



On Some Integral Relations involving I -Function with Applications

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Abstract

In an attempt to give extensions of the results in the theory of special functions we have established two integral relations involving I -function, which was introduced and studied by V.P. Saxena and used them in finding out a number of integrals involving the product of generalized hyper-geometric functions in one and two arguments.

Keywords: I -function, Fox's H -function, Wright's generalized hypergeometric function, associated bessel function.

Introduction

The I -function introduced by Saxena¹ will be represented and defined as follows:

$$I_{p_i, q_i; R}^{m, n}[z] = I_{p_i, q_i; R}^{m, n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{1, n} \\ (b_j, \beta_j)_{1, m} \end{matrix} ; \begin{matrix} (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \phi(\xi) z^\xi d\xi, \quad (1.1)$$

Where

$$\phi(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^R \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} \xi) \right\}}, \quad (1.2)$$

$p_i (i = 1, 2, \dots, R), q_i (i = 1, 2, \dots, R), m, n$ are integers satisfying $0 \leq n \leq p_i, 0 \leq m \leq q_i (i = 1, 2, \dots, R)$; R is finite, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive; a_j, b_j, a_{ji}, b_{ji} are complex numbers and $\mathcal{L}$ is the path of integration separating the increasing and decreasing sequences of poles of the integrand and the convergence, existence conditions and other details of the I -function, one can refer².

The integral converges, if $|\arg x| < \frac{1}{2} \pi \Omega_i$, where

$$\Omega_i = \sum_{j=1}^n \alpha_j - \sum_{j=n+1}^{p_i} \alpha_j + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^{q_i} \beta_j > 0 \quad \text{and} \quad T = \sum_{j=1}^{q_i} b_j - \sum_{j=1}^{p_i} a_j > 0 \quad (1.3)$$

Evidently, by the case $R = 1$ of the function (1.1) would correspond to the well known Fox's H -function³,

$$H_{p, q}^{m, n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{matrix} \right. \right] = \frac{1}{2\pi \omega} \int_{\mathcal{L}} \theta(\xi) z^\xi d\xi, \quad (1.4)$$

where $\omega = \sqrt{-1}, x (\neq 0)$ is a complex variable and $x^\xi = \exp[\xi \{\log|x| + \omega \arg x\}]$ in which $\log|x|$ represents the natural logarithm of $|x|$ and $\arg x$ is not necessarily the principal value. Also,

$$\theta(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j \xi)}. \quad (1.5)$$

The following function which follows as special cases of the I -function will be required in the sequel¹:

$${}_p\Psi_q \left[\begin{matrix} (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{matrix} ; -z \right] = I_{p, q+1; 1}^{1, p} \left[z \left| \begin{matrix} (1 - a_j, \alpha_j)_{1, p} \\ (0, 1), (1 - b_j, \beta_j)_{1, q} \end{matrix} \right. \right], \quad (1.6)$$

where Ψ is known as the Wright's generalized hypergeometric function.

In our investigation we shall need the following known results ^{2,4} respectively.

$$\int_0^{\pi/2} \cos 2u\phi (\sin \phi)^v d\phi = \frac{\sqrt{\pi} \Gamma\left(u - \frac{v}{2}\right) \Gamma\left(\frac{1}{2} + \frac{v}{2}\right)}{2 \Gamma\left(-\frac{v}{2}\right) \Gamma\left(1 + u + \frac{v}{2}\right)}, \quad (1.7)$$

provided $R(v+1) > 0$ and u is a non-negative integer.

$$\int_0^{\pi/2} \sin 2u\phi (\sin 2\phi)^v d\phi = \frac{\sqrt{\pi} \Gamma\left(1 + \frac{v}{2}\right) \Gamma\left(\frac{u}{2} - \frac{v}{2}\right)}{2 \Gamma\left(\frac{1}{2} - \frac{v}{2}\right) \Gamma\left(1 + \frac{u}{2} + \frac{v}{2}\right)}, \quad (1.8)$$

Where $R(v+5) > 0$ and $u = 1, 3, 5, \dots$

The Integral Relations

In this section we establish the following integral relations:

Relation I

$$\int_0^\infty \int_0^\infty \frac{y^{2v}}{(x^2 + y^2)^{-v}} \cos 2u \left(\tan^{-1} \frac{y}{x} \right) I_{p_i, q_i; R}^{m, n} \left[\frac{a(x^2 + y^2)^{\rho + \sigma}}{y^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] \phi(x^2 + y^2) dx dy$$

$$= \frac{\sqrt{\pi}}{4} \int_0^\infty I_{p_i+2, q_i+2; R}^{m+1, n+1} \left[az^\sigma \left| \begin{matrix} P_1 \\ Q_1 \end{matrix} \right. \right] \phi(z) dz \quad (2.1)$$

