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Construction of Balanced Bipartite Block Designs

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Abstract

In this paper some methods of construction of balanced bipartite block (BBPB) designs are obtained which are based on incidence matrices of the known balanced incomplete block (BIB) designs and two-associate-class partially balanced incomplete block (2-PBIB) group divisible (GD) designs. The obtained results are given with examples to show how they can be applied.

Keywords: BBPB designs, Balanced bipartite block designs with unequal block sizes (BBPBUB), BIB designs, GD designs.

Introduction

Consider the experimental setting where v distinct treatments are divided into two disjoint groups of cardinality v_1 and v_2 respectively and the purpose is to compare the set of v_1 test treatments denoted by $1, 2, \ldots, v_1$ to the set of $v_2 (\geq 2)$ control treatments denoted by $v_1 + 1, \ldots, v_1 + v_2 (= v)$. Bechhofer and Tamhane[1] have defined proper balanced treatment incomplete block (BTIB) designs to compare a set of test treatments to a control treatment. As an extension of these designs Kageyama and Sinha[2] have defined balanced bipartite block (BBPB) designs for comparing a set of test treatments to a set of control treatments.

Definition 1: An incomplete block binary design with a set of v_1 treatments occurring r_1 times and another set of v_2 treatments occurring r_2 times $(r_1 \neq r_2)$ arranged into b blocks of constant block size k is said to be a BBPB design if: (i) any two distinct treatments in the i^{th} set occur together in λ_{ii} blocks, $i = 1,2;$ (ii) any two treatments from different sets occur together in $\lambda_{12} = \lambda_{21} (> 0)$ blocks.

Some systematic methods of constructing BBPB designs have given by Kageyama and Sinha[2] and Sinha and Kageyama[3]. As a natural extension of BTIB designs Angelis and Moyssiadis[4] have given the concept of balanced treatment incomplete block designs with unequal block sizes (BTIUB) for comparing a set of test treatments to a single control treatment with unequal blocks. Some methods of construction of A-efficient BTIUB designs have been given by Angelis and Moyssiadis[4] and Angelis, Moyssiadis and Kageyama[5]. Jacroux[6] has given some methods of construction of A- and MV-optimal balanced treatment unequal block designs. Parsad and Gupta[7] have given the structure of optimal designs for comparing v_1 test treatments to a control treatment. Using the definition of BTIUB designs given by Angelis and Moyssiadis[4] and the BBPB designs by Kageyama and Sinha[2], Jaggi, Parsad and Gupta[8] have defined balanced bipartite block designs with unequal block sizes (BBPBUB) for both binary and non-binary block designs.

In following sections, we give some methods of construction of BBPBUB designs for comparing a set of test treatments to a set of control treatments by using BIB designs and GD designs. The definition of these designs can be found in Raghavrao[9].

In what follows, we denote by $\otimes$ the kronecker product of matrices, $\mathbf{1}'_p \otimes N$ the p replications of N, I_p the identity matrix of order p, $J_{p\times q}$ the matrix of ones of order $p \times q$, $\mathbf{1}'_p$ the $1 \times p$ row vector of ones, $0_{p\times q}$ the null matrix of order $p \times q$ and by p_1, p_2, p_3 the positive integers.

Methods of Construction of BBPBUB Designs Using BIB Designs

In this section, we describe some methods of construction of BBPBUB designs making use of the incidence matrices of BIB designs, etc.

Theorem 1: Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters v_L, b_L, r_L, k_L, λ_L such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$, then

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{matrix} \mathbf{1}'_{v_3} \otimes N_2 & I_{v_2} \otimes \mathbf{1}'_{b_3} & \mathbf{1}'_{p_2} \otimes N_4 & 0_{v_2\times p_3 b_5} & I_{v_2} & 0_{v_2\times v_3} \\ I_{v_3} \otimes \mathbf{1}'_{b_2} & \mathbf{1}'_{v_2} \otimes N_3 & 0_{v_3\times p_2 b_4} & \mathbf{1}'_{p_3} \otimes N_5 & 0_{v_3\times v_2} & I_{v_3} \end{matrix}\right] \quad (1)$$

is the incidence matrix of a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + v_3 b_2 + v_2 b_3 + p_2 b_4 + p_3 b_5 + v_2 + v_3$, $\mathbf{r}' = \{(p_1 r_1 + v_3 r_2 + b_3 + p_2 r_4 + 1)\mathbf{1}'_{v_2}, (p_1 r_1 + b_2 + v_2 r_3 + p_3 r_5 + 1)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1 \mathbf{1}'_{p_1 b_1}, (k_2 + 1)\mathbf{1}'_{v_3 b_2}, (k_3 + 1)\mathbf{1}'_{v_2 b_3}, k_4 \mathbf{1}'_{p_2 b_4}, k_5 \mathbf{1}'_{p_3 b_5},$

