Research Journal of Mathematical and Statistical Sciences
Vol. **1(1),** 23-32, February **(2013)** *Res. J. Mathematical & Statistical Sci.*

Asymptotic Behavior of Eigenvalues and Fundamental Solutions of One Discontinuous Fourth-Order Boundary Value Problem

Erdoğan Şen[1], Serkan Araci[2] and Mehmet Acikgoz[2]
[1]Department of Mathematics, Faculty of Science and Letters, Namik Kemal University, 59030 Tekirdağ, TURKEY
[2]University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY

Available online at: www.isca.in
Received 24th August 2012, revised 30th August 2012, accepted 29th November 2012

Abstract

In this work, we study a fourth-order boundary value problem problem with eigenparameter dependent boundary conditions and transmission conditions at a interior point. A self-adjoint linear operator A is defined in a suitable Hilbert space H such that the eigenvalues of such a problem coincide with those of A. We obtain asymptotic formulae for its eigenvalues and fundamental solutions.

2010 Mathematics Subject Classification. 34L20, 35R10.

Keywords: Asymptotics of eigenvalues and eigenfunctions, transmission conditions, fourth-order differential operator, spectrum.

Introduction

It is well-known that many topics in mathematical physics require the investigation of eigenvalues and eigenfunctions of Sturm-Liouville type boundary value problems. In recent years, more and more researches are interested in the discontinuous Sturm-Liouville problem[1-7]. Various physics applications of this kind problem are found in many literatures, including some boundary value problem with transmission conditions that arise in the theory of heat and mass transfer[8,9] . The literature on such results is voluminous[1-11] .

Fourth-order discontinuous boundary value problems with eigen-dependent boundary conditions and with two supplementary transmission conditions at the point of discontinuity have been investigated[12-13]. Note that discontinuous Sturm-Liouville problems with eigen-dependent boundary conditions and with four supplementary transmission conditions at the points of discontinuity have been investigated[4].

In this study, we shall consider a fourth-order differential equation

$$Lu := \left(a(x)u''(x)\right)'' + q(x)u(x) = \lambda u(x) \qquad (1.1)$$

on $I = [-1,0) \cup (0,1]$, with boundary conditions at $x = -1$

$$L_1 u := \alpha_1 u(-1) + \alpha_2 u'''(-1) = 0, \qquad (1.2)$$

$$L_2 u := u''(-1) = 0, \qquad (1.3)$$

with the four transmission conditions at the points of discontinuity $x = 0$,

$$L_3 u := u(0+) - u(0-) = 0, \qquad (1.4)$$

$$L_4 u := u'(0+) - u'(0-) = 0, \qquad (1.5)$$

$$L_5 u := u''(0+) - u''(0-) + \lambda\delta_1 u'(0-) = 0, \qquad (1.6)$$

$$L_6 u := u'''(0+) - u'''(0-) + \lambda\delta_2 u(0-) = 0, \qquad (1.7)$$

and the eigen-dependent boundary conditions at $x = 1$

$$L_7 u := \lambda u(1) + u'''(1) = 0, \qquad (1.8)$$

$$L_8 u := \lambda u'(1) + u''(1) = 0, \qquad (1.9)$$

where $a(x)=a_1^4$, for $x\in[-1,0)$, $a(x)=a_2^4$, for $x\in(0,1]$, $a_1>0$ and $a_2>0$ are given real numbers, $q(x)$ is a given real-valued function continuous in $[-1,0)\cup(0,1]$ and has a finite limit $q(0\pm)=\lim_{x\to\pm0}q(x)$; λ is a complex eigenvalue parameter; α_i,δ_i $(i=1,2)$ are real numbers and $|\alpha_1|+|\alpha_2|\neq 0,\quad |\delta_1|+|\delta_2|\neq 0.$

Preliminaries

Firstly we define the inner product in L^2 for every f , $g\in L^2(I)$ as, $\langle f,g\rangle_1=\dfrac{1}{a_1^4}\int_{-1}^{0}f_1\overline{g_1}dx+\dfrac{1}{a_2^4}\int_0^1 f_2\overline{g_2}dx,$

where $f_1(x)=f(x)\big|_{[-1,0)},\quad f_2(x)=f(x)\big|_{(0,1]}$. It is easy to see that $\left(L^2(I),[\cdot,\cdot]\right)$ is a Hilbert space. Now we define the inner product in the direct sum of spaces $L^2(I)\oplus C\oplus C\oplus C_{\delta_1}\oplus C_{\delta_2}$ by,

$$[F,G]:=\langle f,g\rangle_1+\langle h_1,k_1\rangle+\langle h_2,k_2\rangle+\langle h_3,k_3\rangle+\langle h_4,k_4\rangle$$

for, $F:=(f,h_1,h_2,h_3,h_4),G:=(g,k_1,k_2,k_3,k_4)\in L^2(I)\oplus C\oplus C\oplus C_{\delta_1}\oplus C_{\delta_2}$.