Where P_1 and Q_1 denotes the parameter, $(1 - u + v, \rho), (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i}, (1 + u + v, \rho),$ and $\left(\frac{1}{2} + v, \rho\right), (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i}, (1 + v, \rho)$ respectively and u is a non-negative integer, $T > 0, \Omega_i > 0, |\arg a| < \frac{1}{2}\pi\Omega_i$, and $Re\left(1 + 2v - 2\rho \frac{(\alpha_i - 1)}{\alpha_i}\right) > 0$ and provided that the double integral converges.

Relation II

$$\int_0^\infty \int_0^\infty \left\{ \frac{xy}{x^2 + y^2} \right\}^v \sin 2u \left(\tan^{-1} \frac{y}{x} \right) I_{p_i, q_i; R}^{m, n} \left[\frac{a(x^2 + y^2)^{2\rho + \sigma}}{(2xy)^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] \phi(x^2 + y^2) dx dy$$

$$= \frac{\sqrt{\pi}}{2^{v+2}} \int_0^\infty I_{p_i+2, q_i+2; R}^{m+1, n+1} \left[az^\sigma \left| \begin{matrix} P_2 \\ Q_2 \end{matrix} \right. \right] \phi(z) dz \quad (2.2)$$

Where $P_2 = \left(1 - \frac{u}{2} + \frac{v}{2}, \rho\right), (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i}, \left(1 + \frac{u}{2} + \frac{v}{2}, \rho\right)$ and $Q_2 = \left(1 + \frac{v}{2}, \rho\right), (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i}, \left(\frac{1}{2} + \frac{v}{2}, \rho\right)$ respectively and u is an odd positive integer, $T > 0, \Omega_i > 0, |\arg a| < \frac{1}{2}\pi\Omega_i$, and $Re\left(v + 5 - 2\rho \frac{(\alpha_i - 1)}{\alpha_i}\right) > 0$ and the double integral converges.

To prove (2.1), we first establish the integral

$$\int_0^{\pi/2} \cos 2u\theta (\sin \theta)^{2v} I_{p_i, q_i; R}^{m, n} \left[\frac{az^\sigma}{(\sin \theta)^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] d\theta = \frac{\sqrt{\pi}}{2} I_{p_i+2, q_i+2; R}^{m+1, n+1} \left[az^\sigma \left| \begin{matrix} P_1 \\ Q_1 \end{matrix} \right. \right], \quad (2.3)$$

Which is valid when $T > 0, \Omega_i > 0, |\arg z| < \frac{1}{2}\pi\Omega_i$, and $Re\left(1 + 2v - 2\rho \frac{(\alpha_i - 1)}{\alpha_i}\right) > 0$ and u is a non-negative integer and where P_1 and Q_1 are given with (2.1).

The integral (2.3) is obtained, if we replace the I -function by its equivalent contour integral as given in (1.1) and interchange the order of integration (which is permissible under the conditions stated), evaluate the inner integral with the help of (1.7) and finally interpret it with (1.1).

Now replace z by r^2 in (2.3), multiply both sides by $r\phi(r^2)$ and then integrate the resulting equation with respect to r over the semi-infinite ray $(0, \infty)$, we get

$$\int_0^\infty r\phi(r^2) \int_0^{\pi/2} \cos 2u\theta (\sin \theta)^{2v} I_{p_i, q_i; R}^{m, n} \left[\frac{az^\sigma}{(\sin \theta)^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] d\theta dr = \frac{\sqrt{\pi}}{2} \int_0^\infty r\phi(r^2) I_{p_i+2, q_i+2; R}^{m+1, n+1} \left[az^\sigma \left| \begin{matrix} P_1 \\ Q_1 \end{matrix} \right. \right] dr. \quad (2.4)$$

Again changing the polar coordinates occurring on the left-hand side of (2.4) into Cartesian coordinates by means of the substitutions $x = r \cos \theta, y = r \sin \theta, r^2 = x^2 + y^2, \theta = \tan^{-1} \frac{y}{x}$ and putting $r^2 = z$ in the right-hand side of (2.4) and simplifying further, (2.4) yields (2.1). A similar consequence of the known formula (1.8) provides the integral relation (2.2).

