family of fourth order methods (2), we introduce a new family of seventh order methods without using the derivatives other than first. The convergence analysis is provided to establish its seventh order convergence. In terms of computational cost, it requires the evaluations of three functions and one first derivative per iteration. Thus the new family adds only one evaluation of the function at another point than King's family and order increases from four to seven. The efficacy of the methods is tested on a number of numerical methods by comparing with some well-known methods.

The contents of paper are summarized as follows. In Section 2, we obtain new methods and convergence analysis is carried out to establish seventh order convergence. In Section 3, the method is tested and compared with other well-known methods on a number of problems. Concluding remarks are given in Section 5.

The method and its convergence

Our aim is to develop a scheme that improves the order of convergence of King's methods (2). Thus we consider the following iteration scheme

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)},$$

$$z_i = w_i - \frac{f(x_i) + \beta f(w_i)}{f(x_i) + (\beta - 2) f(w_i)} \frac{f(w_i)}{f'(x_i)},$$

$$x_{i+1} = z_i - \frac{f[x_i, w_i]}{f[x_i, z_i]\, f[w_i, z_i]} f(z_i). \quad (8)$$

where $f[x_i, w_i] = \frac{f(w_i) - f(x_i)}{w_i - x_i}$, $f[x_i, z_i] = \frac{f(z_i) - f(x_i)}{z_i - x_i}$ and $f[w_i, z_i] = \frac{f(z_i) - f(w_i)}{z_i - w_i}$ are Newton divided differences. In fact, the above scheme defines a family of methods depending on the parameter $\beta \in R.$ in the following theorem; we shall prove that the family (8) indeed improves the order of convergence of King's method from four to seven.

Theorem 1. *Let* $f : R \to R$ *has continuous derivatives up to third order in* $R.$ *If* $f(x)$ *has a simple root* r *in* R *and* x_0 *is sufficiently close to* $r,$ *then the error in the method given by* (8) *satisfies the equation*

$$e_{i+1} = A_2^2 (A_2^2 - A_3)((1 + 2\beta) A_2^2 - A_3) e_i^7 + O(e_i^8). \quad (9)$$

where $e_i = x_i - r$ *and* $A_k = \frac{f^{(k)}(r)}{k!\, f'(r)}$ *for* $k \in N,$ *N is the set of natural numbers.*

Proof. Let $e_i = x_i - r$ be the error in the iterate $x_i.$ Using Taylor's series expansion, we get

$$f(x_i) = f'(r)[e_i + A_2 e_i^2 + A_3 e_i^3 + A_4 e_i^4 + A_5 e_i^5 + O(e_i^6)],$$

and

$$f'(x_i) = f'(r)[1 + 2A_2 e_i + 3A_3 e_i^2 + 4A_4 e_i^3 + 5A_5 e_i^4 + O(e_i^5)].$$

where $A_k = \frac{f^{(k)}(r)}{k!\, f'(r)}$ for $k \in N.$

Now,

$$\frac{f(x_i)}{f'(x_i)}=e_i-A_2e_i^2+2(A_2^2-A_3)e_i^3+(-4A_2^3+7A_2A_3-3A_4)e_i^4+(8A_2^4-20A_2^2A_3+6A_3^2+10A_2A_4-4A_5)e_i^5+O(e_i^6).$$

For,

$$w_i=x_i-\frac{f(x_i)}{f'(x_i)}=r+A_2e_i^2-2(A_2^2-A_3)e_i^3+(4A_2^3-7A_2A_3+3A_4)e_i^4$$
$$-(8A_2^4-20A_2^2A_3+6A_3^2+10A_2A_4-4A_5)e_i^5+O(e_i^6).$$

Expanding $f(w_i)$, we obtain

$$f(w_i)=f'(r)[A_2e_i^2-2(A_2^2-A_3)e_i^3+(5A_2^3-7A_2A_3+3A_4)e_i^4$$
$$+(-12A_2^4+24A_2^2A_3-6A_3^2-10A_2A_4+4A_5)e_i^5+O(e_i^6)].$$

Therefore,

$$\frac{f(x_i)+\beta f(w_i)}{f(x_i)+(\beta-2)f(w_i)}=1+2A_2e_i-2((1+\beta)A_2^2-2A_3)e_i^2+2((2\beta+\beta^2)A_2^3$$
$$-(2+4\beta)A_2A_3+3A_4)e_i^3-2((-2+\beta+3\beta^2+\beta^3)A_2^4$$
$$+(3-8\beta-6\beta^2)A_2^2A_3+4\beta A_3^2+(2+6\beta)A_2A_4-6A_5)e_i^4+O(e_i^5).$$