Then $Z:=\left(L^2(I)\oplus C\oplus C\oplus C_{\delta_1}\oplus C_{\delta_2},[\cdot,\cdot]\right)$ is the direct sum of modified Krein spaces. A fundamental symmetry on the Krein space is given by

$$J:=\begin{bmatrix} J_0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \operatorname{sgn}\delta_1 & 0\\ 0 & 0 & 0 & 0 & \operatorname{sgn}\delta_2 \end{bmatrix},$$

where, $J_0\ :\ L^2(I)\to L^2(I)$

is defined by $(J_0f)(x)=f(x)$. We define a linear operator A in Z by the domain of definition

$$D(A):=\{(f,h_1,h_2,h_3,h_4)\in Z\mid f_1^{(i)}\in AC_{loc}((-1,0)),\ f_2^{(i)}\in AC_{loc}((0,1)),\ i=\overline{0,3},$$
$$Lf\in L^2(I),\ L_kf=0,\ k=\overline{1,4},\ h_1=f(1),\ h_2=f'(1),\ h_3=-\delta_1f'(0),\ h_4=-\delta_2f(0)\}$$
$$AF=(Lf,\ -f'''(1),\ -f''(1),\ f''(0+)-f''(0-),\ f'''(0+)-f'''(0-)),$$
$$F=(f,\ f(1),\ f'(1),-\delta_1f'(0),\ -\delta_2f(0))\in D(A).$$

Consequently, the considered problem (1.1)-(1.9) can be rewritten in operator form as

$$AF=\lambda F,$$

i.e., the problem (1.1)-(1.9) can be considered as the eigenvalue problem for the operator A. Then, we can write the following conclusions:

Theorem 2.1. *The eigenvalues and eigenfunctions of the problem (1.1)-(1.9) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator* A *respectively.*

Theorem 2.2. *The operator* A *is self-adjoint in Krein space* Z.

Fundamental Solutions

Lemma 3.1. *Let the real-valued function* $q(x)$ *be continuous in* $[-1,1]$ *and* $f_i(\lambda)$ $(i=\overline{1,4})$ *are given entire functions. Then for any* $\lambda\in C$ *the equation,* $(a(x)u''(x))''+q(x)u(x)=\lambda u(x),\quad x\in I$
has a unique solution $u=u(x,\lambda)$ *such that*

$$u(-1)= f_1(\lambda),\ u'(-1)= f_2(\lambda),\ u''(-1)= f_3(\lambda),\ u'''(-1)= f_4(\lambda)$$
$$\left(\text{or } u(1)= f_1(\lambda),\ u'(1)= f_2(\lambda),\ u''(1)= f_3(\lambda),\ u'''(1)= f_4(\lambda)\right).$$

and for each $x\in[-1,1]$, $u(x,\lambda)$ *is an entire function of* λ.

Let $\phi_{11}(x,\lambda)$ be the solution of Eq. (1.1) on $[-1,0)$ which satisfies the initial conditions

$$\phi_{11}(-1)=\alpha_2,\ \phi_{11}'(-1)=\phi_{11}''(-1)=0,\ \phi_{11}'''(-1)=-\alpha_1.$$

By virtue of Lemma 3.1, after defining this solution, we may define the solution $\phi_{12}(x,\lambda)$ of Eq. (1.1) on $(0,1]$ by means of the solution $\phi_{11}(x,\lambda)$ by the initial conditions

$$\phi_{12}(0)=\phi_{11}(0),\ \phi_{12}'(0)=\phi_{11}'(0),\ \phi_{12}''(0)=\phi_{11}''(0)-\lambda\delta_1\phi_{11}'(0),\ \phi_{12}'''(0)=\phi_{11}'''(0)-\lambda\delta_2\phi_{11}(0). \quad (3.1)$$

After defining this solution, we may define the solution $\phi_{21}(x,\lambda)$ of equation (1.1) on $[-1,0)$ which satisfies the initial conditions

$$\phi_{21}(-1)=0,\ \phi_{21}'(-1)=\beta_2,\ \phi_{21}''(-1)=-\beta_1,\ \phi_{21}'''(-1)=0. \quad (3.2)$$

After defining this solution, we may define the solution $\phi_{22}(x,\lambda)$ of Eq. (1.1) on $(0,1]$ by means of the solution $\phi_{21}(x,\lambda)$ by the initial conditions

$$\phi_{22}(0)=\phi_{21}(0),\ \phi_{22}'(0)=\phi_{21}'(0),\ \phi_{22}''(0)=\phi_{21}''(0)-\lambda\delta_1\phi_{21}'(0),\phi_{22}'''(0)=\phi_{21}'''(0)-\lambda\delta_2\phi_{21}(0). \quad (3.3)$$

Analogically we shall define the solutions $\chi_{11}(x,\lambda)$ and $\chi_{12}(x,\lambda)$ by the initial conditions

$$\begin{aligned}&\chi_{12}(1)=-1,\ \chi_{12}'(1)=\chi_{12}''(1)=0,\chi_{12}'''(1)=\lambda,\ \chi_{11}(0)=\chi_{12}(0),\ \chi_{11}'(0)=\chi_{12}'(0),\\ &\chi_{11}''(0)=\chi_{12}''(0)+\lambda\delta_1\chi_{12}'(0),\ \chi_{11}'''(0)=\chi_{12}'''(0)+\lambda\delta_2\chi_{12}(0).\end{aligned} \quad (3.4)$$

Moreover, we shall define the solutions $\chi_{21}(x,\lambda)$ and $\chi_{22}(x,\lambda)$ by the initial conditions

$$\begin{aligned}&\chi_{22}(1)=0,\ \chi_{22}'(1)=-1,\ \chi_{22}''(1)=\lambda,\ \chi_{22}'''(1)=0,\ \chi_{21}(0)=\chi_{22}(0),\chi_{21}'(0)=\chi_{22}'(0),\\ &\chi_{21}''(0)=\chi_{22}''(0)+\lambda\delta_1\chi_{22}'(0),\ \chi_{21}'''(0)=\chi_{22}'''(0)+\lambda\delta_2\chi_{22}(0).\end{aligned} \quad (3.5)$$