Applications

On choosing $\phi(z)$ in convenient and suitable form, a large number of interesting double integrals of general nature can be evaluated very easily. The same has been shown in the following examples:

Example 1: Choosing $\phi(z) = z^{\mu-1} e^{-z} I_\lambda(z)$ in (2.1), we get

$$\int_0^\infty \int_0^\infty y^v (x^2 + y^2)^{\mu-v-1} e^{-(x^2+y^2)} I_\lambda(x^2 + y^2) \cos 2u \left(\tan^{-1} \frac{y}{x} \right) \\ \cdot I_{p_i, q_i; R}^{m, n} \left[\frac{a(x^2+y^2)^{\rho+\sigma}}{y^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] dx dy = \frac{\sqrt{\pi}}{4} \int_0^\infty z^{\mu-1} e^{-z} I_\lambda(z) I_{p_i+2, q_i+2; R}^{m+1, n+1} [az^\sigma |_{Q_1}^{P_1}] dz. \quad (3.1)$$

On evaluating the integral on the right-hand side of (3.1), with the help of Tables of Integral Transforms, Vol. 1⁵ one gets

$$\int_0^\infty \int_0^\infty y^v (x^2 + y^2)^{\mu-v-1} e^{-(x^2+y^2)} I_\lambda(x^2 + y^2) \cos 2u \left(\tan^{-1} \frac{y}{x} \right) \\ \cdot I_{p_i, q_i; R}^{m, n} \left[\frac{a(x^2+y^2)^{\rho+\sigma}}{y^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] dx dy = \frac{1}{2^{\mu+2}} I_{p_i+4, q_i+3; R}^{m+2, n+2} \left[\frac{a}{2^\sigma} \left|_{Q_3}^{P_3} \right. \right], \quad (3.2)$$

Where,

$$P_3 = (1 - \mu - \lambda, \sigma), (1 - u + v, \rho), (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i}, (1 - \mu + \lambda, \sigma), (1 + u + v, \rho)$$

$$Q_3 = \left(\frac{1}{2} + v, \rho \right), \left(\frac{1}{2} - \mu, \sigma \right) (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i}, (1 + v, \rho).$$

and the result (3.2) is valid under the conditions $R \left(\lambda + \mu + \sigma \frac{a_i}{\alpha_i} \right) > 0, (i = 1, 2, \dots, m), \operatorname{Re} \left(\mu + \sigma \frac{(a_i-1)}{\alpha_i} \right) < \frac{1}{2}$, along with the conditions given in (2.1).

Example 2: If in (2.1), we set $\phi(z) = z^{\mu-1} H_{P, Q}^{M, N} \left[bz \left| \begin{matrix} (e_j, E_j)_{1, P} \\ (f_j, F_j)_{1, Q} \end{matrix} \right. \right]$ and making use of the known result¹, we obtain

$$\int_0^\infty \int_0^\infty y^{2v} (x^2 + y^2)^{\mu-v-1} \cos 2u \left(\tan^{-1} \frac{y}{x} \right) H_{P, Q}^{M, N} \left[b(x^2 + y^2) \left| \begin{matrix} (e_j, E_j)_{1, P} \\ (f_j, F_j)_{1, Q} \end{matrix} \right. \right] \\ \cdot I_{p_i, q_i; R}^{m, n} \left[\frac{a(x^2+y^2)^{\rho+\sigma}}{y^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] \phi(x^2 + y^2) dx dy = \frac{\sqrt{\pi}}{4b^\mu} I_{p_i+Q+2, q_i+P+2; R}^{m+N+1, n+P+1} \left[\frac{a}{b^\sigma} \left|_{Q_4}^{P_4} \right. \right], \quad (3.3)$$

Where,

$$P_4 = (1 - u + v, \rho), (a_j, \alpha_j)_{1, n}, \left(1 - f_j - \frac{\mu}{2} F_j, \sigma F_j \right)_{1, Q}, (a_{ji}, \alpha_{ji})_{n+1, p_i}, (1 + u + v, \rho) \text{ and}$$

$Q_4 = \left(\frac{1}{2} + v, \rho \right), (b_j, \beta_j)_{1, m}, \left(1 - e_j - \frac{\mu}{2} E_j, \sigma E_j \right)_{1, P}, (b_{ji}, \beta_{ji})_{m+1, q_i}, (1 + v, \rho)$, provided that i. Conditions mentioned along with (2.1) are satisfied, ii. $\tau = \sum_{j=1}^N E_j - \sum_{j=N+1}^P E_j + \sum_{j=1}^M F_j - \sum_{j=M+1}^Q F_j > 0$, iii. $|\arg b| < \frac{1}{2} \tau \pi$ and $\sum_{j=1}^Q F_j - \sum_{j=1}^P E_j > 0$.