$\mathbf{1}\mathbf{1}'_{v_2}, \mathbf{1}\mathbf{1}'_{v_3}\}$ if and only if the positive integers p_1, p_2 and p_3 satisfy

$$\frac{p_1 r_1(k_1-1)}{k_1} + \frac{v_3 r_2 k_2}{(k_2+1)} + \frac{b_3 k_3}{(k_3+1)} + \frac{p_2 r_4(k_4-1)}{k_4} - (v_2-1)\left\{\frac{p_1\lambda_1}{k_1} + \frac{v_3\lambda_2}{(k_2+1)} + \frac{p_2\lambda_4}{k_4}\right\} - v_3\left\{\frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)}\right\} = 0$$

and

$$\frac{p_1 r_1(k_1-1)}{k_1} + \frac{b_2 k_2}{(k_2+1)} + \frac{v_2 r_3 k_3}{(k_3+1)} + \frac{p_3 r_5(k_5-1)}{k_5} - (v_3-1)\left\{\frac{p_1\lambda_1}{k_1} + \frac{v_2\lambda_3}{(k_3+1)} + \frac{p_3\lambda_5}{k_5}\right\} - v_2\left\{\frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)}\right\} = 0.$$

Proof: For the block design with incidence matrix N given in (1) we have

$$C = \begin{bmatrix} (a_1+s_1)I_{v_2} - s_1\mathbf{1}_{v_2}\mathbf{1}'_{v_2} & -s_0\mathbf{1}_{v_2}\mathbf{1}'_{v_3} \\ -s_0\mathbf{1}_{v_3}\mathbf{1}'_{v_2} & (a_2+s_2)I_{v_3} - s_2\mathbf{1}_{v_3}\mathbf{1}'_{v_3} \end{bmatrix}$$

where the off-diagonal elements of $C(=c_{ij})$ matrix are:

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{v_3\lambda_2}{(k_2+1)} + \frac{p_2\lambda_4}{k_4} = s_1(\text{say}); i,j \le v_2 \;\&\; i \ne j$$

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)} = s_0(\text{say})\,; i \le v_2, j \ge (v_2+1)$$

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{v_2\lambda_3}{(k_3+1)} + \frac{p_3\lambda_5}{k_5} = s_2(\text{say}); i,j \ge (v_2+1) \;\&\; i \ne j$$

and the diagonal elements of C matrix are:

$$\frac{p_1 r_1(k_1-1)}{k_1} + \frac{v_3 r_2 k_2}{(k_2+1)} + \frac{b_3 k_3}{(k_3+1)} + \frac{p_2 r_4(k_4-1)}{k_4} = a_1(\text{say})$$

and

$$\frac{p_1 r_1(k_1-1)}{k_1} + \frac{b_2 k_2}{(k_2+1)} + \frac{v_2 r_3 k_3}{(k_3+1)} + \frac{p_3 r_5(k_5-1)}{k_5} = a_2(\text{say})$$

Then by Jaggi, Parsad and Gupta[8], $a_1 - (v_2-1)s_1 - v_3 s_0 = 0$ and $a_2 - (v_3-1)s_2 - v_2 s_0 = 0$ i.e.

$$\frac{p_1 r_1(k_1-1)}{k_1} + \frac{v_3 r_2 k_2}{(k_2+1)} + \frac{b_3 k_3}{(k_3+1)} + \frac{p_2 r_4(k_4-1)}{k_4} - (v_2-1)\left\{\frac{p_1\lambda_1}{k_1} + \frac{v_3\lambda_2}{(k_2+1)} + \frac{p_2\lambda_4}{k_4}\right\} - v_3\left\{\frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)}\right\} = 0$$

and

$$\frac{p_1 r_1(k_1-1)}{k_1} + \frac{b_2 k_2}{(k_2+1)} + \frac{v_2 r_3 k_3}{(k_3+1)} + \frac{p_3 r_5(k_5-1)}{k_5} - (v_3-1)\left\{\frac{p_1\lambda_1}{k_1} + \frac{v_2\lambda_3}{(k_3+1)} + \frac{p_3\lambda_5}{k_5}\right\} - v_2\left\{\frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)}\right\} = 0.$$

Hence the proof.

Example 1: Consider five BIB designs with parameters (11,11,5,5,2), (7,7,3,3,1), (4,6,3,2,1), (7,7,4,4,2) and (4,4,3,3,2) respectively. Then taking $p_1 = p_2 = p_3 = 1$, the design D with incidence matrix N as in (1) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 7$, $v_2^* = 4$, $b = 103$, $\mathbf{r}' = \{28\mathbf{1}'_7, 37\mathbf{1}'_4\}$, $\mathbf{k}' = \{5\mathbf{1}'_{11}, 4\mathbf{1}'_{28}, 3\mathbf{1}'_{42}, 4\mathbf{1}'_7, 3\mathbf{1}'_4, 1\mathbf{1}'_7, 1\mathbf{1}'_4\}$.