Let us consider the Wronskians

$$W_1(\lambda):=\begin{vmatrix}\phi_{11}(x,\lambda) & \phi_{21}(x,\lambda) & \chi_{11}(x,\lambda) & \chi_{21}(x,\lambda)\\ \phi_{11}'(x,\lambda) & \phi_{21}'(x,\lambda) & \chi_{11}'(x,\lambda) & \chi_{21}'(x,\lambda)\\ \phi_{11}''(x,\lambda) & \phi_{21}''(x,\lambda) & \chi_{11}''(x,\lambda) & \chi_{21}''(x,\lambda)\\ \phi_{11}'''(x,\lambda) & \phi_{21}'''(x,\lambda) & \chi_{11}'''(x,\lambda) & \chi_{21}'''(x,\lambda)\end{vmatrix}$$

and

$$W_2(\lambda):=\begin{vmatrix}\phi_{12}(x,\lambda) & \phi_{22}(x,\lambda) & \chi_{12}(x,\lambda) & \chi_{22}(x,\lambda)\\ \phi_{12}'(x,\lambda) & \phi_{22}'(x,\lambda) & \chi_{12}'(x,\lambda) & \chi_{22}'(x,\lambda)\\ \phi_{12}''(x,\lambda) & \phi_{22}''(x,\lambda) & \chi_{12}''(x,\lambda) & \chi_{22}''(x,\lambda)\\ \phi_{12}'''(x,\lambda) & \phi_{22}'''(x,\lambda) & \chi_{12}'''(x,\lambda) & \chi_{22}'''(x,\lambda)\end{vmatrix},$$

which are independent of x and entire functions. This sort of calculation gives $W_1(\lambda)=W_2(\lambda)$. Now we may introduce in consideration the characteristic function $W(\lambda)$ as $W(\lambda)=W_1(\lambda)$.

Theorem 3.2. *The eigenvalues of the problem (1.1)-(1.9) are the zeros of the function* $W(\lambda)$.

Proof. Let $W(\lambda)=0$. Then the functions $\phi_{11}(x,\lambda)$, $\phi_{21}(x,\lambda)$ and $\chi_{11}(x,\lambda)$, $\chi_{21}(x,\lambda)$ are linearly dependent, i.e.,

$$k_1\phi_{11}(x,\lambda)+k_2\phi_{21}(x,\lambda)+k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda)=0$$

for some $k_1 \neq 0$ or $k_2 \neq 0$ or $k_3 \neq 0$ or $k_4 \neq 0$. From this, it follows that $k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda)$ satisfies the boundary conditions (1.2)-(1.3). Therefore

$$\begin{cases} k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda), & x\in[-1,0) \\ k_3\chi_{12}(x,\lambda)+k_4\chi_{22}(x,\lambda), & x\in(0,1] \end{cases}$$

is an eigenfunction of the problem (1.1)-(1.9) corresponding to eigenvalue λ.

Now we let $u(x)$ be any eigenfunction corresponding to eigenvalue λ, but $W(\lambda)\neq 0$. Then the functions ϕ_{11}, ϕ_{21}, χ_{11}, χ_{21} would be linearly independent on $(0,1]$. Therefore $u(x)$ may be represented as

$$u(x)=\begin{cases} c_1\phi_{11}(x,\lambda)+c_2\phi_{21}(x,\lambda)+c_3\chi_{11}(x,\lambda)+c_4\chi_{21}(x,\lambda), & x\in[-1,0) \\ c_5\phi_{12}(x,\lambda)+c_6\phi_{22}(x,\lambda)+c_7\chi_{12}(x,\lambda)+c_8\chi_{22}(x,\lambda), & x\in(0,1], \end{cases}$$

where at least one of the constants c_1, c_2, c_3, c_4, c_5, c_6, c_7 and c_8 is not zero. Considering the equations

$$L_v(u(x))=0, \quad v=\overline{1,8} \tag{3.6}$$

as a system of linear equations of the variables c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 and taking (3.1)-(3.5) into account, it follows that the determinant of this system is

$$\begin{vmatrix} 0 & 0 & L_1\chi_{11} & L_1\chi_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & L_2\chi_{11} & L_2\chi_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_3\phi_{12} & L_3\phi_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_4\phi_{12} & L_4\phi_{22} & 0 & 0 \\ -\phi_{12}(0) & -\phi_{22}(0) & -\chi_{12}(0) & -\chi_{22}(0) & \phi_{12}(0) & \phi_{22}(0) & \chi_{12}(0) & \chi_{22}(0) \\ -\phi'_{12}(0) & -\phi'_{22}(0) & -\chi'_{12}(0) & -\chi'_{22}(0) & \phi'_{12}(0) & \phi'_{22}(0) & \chi'_{12}(0) & \chi'_{22}(0) \\ -\phi''_{12}(0) & -\phi''_{22}(0) & -\chi''_{12}(0) & -\chi''_{22}(0) & \phi''_{12}(0) & \phi''_{22}(0) & \chi''_{12}(0) & \chi''_{22}(0) \\ -\phi'''_{12}(0) & -\phi'''_{22}(0) & -\chi'''_{12}(0) & -\chi'''_{22}(0) & \phi'''_{12}(0) & \phi'''_{22}(0) & \chi'''_{12}(0) & \chi'''_{22}(0) \end{vmatrix}$$

$$= -W(\lambda)^3 \neq 0.$$

Therefore, the system (3.6) has only the trivial solution $c_i=0$ $\left(i=\overline{1,8}\right)$. Thus we get a contradiction, which completes the proof.

Asymptotic formulae for eigenvalues and fundamental solutions

We start by proving some lemmas.