Example 3: If we take $\phi(z) = z^{\lambda+\mu-1} e^{-2z} F \left[\begin{matrix} (a): (c): (c'); \\ (b): (d): (d'); \end{matrix} ; 2\alpha z, 2\beta z \right]$ in (2.1) and proceed on the similar lines as above and evaluate the right-hand side integral in the light of Melline Transform⁵ of the function $z^\lambda e^{-z} F \left[\begin{matrix} (a): (c): (c'); \\ (b): (d): (d'); \end{matrix} ; \alpha z, \beta z \right]$, we find the following double integral:

$$\int_0^\infty \int_0^\infty y^v (x^2 + y^2)^{\lambda+\mu-v-1} e^{-2(x^2+y^2)} \cos 2u \left(\tan^{-1} \frac{y}{x} \right) F \left[\begin{matrix} (a): (c): (c'); \\ (b): (d): (d'); \end{matrix} ; 2\alpha(x^2 + y^2), 2\beta(x^2 + y^2) \right]$$

$$\begin{aligned} & \bullet I_{p_i, q_i; R}^{m, n} \left[\frac{a(x^2 + y^2)^{\rho + \sigma}}{y^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n} \\ (b_j, \beta_j)_{1, m} \end{matrix} ; \begin{matrix} (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] dx dy \\ &= \frac{2^{\lambda - \mu - 2} \Gamma(\lambda)}{\Gamma(2\lambda)} \sum_{s=0}^{\infty} \frac{(-1)^s (2\lambda + s) \Gamma(\lambda + s)}{(s)!} \\ & \bullet F \left[\begin{matrix} -s, 2\lambda + s, (a): (c): (c'); \\ \lambda + \frac{1}{2}, (b): (d): (d'); \end{matrix} \alpha, \beta \right] I_{p_i+4, q_i+3; R}^{m+2, n+2} \left[\frac{a}{2^\sigma} \left| \begin{matrix} P_5 \\ Q_5 \end{matrix} \right. \right], \end{aligned} \quad (3.4)$$

Where parameters P_5 and Q_5 stands for $(1 - s - \mu - \lambda, \sigma), (1 - u + v, \rho), (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i}, (1 + s - \mu + \lambda, \sigma), (1 + u + v, \rho)$ and $(\frac{1}{2} + v, \rho), (\frac{1}{2} - \mu, \sigma) (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i}, (1 + v, \rho)$ respectively and the integral (3.4) holds when $A + C < B + D, A + C' < B + D', R \left(\lambda + \mu + \sigma \frac{\alpha_i}{\alpha_i} \right) > 0, (i = 1, 2, \dots, m)$ together with the conditions already given in (2.1).

Special cases

On account of the most general nature of I -function, a large number of integrals and integral relations involving simpler functions can be easily obtained as special cases. The same has been shown by the way of following illustrations:

By putting $R = 1$ in (2.1) and (2.2), I -function reduces to Fox's H -function and we arrive at known results ⁶

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{y^{2v}}{(x^2 + y^2)^{-v}} \cos 2u \left(\tan^{-1} \frac{y}{x} \right) H_{p, q}^{m, n} \left[\frac{a(x^2 + y^2)^{\rho + \sigma}}{y^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{matrix} \right. \right] \phi(x^2 + y^2) dx dy \\ &= \frac{\sqrt{\pi}}{4} \int_0^\infty H_{p+2, q+2}^{m+1, n+1} \left[az^\sigma \left| \begin{matrix} (1 - u + v, \rho), (a_j, \alpha_j)_{1, p}, (1 + u + v, \rho), \\ (\frac{1}{2} + v, \rho), (b_j, \beta_j)_{1, q}, (1 + v, \rho) \end{matrix} \right. \right] \phi(z) dz \end{aligned} \quad (4.1)$$

Where u is a non-negative integer, $T > 0, \Omega > 0, |\arg a| < \frac{1}{2} \pi \Omega$, and $Re \left(1 + 2v - 2\rho \frac{(\alpha_i - 1)}{\alpha_i} \right) > 0$ and provided that the double integral converges.