Corollary 1: In theorem 1, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + v_3 b_2 + v_2 b_3 + p_2 b_4 + p_3 b_5$, $\mathbf{r}' = \{(p_1 r_1 + v_3 r_2 + b_3 + p_2 r_4)\mathbf{1}'_{v_2}, (p_1 r_1 + b_2 + v_2 r_3 + p_3 r_5)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1 b_1}, (k_2+1)\mathbf{1}'_{v_3 b_2}, (k_3+1)\mathbf{1}'_{v_2 b_3}, k_4\mathbf{1}'_{p_2 b_4}, k_5\mathbf{1}'_{p_3 b_5}\}$.

Example 2: In example 1, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = p_2 = p_3 = 1$. The parameters of the design are $v_1^* = 7$, $v_2^* = 4$, $b = 92$, $\mathbf{r}' = \{27\mathbf{1}'_7, 36\mathbf{1}'_4\}$, $\mathbf{k}' = \{5\mathbf{1}'_{11}, 4\mathbf{1}'_{28}, 3\mathbf{1}'_{42}, 4\mathbf{1}'_7, 3\mathbf{1}'_4\}$.

Corollary 2: In theorem 1, if we remove last $p_2 b_4$, $p_3 b_5$, v_2 and v_3 blocks, then again we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + v_3 b_2 + v_2 b_3$, $\mathbf{r}' = \{(p_1 r_1 + v_3 r_2 + b_3)\mathbf{1}'_{v_2}, (p_1 r_1 + b_2 + v_2 r_3)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1 b_1}, (k_2+1)\mathbf{1}'_{v_3 b_2}, (k_3+1)\mathbf{1}'_{v_2 b_3}\}$.

Example 3: Consider three BIB designs with parameters (9,12,4,3,1), (5,10,4,2,1) and (4,4,3,3,2) respectively. Then using Corollary 2 and taking $p_1 = 1$, we get a non-proper non-equireplicate BBPB design D with parameters $v_1^* = 5$, $v_2^* = 4$, $b = 72$, $\mathbf{r}' = \{24\mathbf{1}'_5, 29\mathbf{1}'_4\}$, $\mathbf{k}' = \{3\mathbf{1}'_{12}, 3\mathbf{1}'_{40}, 4\mathbf{1}'_{20}\}$.

Remark 1: In corollary 2, if $k_2 = k_1 - 1$ and $k_3 = k_1 - 1$, then we get a proper BBPB design D with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + v_3 b_2 + v_2 b_3$, $\mathbf{r}' = \{(p_1 r_1 + v_3 r_2 + b_3)\mathbf{1}'_{v_2}, (p_1 r_1 + b_2 + v_2 r_3)\mathbf{1}'_{v_3}\}$, $k = k_1$.

Example 4: Consider three BIB designs with parameters (9,12,4,3,1), (5,10,4,2,1) and (4,6,3,2,1) respectively. Then using remark 1 and taking $p_1 = 1$, we get a proper non-equireplicate BBPB design D with parameters $v_1^* = 5$, $v_2^* = 4$, $b = 82$, $\mathbf{r}' = \{26\mathbf{1}'_5, 29\mathbf{1}'_4\}$, $k = 3$.

Remark 2: In corollary 2, if $v_2 = v_3 = v$ and $v_1 = 2v$, then we get a non-proper non-equireplicate BBPB design D with parameters $v_1^* = v$, $v_2^* = v$, $b = p_1 b_1 + v b_2 + v b_3$, $\mathbf{r}' = \{(p_1 r_1 + v r_2 + b_3)\mathbf{1}'_v, (p_1 r_1 + b_2 + v r_3)\mathbf{1}'_v\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1 b_1}, (k_2+1)\mathbf{1}'_{v b_2}, (k_3+1)\mathbf{1}'_{v b_3}\}$.

Example 5: Consider three BIB designs with parameters (8,14,7,4,3), (4,6,3,2,1) and (4,4,3,3,2) respectively. Then using remark 2 and taking $p_1 = 1$ and $v = 4$, we get a non-proper non-equireplicate BBPB design D with parameters $v_1^* = 4$, $v_2^* = 4$, $b = 54$, $\mathbf{r}' = \{23\mathbf{1}'_4, 25\mathbf{1}'_4\}$, $\mathbf{k}' = \{4\mathbf{1}'_{14}, 3\mathbf{1}'_{24}, 4\mathbf{1}'_{16}\}$.