Lemma 4.1. *Let* $\phi(x,\lambda)$ *be the solution of Eq. (1.1) defined in Section 3, and let* $\lambda=s^4$, $s=\sigma+it$. *Then the following integral equations hold for* $k=\overline{0,3}$:

$$\frac{d^k}{dx^k}\phi_{11}(x,\lambda)=\frac{\alpha_2}{2}\frac{d^k}{dx^k}\cos\frac{s(x+1)}{a_1}+\frac{\alpha_1 a_1^3}{2s^3}\frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1}+\left(\frac{\alpha_2}{4}-\frac{\alpha_1 a_1^3}{4s^3}\right)\frac{d^k}{dx^k}e^{\frac{s(x+1)}{a_1}} \tag{4.1}$$
$$+\left(\frac{\alpha_2}{4}+\frac{\alpha_1 a_1^3}{4s^3}\right)\frac{d^k}{dx^k}e^{-\frac{s(x+1)}{a_1}}+\frac{a_1^3}{2s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin\frac{s(x-y)}{a_1}-e^{\frac{s(x-y)}{a_1}}+e^{-\frac{s(x-y)}{a_1}}\right)q(y)\phi_{11}(y,\lambda)dy.$$

$$\frac{d^k}{dx^k}\phi_{12}(x,\lambda)=\left(\frac{\phi_{12}(0)}{2}-\frac{a_2^2\phi''_{12}(0)}{2s^2}\right)\frac{d^k}{dx^k}\cos\frac{sx}{a_2}+\left(\frac{a_2\phi'_{12}(0)}{2s}-\frac{a_2^3\phi'''_{12}(0)}{2s^3}\right)\times\frac{d^k}{dx^k}\sin\frac{sx}{a_2}$$
$$+\left(\frac{\phi_{12}(0)}{4}+\frac{a_2\phi'_{12}(0)}{4s}+\frac{a_2^2\phi''_{12}(0)}{4s^2}+\frac{a_2^3\phi'''_{12}(0)}{4s^3}\right)\times\frac{d^k}{dx^k}e^{\frac{sx}{a_2}}+\left(\frac{\phi_{12}(0)}{4}-\frac{a_2\phi'_{12}(0)}{4s}+\frac{a_2^2\phi''_{12}(0)}{4s^2}-\frac{a_2^3\phi'''_{12}(0)}{4s^3}\right)\frac{d^k}{dx^k}e^{-\frac{sx}{a_2}} \tag{4.2}$$
$$+\frac{a_2^3}{2s^3}\int_0^x\frac{d^k}{dx^k}\left(\sin\frac{s(x-y)}{a_2}-e^{\frac{s(x-y)}{a_2}}+e^{-\frac{s(x-y)}{a_2}}\right)q(y)\phi_{12}(y,\lambda)dy.$$

$$\frac{d^k}{dx^k}\phi_{21}(x,\lambda)=\frac{a_1}{2s}\frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1}+\frac{a_1}{4s}\frac{d^k}{dx^k}e^{\frac{s(x+1)}{a_1}}-\frac{a_1}{4s}\frac{d^k}{dx^k}e^{-\frac{s(x+1)}{a_1}}$$
$$+\frac{a_1^3}{2s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin\frac{s(x-y)}{a_1}-e^{\frac{s(x-y)}{a_1}}+e^{-\frac{s(x-y)}{a_1}}\right)q(y)\phi_{21}(y,\lambda)dy. \quad (4.3)$$

$$\frac{d^k}{dx^k}\phi_{22}(x,\lambda)=\left(\frac{\phi_{22}(0)}{2}-\frac{a_2^2\phi_{22}''(0)}{2s^2}\right)\frac{d^k}{dx^k}\cos\frac{sx}{a_2}+\left(\frac{a_2\phi_{22}'(0)}{2s}-\frac{a_2^3\phi_{22}'''(0)}{2s^3}\right)\times\frac{d^k}{dx^k}\sin\frac{sx}{a_2}$$
$$+\left(\frac{\phi_{22}(0)}{4}+\frac{a_2\phi_{22}'(0)}{4s}+\frac{a_2^2\phi_{22}''(0)}{4s^2}+\frac{a_2^3\phi_{22}'''(0)}{4s^3}\right)\times\frac{d^k}{dx^k}e^{\frac{sx}{a_2}}+\left(\frac{\phi_{22}(0)}{4}-\frac{a_2\phi_{22}'(0)}{4s}+\frac{a_2^2\phi_{22}''(0)}{4s^2}-\frac{a_2^3\phi_{22}'''(0)}{4s^3}\right)\frac{d^k}{dx^k}e^{-\frac{sx}{a_2}} \quad (4.4)$$
$$+\frac{a_2^3}{2s^3}\int_{0}^{x}\frac{d^k}{dx^k}\left(\sin\frac{s(x-y)}{a_2}-e^{\frac{s(x-y)}{a_2}}+e^{-\frac{s(x-y)}{a_2}}\right)q(y)\phi_{22}(y,\lambda)dy.$$

Proof. Regard $\phi_{11}(x,\lambda)$ as the solution of the following non-homogeneous Cauchy problem:

$$\begin{cases}-\left(a(x)\phi_{11}''(x)\right)''+s^4\phi_{11}(x)=q(x)\phi_{11}(x,\lambda),\\ \phi_{11}(-1,\lambda)=1,\ \phi_{11}'(-1,\lambda)=0,\\ \phi_{11}''(-1,\lambda)=0,\ \phi_{11}'''(-1,\lambda)=0.\end{cases}$$

Using the method of constant changing, $\phi_{11}(x,\lambda)$ satisfies

$$\phi_{11}(x,\lambda)=\frac{\alpha_2}{2}\cos\frac{s(x+1)}{a_1}+\frac{\alpha_1a_1^3}{2s^3}\sin\frac{s(x+1)}{a_1}+\left(\frac{\alpha_2}{4}-\frac{\alpha_1a_1^3}{4s^3}\right)e^{\frac{s(x+1)}{a_1}}$$
$$+\left(\frac{\alpha_2}{4}+\frac{\alpha_1a_1^3}{4s^3}\right)e^{-\frac{s(x+1)}{a_1}}+\frac{a_1^3}{2s^3}\int_{-1}^{x}\left(\sin\frac{s(x-y)}{a_1}-e^{\frac{s(x-y)}{a_1}}+e^{-\frac{s(x-y)}{a_1}}\right)q(y)\phi_{11}(y,\lambda)dy.$$

Then differentiating it with respect to x, we have (4.1). The proof for (4.2), (4.3) and (4.4) is similar.