$$\begin{aligned} & \int_0^\infty \int_0^\infty \left\{ \frac{xy}{x^2 + y^2} \right\}^v \sin 2u \left(\tan^{-1} \frac{y}{x} \right) H_{p, q}^{m, n} \left[\frac{a(x^2 + y^2)^{2\rho + \sigma}}{(2xy)^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{matrix} \right. \right] \phi(x^2 + y^2) dx dy \\ &= \frac{\sqrt{\pi}}{2^{v+2}} \int_0^\infty H_{p+2, q+2}^{m+1, n+1} \left[az^\sigma \left| \begin{matrix} \left(1 - \frac{u}{2} + \frac{v}{2}, \rho \right), (a_j, \alpha_j)_{1, p}, \left(1 + \frac{u}{2} + \frac{v}{2}, \rho \right) \\ \left(1 + \frac{v}{2}, \rho \right), (b_j, \beta_j)_{1, q}, \left(\frac{1}{2} + \frac{v}{2}, \rho \right) \end{matrix} \right. \right] \phi(z) dz \end{aligned} \quad (4.2)$$

where u is an odd positive integer, $T > 0, \Omega > 0, |\arg a| < \frac{1}{2} \pi \Omega$, and $Re \left(v + 5 - 2\rho \frac{(\alpha_i - 1)}{\alpha_i} \right)$.

If we take $R = m = 1, n = p_i = p, q_i = q + 1, b_1 = 0, \beta_1 = 1, a_j = 1 - a_j, b_{ji} = 1 - b_j, \beta_{ji} = \beta_j$ in (2.1) then I -function occurring therein reduce into Wright's generalized hypergeometric function ${}_p\Psi_q$ given by (1.6) and the integral relation (3.1) takes the form

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{y^{2v}}{(x^2 + y^2)^{-v}} \cos 2u \left(\tan^{-1} \frac{y}{x} \right) {}_p\Psi_q \left[\frac{-a(x^2 + y^2)^{\rho + \sigma}}{y^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{matrix} \right. \right] \phi(x^2 + y^2) dx dy \\ &= \frac{\sqrt{\pi}}{4} \int_0^\infty {}_p\Psi_{q+2} \left[az^\sigma \left| \begin{matrix} (1 - u + v, \rho), (1 + u + v, \rho), (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q}, \left(\frac{1}{2} + v, \rho \right), (1 + v, \rho) \end{matrix} \right. \right] \phi(z) dz \end{aligned} \quad (4.3)$$

The conditions of validity of (4.3) easily follow from those given in (2.1).

On the other hand, reducing the Fox's H -function occurring in the integral (3.3) to the Bessel function $J_\lambda(z)$ and using the Hankel transform of the I -function we arrive at the result:

$$\int_0^\infty \int_0^\infty y^{2v} (x^2 + y^2)^{\mu-v-1} \cos 2u \left(\tan^{-1} \frac{y}{x} \right) J_\lambda(b(x^2 + y^2)) \cdot I_{p_i, q_i; R}^{m, n} \left[\frac{a(x^2 + y^2)^{\rho+\sigma}}{y^{2\rho}} \left| \begin{matrix} (a_j, \alpha_j)_{1, n}; (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}; (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] \phi(x^2 + y^2) dx dy = \frac{\sqrt{\pi}}{4b^\mu} I_{p_i+3, q_i+2; R}^{m+1, n+2} \left[\frac{a}{b^\sigma} \middle| Q_6 \right], \quad (4.4)$$

Where, $P_6 = (1 - u + v, \rho), \left(1 - \frac{(\mu+\lambda)}{2}, \sigma\right), (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i}, \left(1 - \frac{(\mu+\lambda)}{2}, \sigma\right), (1 + u + v, \rho)$ and $Q_6 = \left(\frac{1}{2} + v, \rho\right), (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i}, (1 + v, \rho)$. The condition of validity of (4.4) easily follows from those given in (2.1) and (3.3).

Conclusion

In this way a number of integrals may be evaluated choosing $\phi(z)$ suitably. We conclude this paper with the remark that the I -function is the most generalized form of the hypergeometric function in one argument and it contains an important class of symmetric Fourier kernel of a very general nature and a vast number of well known analytic functions as special cases like Fox's H -function and Meijer's G -function which are also generalization of many higher transcendental functions⁷. Therefore the results given here are useful in obtaining many new results involving the products of simple commonly used functions appearing in mathematical analysis both pure and applied.

References

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