Also,

$$\frac{f(w_i)}{f'(x_i)}=A_2e_i^2-2(2A_2^2-A_3)e_i^3+(13A_2^3-14A_2A_3+3A_4)e_i^4$$
$$-2(19A_2^4-32A_2^2A_3+6A_3^2+10A_2A_4-2A_5)e_i^5+O(e_i^6),$$

For,

$$z_i=w_i-\frac{f(x_i)+\beta f(w_i)}{f(x_i)+(\beta-2)f(w_i)}\frac{f(w_i)}{f'(x_i)}$$
$$=r+((1+2\beta)A_2^3-A_2A_3)e_i^4-2((2+6\beta+\beta^2)A_2^4-(4+6\beta)A_2^2A_3+A_3^2+A_2A_4)e_i^5+O(e_i^6).$$

Expanding $f(z_i)$, we get

$$f(z_i)=f'(r)[((1+2\beta)A_2^3-A_2A_3)e_i^4-2((2+6\beta+\beta^2)A_2^4$$
$$-(4+6\beta)A_2^2A_3+A_3^2+A_2A_4)e_i^5+O(e_i^6)].$$

Furthermore,

$$f[x_i,w_i]=\frac{f(w_i)-f(x_i)}{w_i-x_i}=f'(r)[1+A_2e_i+(A_2^2+A_3)e_i^2+(-2A_2^3+3A_2A_3+A_4)e_i^3$$
$$(4A_2^4-8A_2^2A_3+2A_3^2+4A_2A_4+A_5)e_i^4+O(e_i^5)],$$

$$f[x_i, z_i] = \frac{f(z_i) - f(x_i)}{z_i - x_i} = f'(r)[1 + A_2 e_i + A_3 e_i^2 + A_4 e_i^3 + ((1+2\beta)A_2^4 - A_2^2 A_3 + A_5)e_i^4 + O(e_i^5)],$$

and $$f[w_i, z_i] = \frac{f(z_i) - f(w_i)}{z_i - w_i} = f'(r)[1 + A_2^2 e_i^2 - 2A_2(A_2^2 - A_3)e_i^3 + O(e_i^4)].$$

Using the above results, we obtain $\frac{f[x_i, w_i]}{f[x_i, z_i]\, f[w_i, z_i]} = \frac{1}{f'(r)}[1 + A_2(-A_2^2 + A_3)e_i^3 + O(e_i^4)].$

Using this result in $x_{i+1} = z_i - \frac{f[x_i, w_i]}{f[x_i, z_i]\, f[w_i, z_i]} f(z_i),$
we get $e_{i+1} = A_2^2(A_2^2 - A_3)((1+2\beta)A_2^2 - A_3)e_i^7 + O(e_i^8).$

Thus, the seventh order convergence is established.

Remark 1. The proposed method requires three evaluations of the function and one evaluation of first derivative. The efficiency index (E) defined by (7) of the present methods (8) is $E = \sqrt[4]{7} \approx 1.627$ which is better than $E = \sqrt{2} \approx 1.414$ of Newton's method (1), $E = \sqrt[3]{4} \approx 1.587$ of King's methods (2) and $E = \sqrt[4]{6} \approx 1.565$ of sixth order methods[6-8].

Remark 2. The modified third-step of the present methods is simple and interesting, since the weight function, that is the function multiplied by $f(z)$, is completely expressed in terms of first order Newton divided differences.

Remark 3. By theorem 1 of [9], the family of methods (3) satisfies the error equation
$e_{i+1} = -2(A_2^2 - A_3)(A_3^2 - 2A_2^2 A_3 + A_2 A_4)e_i^7 + O(e_i^8).$

Also, by theorem 1 of [10], the family of methods (4) satisfies the error equation
$e_{i+1} = 2A_2^2 A_3[A_3 - (1+2\beta)A_2^2]e_i^7 + O(e_i^8).$

We can see from the iteration schemes and the error equations that the iterative methods (3), (4) and (8) are three different and independent methods.

Numerical examples

We employ the present methods (8) designated as M7 to solve some nonlinear equations and compare it with Newton's method (1), King's methods (2), the method developed by Grau et al. [6], the method developed by Kou et al. (3) and the method by Bi et al. (4) which are designated as M2, M4, M6, MK7 and MB7, respectively.