Lemma 4.2. *Let* $\lambda=s^4$, $s=\sigma+it$ *. Then the following asymptotic formulae hold for* $k=\overline{0,3}$:

$$\frac{d^k}{dx^k}\phi_{11}(x,\lambda)=\frac{\alpha_2}{2}\frac{d^k}{dx^k}\cos\frac{s(x+1)}{a_1}+\frac{\alpha_2}{4}\frac{d^k}{dx^k}\left(e^{\frac{s(x+1)}{a_1}}+e^{-\frac{s(x+1)}{a_1}}\right)+O\left(|s|^{k-1}e^{|s|\frac{(x+1)}{a_1}}\right). \quad (4.5)$$

$$\frac{d^k}{dx^k}\phi_{12}(x,\lambda)=\frac{a_2^2s^2\delta_1\phi_{11}'(0)}{2}\frac{d^k}{dx^k}\cos\frac{sx}{a_2}+\frac{a_2^3s\delta_2\phi_{11}(0)}{2}\frac{d^k}{dx^k}\sin\frac{sx}{a_2}-\frac{a_2^2s^2\delta_1\phi_{11}'(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_2}}+e^{-\frac{sx}{a_2}}\right)$$
$$-\frac{a_2^3s\delta_2\phi_{11}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_2}}-e^{-\frac{sx}{a_2}}\right)+O\left(e^{|s|^k\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right). \quad (4.6)$$

$$\frac{d^k}{dx^k}\phi_{21}(x,\lambda)=\frac{a_1}{2s}\frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1}+\frac{a_1}{4s}\frac{d^k}{dx^k}\left(e^{\frac{s(x+1)}{a_1}}-e^{-\frac{s(x+1)}{a_1}}\right)+O\left(|s|^{k-2}e^{|s|\frac{x+1}{a_1}}\right).$$

$$\frac{d^k}{dx^k}\phi_{22}(x,\lambda)=\frac{a_2^2s^2\delta_1\phi_{21}'(0)}{2}\frac{d^k}{dx^k}\cos\frac{sx}{a_2}+\frac{a_2^3s\delta_2\phi_{21}(0)}{2}\frac{d^k}{dx^k}\sin\frac{sx}{a_2}-\frac{a_2^2s^2\delta_1\phi_{21}'(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_2}}+e^{-\frac{sx}{a_2}}\right)$$
$$-\frac{a_2^3s\delta_2\phi_{21}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_2}}-e^{-\frac{sx}{a_2}}\right)+O\left(e^{|s|^{k-1}\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right).$$

Each of these asymptotic formulae holds uniformly for x *as* $|\lambda|\to\infty.$

Proof. Let $F_{11}(x,\lambda)=e^{-|s|\frac{x+1}{a_1}}\phi_{11}(x,\lambda).$ It is easy to see that $F_{11}(x,\lambda)$ is bounded. Therefore $\phi_{11}(x,\lambda)=O(e).$ Substituting it into (4.1) and differentiating it with respect to x for $k=\overline{0,3},$ we obtain $(4.5).$ According to transmission conditions (1.4)-(1.7) as $|\lambda|\to\infty,$ we get

$$\phi_{12}(0)=\phi_{11}(0),\ \phi_{12}'(0)=\phi_{11}'(0),\ \phi_{12}''(0)=-s^4\delta_1\phi_{11}'(0),\ \phi_{12}'''(0)=-s^4\delta_2\phi_{11}(0).$$

Substituting these asymptotic formulae into (4.2) for $k=0,$ we obtain

$$\phi_{12}(x,\lambda)=\frac{a_2^2s^2\delta_1\phi_{11}'(0)}{2}\cos\frac{sx}{a_2}+\frac{a_2^3s\delta_2\phi_{11}(0)}{2}\sin\frac{sx}{a_2}-\frac{a_2^2s^2\delta_1\phi_{11}'(0)}{4}\left(e^{\frac{sx}{a_2}}+e^{-\frac{sx}{a_2}}\right)-\frac{a_2^3s\delta_2\phi_{11}(0)}{4}\left(e^{\frac{sx}{a_2}}-e^{-\frac{sx}{a_2}}\right)$$
$$+\frac{a_2^3}{2s^3}\int_0^x\left(\sin\frac{s(x-y)}{a_2}-e^{\frac{s(x-y)}{a_2}}+e^{-\frac{s(x-y)}{a_2}}\right)q(y)\phi_{12}(y,\lambda)dy+O\left(e^{|s|\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right).\quad(4.7)$$

Multiplying through by $|s|^{-3}e^{-|s|\left(\frac{a_1x+a_2}{a_1a_2}\right)},$ and denoting, $F_{12}(x,\lambda):=O\left(|s|^{-3}e^{-|s|\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right)\phi_{12}(x,\lambda).$

Denoting $M:=\max_{x\in[0,1]}|F_{12}(x,\lambda)|$ from the last formula, it follows that

$$M(\lambda)\le\frac{3|\alpha_2\delta_1|}{4a_1}+\frac{|\alpha_2\delta_2|}{4|s|^2}+\frac{M(\lambda)}{2|s|^3}\int_0^x q(y)dy+M_0$$

for some $M_0>0$. From this, it follows that $M(\lambda)=O(1)$ as $|\lambda|\to\infty$, so

$$\phi_{12}(x,\lambda)=O\left(|s|^3e^{|s|\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right).$$

Substituting this back into the integral on the right side of $(4.7$) yields (4.6) for $k=0$. The other cases may be considered analogically.
Similarly one can establish the following lemma. for $\chi_{ij}(x,\lambda)\quad(i=1,2,\ j=1,2).$

Lemma 4.3. *Let* $\lambda=s^4$, $s=\sigma+it$. *Then the following asymptotic formulae hold for* $k=\overline{0,3}$:

$$\frac{d^k}{dx^k}\chi_{11}(x,\lambda)=-\frac{a_1^2s^2\delta_1\chi_{12}'(0)}{2}\frac{d^k}{dx^k}\cos\frac{sx}{a_1}+\frac{a_1^3s\delta_2\chi_{12}(0)}{2}\frac{d^k}{dx^k}\sin\frac{sx}{a_1}+\frac{a_1^2s^2\delta_1\chi_{12}'(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_1}}+e^{-\frac{sx}{a_1}}\right)$$
$$+\frac{a_1^3s\delta_2\chi_{12}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_1}}-e^{-\frac{sx}{a_1}}\right)+O\left(|s|^{k+1}e^{|s|\left(\frac{a_1-a_2x}{a_1a_2}\right)}\right).$$

$$\frac{d^k}{dx^k}\chi_{12}(x,\lambda)=-\frac{a_2^3s}{2}\frac{d^k}{dx^k}\sin\frac{s(x-1)}{a_2}+\frac{a_1^3s\delta_2}{4}\frac{d^k}{dx^k}\left(e^{\frac{s(x-1)}{a_2}}-e^{-\frac{s(x-1)}{a_2}}\right)+O\left(|s|^{k+1}e^{|s|\left(\frac{1-x}{a_2}\right)}\right).$$

$$\frac{d^k}{dx^k}\chi_{21}(x,\lambda)=-\frac{a_1^2s^2\delta_1\chi'_{22}(0)}{2}\frac{d^k}{dx^k}\cos\frac{sx}{a_1}+\frac{a_1^3s\delta_2\chi_{22}(0)}{2}\frac{d^k}{dx^k}\sin\frac{sx}{a_1}+\frac{a_1^2s^2\delta_1\chi'_{22}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_1}}+e^{-\frac{sx}{a_1}}\right)$$

$$+\frac{a_1^3s\delta_2\chi_{22}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_1}}-e^{-\frac{sx}{a_1}}\right)+O\left(|s|^{k+2}e^{|s|\left(\frac{a_1-a_2x}{a_1a_2}\right)}\right).$$

$$\frac{d^k}{dx^k}\chi_{22}(x,\lambda)=-\frac{a_2^2s^2}{2}\frac{d^k}{dx^k}\sin\frac{s(x-1)}{a_2}+\frac{a_2^2s^2}{4}\frac{d^k}{dx^k}\left(e^{\frac{s(x-1)}{a_2}}-e^{-\frac{s(x-1)}{a_2}}\right)+O\left(|s|^{k+1}e^{|s|\left(\frac{1-x}{a_2}\right)}\right).$$

where $k=\overline{0,3}.$ *Each of these asymptotic formulae holds uniformly for* x.

Theorem 4.4. *Let* $\lambda=s^4,\quad s=\sigma+it$ *. Then the characteristic functions* $W_i(\lambda)\quad(i=1,2)$ *have the following asymptotic formulae:*

$$W_1(\lambda)=-\frac{a_2^4\delta_1\delta_2\alpha_2s^{12}}{16}\left(2+\cos\frac{s\left(e^{-\frac{s}{a_2}}+e^{\frac{s}{a_2}}\right)}{a_2}\right)\left(e^{-\frac{s}{a_1}}+e^{\frac{s}{a_1}}\right)\cos\frac{s}{a_1}+O\left(|s|^{11}e^{2|s|\left(\frac{a_1+a_2}{a_1a_2}\right)}\right).$$

$$W_2(\lambda)=-\frac{a_2^4\delta_1\delta_2\alpha_2s^{12}}{16}\left(2+\cos\frac{s\left(e^{-\frac{s}{a_1}}+e^{\frac{s}{a_1}}\right)}{a_1}\right)\left(e^{-\frac{s}{a_2}}+e^{\frac{s}{a_2}}\right)\cos\frac{s}{a_2}+O\left(|s|^{11}e^{2|s|\left(\frac{a_1+a_2}{a_1a_2}\right)}\right).$$

Proof. Substituting the asymptotic equalities $\frac{d^k}{dx^k}\chi_{11}(-1,\lambda)$ and $\frac{d^k}{dx^k}\chi_{21}(-1,\lambda)$ into the representation of $W_1(\lambda),$ we get

$$W_1(\lambda)=\begin{vmatrix}\alpha_2 & 0 & \chi_{11}(-1,\lambda) & \chi_{21}(-1,\lambda)\\ 0 & 1 & \chi'_{11}(-1,\lambda) & \chi'_{21}(-1,\lambda)\\ 0 & 0 & \chi''_{11}(-1,\lambda) & \chi''_{21}(-1,\lambda)\\ -\alpha_1 & 0 & \chi'''_{11}(-1,\lambda) & \chi'''_{21}(-1,\lambda)\end{vmatrix}$$