The test functions and root r correct up to 16 decimal places are displayed in table 1. The functions we have selected are same as in[9,10]. In table 2, we exhibit the absolute values of the difference of root *r* and its approximation x_i, where *r* is computed with 350 significant digits and x_i is calculated by costing the same total number of function evaluations (TFE) for each method. The TFE is counted as sum of the number of evaluations of the function itself plus the number of evaluations of the derivatives. In the calculations, 12 TFE are used by each method. That means 6 iterations are used for M2, 4 iterations for M4 and 3 iterations for the rest. The absolute values of the function $|f(x_i)|$ are also displayed in table 2. It can be observed that the computed results, displayed in table 2, overwhelmingly support the theory of efficiency analysis discussed in the previous section.

Table-1
Test functions

$f(x)$	r
$f_1(x)=x^3+4x^2-15$	1.6319808055660635
$f_2(x)=xe^{x^2}-\sin^2 x+3\cos x+5$	–1.2076478271309189
$f_3(x)=\sin(x)-x/2$	1.8954942670339809
$f_4(x)=10xe^{-x^2}-1$	1.6796306104284499
$f_5(x)=\cos(x)-x$	0.7390851332151606
$f_6(x)=\sin^2(x)-x^2+1$	1.4044916482153412
$f_7(x)=e^{-x}+\cos(x)$	1.7461395304080124

Table-2
Comparison of methods using same total number of function evaluations for all methods (TFE = 12)

	M2	**M4** ($\beta=0$)	**M6**	**MK7** ($\alpha=0$)	**MB7** ($\beta=0$)	**M7** ($\beta=0$)
f_1, $x_0=2$ $\lvert x_i-r\rvert$ $\lvert f(x_i)\rvert$	 3.91e – 55 8.23e-54	 4.87e – 230 1.03e – 228	 2.12e – 180 4.46e – 179	 5.03e – 276 1.06e – 274	 4.18e – 320 8.79e – 319	 9.52e – 306 2.00e – 304
f_2, $x_0=-1$ $\lvert x_i-r\rvert$ $\lvert f(x_i)\rvert$	 8.63e – 33 1.75e – 31	 4.34e – 224 8.82e – 223	 1.25e – 156 2.54e – 155	 5.92e – 266 1.20e – 264	 2.23e – 226 4.52e – 225	 4.74e – 301 9.62e – 300
f_3, $x_0=2$ $\lvert x_i-r\rvert$ $\lvert f(x_i)\rvert$	 1.89e – 80 1.54e – 80	 6.25e – 313 5.12e – 313	 1.03e – 251 8.44e – 252	 0.00e + 00 0.00e + 00	 0.00e + 00 0.00e + 00	 0.00e + 00 0.00e + 00
f_4, $x_0=1.8$ $\lvert x_i-r\rvert$ $\lvert f(x_i)\rvert$	 4.41e – 58 1.22e – 57	 4.20e – 237 1.16e – 236	 3.39e – 187 9.37e – 187	 4.84e – 282 1.34e – 281	 1.73e – 337 4.77e – 337	 1.78e – 319 4.92e – 319
f_5, $x_0=1$ $\lvert x_i-r\rvert$ $\lvert f(x_i)\rvert$	 1.80e – 83 3.00e – 83	 4.21e – 296 7.05e – 296	 2.46e – 237 4.12e – 237	 0.00e + 00 0.00e + 00	 0.00e + 00 0.00e + 00	 0.00e + 00 0.00e + 00
f_6, $x_0=1.6$ $\lvert x_i-r\rvert$ $\lvert f(x_i)\rvert$	 2.00e – 56 4.98e – 56	 1.31e – 226 3.26e – 226	 3.04e – 178 7.54e – 178	 2.52e – 271 6.26e – 271	 0.00e + 00 0.00e + 00	 1.95e – 301 4.84e – 301
f_7, $x_0=2$ $\lvert x_i-r\rvert$ $\lvert f(x_i)\rvert$	 7.97e – 85 9.24e – 85	 9.03e – 280 1.05e – 279	 1.37e – 223 1.58e – 223	 1.11e – 338 1.29e – 338	 0.00e + 00 0.00e + 00	 0.00e + 00 0.00e + 00

Conclusion

In this work, we have obtained a simple and elegant family of three-step methods of order seven by using an additional evaluation of function at the point iterated by King's methods of order four for solving nonlinear equations. The theoretical results have been checked with some numerical examples. The superiority of present methods is also corroborated by numerical results displayed in the table 2. Finally, we conclude that the methods presented in this paper are preferable to other recognized efficient equation solvers, namely Newton, King, Ostrowski etc.

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