$$=\frac{a_1^5\delta_1\delta_2s^3}{8}\left(\chi'_{12}(0)\chi_{22}(0)-\chi_{12}(0)\chi'_{22}(0)\right)\times\left(\begin{vmatrix}\alpha_2 & 0 & \cos\frac{s}{a_1} & e^{-\frac{s}{a_1}}-e^{\frac{s}{a_1}}\\ 0 & 1 & -\frac{s}{a_1}\sin\frac{s}{a_1} & \frac{s}{a_1}\left(-e^{-\frac{s}{a_1}}-e^{\frac{s}{a_1}}\right)\\ 0 & 0 & -\frac{s^2}{a_1^2}\cos\frac{s}{a_1} & \frac{s^2}{a_1^2}\left(e^{\frac{s}{a_1}}-e^{\frac{s}{a_1}}\right)\\ -\alpha_1 & 0 & -\frac{s^3}{a_1^3}\sin\frac{s}{a_1} & \frac{s^3}{a_1^3}\left(-e^{-\frac{s}{a_1}}-e^{\frac{s}{a_1}}\right)\end{vmatrix}\right)$$

$$+\begin{vmatrix}1 & 0 & \sin\frac{s}{a_1} & e^{-\frac{s}{a_1}}+e^{\frac{s}{a_1}}\\ 0 & 0 & \frac{s}{a_1}\cos\frac{s}{a_1} & s\left(-e^{-\frac{s}{a_1}}+e^{\frac{s}{a_1}}\right)\\ 0 & -1 & -\frac{s^2}{a_1^2}\sin\frac{s}{a_1} & s^2\left(e^{\frac{s}{a_1}}+e^{\frac{s}{a_1}}\right)\\ 0 & 0 & -\frac{s^3}{a_1^3}\sin\frac{s}{a_1} & s^3\left(-e^{-\frac{s}{a_1}}+e^{\frac{s}{a_1}}\right)\end{vmatrix}+O\left(|s|^{15}e^{2|s|\left(\frac{a_1+a_2}{a_1a_2}\right)}\right)=0.$$

Analogically, we can obtain the asymptotic formulae of $W_2(\lambda)$.

Corollary 4.5. *The real eigenvalues of the problem (1.1)-(1.9) are bounded below.*

Proof. Putting $s^2 = it^2 \quad (t > 0)$ in the above formulas, it follows that , $W(-t^2) \to \infty$ as $t \to \infty$.
Therefore, $W(\lambda) \neq 0$ for λ negative and sufficiently large in modulus.

Now we can obtain the asymptotic approximation formulae for the eigenvalues of the considered problem (1.1)-(1.9).
Since the eigenvalues coincide with the zeros of the entire function $W(\lambda)$, it follows that they have no finite limit. Moreover, we know from Corollary 4.5 that all real eigenvalues are bounded below. Hence, we may renumber them as $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$, listed according to their multiplicity.

Theorem 4.7. *The eigenvalues* $\lambda_n = s_n^4$, $n = 0,1,2,...$ *of the problem (1.1)-(1.9) have the following asymptotic formulae for* $n \to \infty$:

$$\sqrt[4]{\lambda_n'} = \frac{a_1\pi(2n-1)}{2} + O\left(\frac{1}{n}\right), \ \sqrt[4]{\lambda_n''} = \frac{a_2\pi(2n+1)}{2} + O\left(\frac{1}{n}\right).$$

Proof. By applying the well-known Rouché's theorem, which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour C , and $|g(s)| < |f(s)|$ on C , then $f(s)$ and $f(s) + g(s)$ have the same number zeros inside C provided that each zero is counted according to their multiplicity, we can obtain these conclusions.

Theorem 4.8. *The residual spectrum of the operator* A *is empty, i.e.,* $\sigma_r(A) = \varnothing$.

Proof. It sufficies to prove that if γ is not an eigenvalue of A, then $(A - \gamma I)^{-1}$ is dense in Z. Therefore we examine the equation $(A - \gamma I)Y = F \in Z$, where $F = (f, f_1, f_2, f_3, f_4)$.

Since γ is not an eigenvalue of (1.1)-(1.9) , we have

$$\gamma u(1) + u'''(1) = f_1 \neq 0,$$

or, $$\gamma u'(1) + u''(1) = f_2 \neq 0,$$

or, $$u''(0+) - u''(0-) + \gamma\delta_1 u'(0-) = f_3 \neq 0, \qquad (4.8)$$

or, $$u'''(0+) - u'''(0-) + \gamma\delta_2 u(0-) = f_4 \neq 0. \qquad (4.9)$$

For convenience, we assume that the (4.8) or (4.9) be true.
Consider the initial-value problem

$$\begin{cases} Ly - \gamma y = f, x \in I, \\ \alpha_1 y(-1) + \alpha_2 y'''(-1) = 0, \\ y''(-1) = 0, \\ y(0+) - y(0-) = 0, \\ y'(0+) - y'(0-) = 0, \\ y''(0+) - y''(0-) + \gamma\delta_1 y'(0-) = f_3, \\ y'''(0+) - y'''(0-) + \gamma\delta_2 y(0-) = f_4. \end{cases} \qquad (4.10)$$

Let $u(x)$ be the solution of the equation

$Lu-\gamma u=0$ satisfying

$$u(-1)=\alpha_2, u'(-1)=1, u''(-1)=0, u'''(-1)=-\alpha_1,$$

$$u(0+)-u(0-)=0, u'(0+)-u'(0-)=0, u''(0+)-u''(0-)+\gamma\delta_1 u'(0-)=f_3, u'''(0+)-u'''(0-)+\gamma\delta_2 u(0-)=f_4.$$

In fact

$$u(x)=\begin{cases} u_1(x), & x\in[-1,0), \\ u_2(x), & x\in(0,1], \end{cases}$$

where $u_1(x)$ is the unique solution of the initial-value problem

$$\begin{cases} a_1^4 u_1^{(4)}+q(x)u_1=\gamma u_1,\ x\in[-1,0), \\ u_1(-1)=\alpha_2,\ u_1'(-1)=1, \\ u_1''(-1)=0,\ u_1'''(-1)=-\alpha_1, \end{cases}$$

$u_2(x)$ is the unique solution of the problem

$$\begin{cases} -a_2{}^4 u_2{}^{(4)}+q(x)u_2=\gamma u_2, x\in(0,1] \\ u_2(0+)-u_1(0-)=0, \\ u_2'(0+)-u_1'(0-)=0, \\ u_2''(0+)-u_1''(0-)+\gamma\delta_1 u_1'(0-)=f_3, \\ u_2'''(0+)-u_1'''(0-)+\gamma\delta_2 u_1'(0-)=f_4. \end{cases}$$

Let

$$\omega(x)=\begin{cases} \omega_1(x), & x\in[-1,0), \\ \omega_2(x), & x\in(0,1] \end{cases}$$

be a solution of $L\omega-\gamma\omega=f$ satisfying

$$\alpha_1 w(-1)+\alpha_2 w'''(-1)=0, w''(-1)=0,$$

$$w(0+)-w(0-)=0, w'(0+)-w'(0-)=0,$$

$$w''(0+)-w''(0-)+\gamma\delta_1 w'(0-)=f_3, w'''(0+)-w'''(0-)+\gamma\delta_2 w(0-)=f_4.$$

Then (4.10) has the general solution

$$y(x)=\begin{cases} du_1+\omega_1, & x\in[-1,0) \\ du_2+\omega_2, & x\in(0,1], \end{cases} \qquad (4.11)$$

where $d\in\mathbb{C}$.

Sİnce γ is not an eigenvalue of (1.1) - (1.9) , we have

$$\gamma u_2(1)+u_2'''(1)\neq 0 \qquad (4.12)$$

or

$$\gamma u_2'(1)+u_2''(1)\neq 0. \qquad (4.13)$$

The second component of $(A-\gamma)Y=F$ involves the equation

$y'''(1)+\gamma y(1)=h.$ (4.14)

Substituting (4.11) into (4.14) , we get

$d\left(u_2'''(1)+\gamma u_2(1)\right)=h-\omega_2'''(1)-\gamma\omega_2(1).$

In view of (4.12) , we know that d is a unique solution.

The third component of $(A-\gamma)Y=F$ involves the equation

$y''(1)+\gamma y'(1)=-k$. (4.15)

Substituting (4.11) into (4.15) , we get, $d\left(u_2''(1)+\gamma u_2'(1)\right)=-k-\omega_2''(1)-\gamma\omega_2'(1).$ In view of (4.13) , we know that d is a unique solution. Thus if γ is not an eigenvalue of $(1.1)-(1.9)$, d is uniquely solvable. Hence y is uniquely determined. The above arguments show that $(A-\gamma I)^{-1}$ is defined on all of Z. So $\gamma\notin\sigma_r(A)$, i.e., $\sigma_r(A)=\varnothing$.

Conclusion

In this work firstly we constructed operator formulation of the given boundary value problem with eigenparameter-dependent boundary conditions. And then we obtained asymptotic formulas for eigenvalues and fundamental solutions. Finally, we investigated the spectrum.

References

1. Kim T., Identities involving Frobenius-Euler polynomials arising from non-linear differential equations, *Journal of Number Theory*, **132 (12)**, 2854-2865 **(2012)**

2. Demirci M., Akdoğan Z., Mukhtarov O.Sh., Asymptotic behavior of eigenvalues and eigenfuctions of one discontinuous boundary-value problem, *Intenational Jounal of Computational Cognition* **2(3)**, 101-113 **(2004)**

3. Buschmann D., Stolz G., Weidmann J., One-dimensional Schrödinger operators with local point interactions, *Journal für die Reine und Angewandte Mathematik* **467**, 169-186 **(1995)**

4. Kadakal M., Mukhtarov O.Sh., Sturm-Liouville problems with discontinuities at two points, *Comput. Math. Appl*. **54**, 1367-1379 **(2007)**

5. Yang Q., Wang W., Asymptotic behavior of a differential operator with discontinuities at two points, *Mathematical Methods in the Applied Sciences* **34**, 373-383 **(2011)**

6. Şen E., Bayramov A., Calculation of eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument which contains a spectral parameter in the boundary condition, *Mathematical and Computer Modelling* **54,** 3090-3097 **(2011)**

7. Mukhtarov O.Sh., Tunç E., Eigenvalue problems for Sturm Liouville equations with transmission conditions, *Israel Journal of Mathematics* **144(2)**, 367-380 **(2004)**

8. Fulton C.T., Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh Sect.* **A77**, 293-308 **(1977)**

9. Titeux I., Yakubov Ya., Completeness of root functions for thermal conduction in a strip with piecewise continuous coefficients, *Math. Models Methods Appl.* **7 (7)**, 1035-1050 **(1997)**

10. Binding P.A., Browne P.J., Oscillation theory for indefinite Sturm-Liouville problems with eigen-parameter-dependent boundary conditions, *Proc. Roy. Soc. Edinburgh Sect.* **A127**, 1123-1136 **(1997)**

11. Kerimov N.B., Mamedov Kh.R., On a boundary value problem with a spectral parameter in the boundary conditions, *Sibirsk. Mat. Zh.* **40(2)**, 325-335 **(1999)**

12. Yang Q., Wang W., A class of fourth-order differential operators with transmission conditions, *Iranian Journal of Science and Technology* Transaction A. **A4**, 323-332 **(2011)**

13. Yang Q., Spectrum of a fourth order differential operator with discontinuities at two points, *International Journal of Modern Mathematical Sciences* **1(3)**, 134-142 **(2012)**