Remark 3: Following theorems can be proved on the similar lines of theorem 1. So we avoided proofs of the theorems.

Theorem 2: Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters v_L, b_L, r_L, k_L, λ_L such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$, then

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{array}{cccccc} \mathbf{1}'_{v_3} \otimes N_2 & I_{v_2} \otimes \mathbf{1}'_{b_3} & \mathbf{1}'_{p_2} \otimes N_4 & J_{v_2 \times p_3 b_5} & I_{v_2} & O_{v_2 \times v_3} \\ I_{v_3} \otimes \mathbf{1}'_{b_2} & \mathbf{1}'_{v_2} \otimes N_3 & J_{v_3 \times p_2 b_4} & \mathbf{1}'_{p_3} \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{array}\right] \quad (2)$$

is the incidence matrix of a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + v_3b_2 + v_2b_3 + p_2b_4 + p_3b_5 + v_2 + v_3$, $\mathbf{r}' = \{(p_1r_1 + v_3r_2 + b_3 + p_2r_4 + p_3b_5 + 1)\mathbf{1}'_{v_2}, (p_1r_1 + b_2 + v_2r_3 + p_2b_4 + p_3r_5 + 1)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1b_1}, (k_2+1)\mathbf{1}'_{v_3b_2}, (k_3+1)\mathbf{1}'_{v_2b_3}, (k_4+v_3)\mathbf{1}'_{p_2b_4}, (k_5+v_2)\mathbf{1}'_{p_3b_5}, 1\mathbf{1}'_{v_2}, 1\mathbf{1}'_{v_3}\}$ having off-diagonal elements of its C matrix as

$$s_1 = \frac{p_1\lambda_1}{k_1} + \frac{v_3\lambda_2}{(k_2+1)} + \frac{p_2\lambda_4}{(k_4+v_3)} + \frac{p_3b_5}{(k_5+v_2)},$$
$$s_0 = \frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)} + \frac{p_2r_4}{(k_4+v_3)} + \frac{p_3r_5}{(k_5+v_2)},$$
$$s_2 = \frac{p_1\lambda_1}{k_1} + \frac{v_2\lambda_3}{(k_3+1)} + \frac{p_2b_4}{(k_4+v_3)} + \frac{p_3\lambda_5}{(k_5+v_2)},$$

and diagonal elements of C matrix as

$$a_1 = \frac{p_1r_1(k_1-1)}{k_1} + \frac{v_3r_2k_2}{(k_2+1)} + \frac{b_3k_3}{(k_3+1)} + \frac{p_2r_4(k_4+v_3-1)}{(k_4+v_3)} + \frac{p_3b_5(k_5+v_2-1)}{(k_5+v_2)},$$
$$a_2 = \frac{p_1r_1(k_1-1)}{k_1} + \frac{b_2k_2}{(k_2+1)} + \frac{v_2r_3k_3}{(k_3+1)} + \frac{p_2b_4(k_4+v_3-1)}{(k_4+v_3)} + \frac{p_3r_5(k_5+v_2-1)}{(k_5+v_2)}.$$

Example 6: Consider five BIB designs with parameters (9,12,4,3,1), (5,10,6,3,3), (4,4,3,3,2), (5,10,4,2,1) and (4,6,3,2,1) respectively. Then taking $p_1 = 2$, $p_2 = 1$ and $p_3 = 3$, the design D with incidence matrix N as in (2) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 5$, $v_2^* = 4$, $b = 121$, $\mathbf{r}' = \{59\mathbf{1}'_5, 53\mathbf{1}'_4\}$, $\mathbf{k}' = \{3\mathbf{1}'_{24}, 4\mathbf{1}'_{40}, 4\mathbf{1}'_{20}, 6\mathbf{1}'_{10}, 7\mathbf{1}'_{18}, 1\mathbf{1}'_5, 1\mathbf{1}'_4\}$.

Corollary 3: In theorem 2, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + v_3b_2 + v_2b_3 + p_2b_4 + p_3b_5$, $\mathbf{r}' = \{(p_1r_1 + v_3r_2 + b_3 + p_2r_4 + p_3b_5)\mathbf{1}'_{v_2}, (p_1r_1 + b_2 + v_2r_3 + p_2b_4 + p_3r_5)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1b_1}, (k_2+1)\mathbf{1}'_{v_3b_2}, (k_3+1)\mathbf{1}'_{v_2b_3}, (k_4+v_3)\mathbf{1}'_{p_2b_4}, (k_5+v_2)\mathbf{1}'_{p_3b_5}\}$.

Example 7: In example 6, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = 2$, $p_2 = 1$ and $p_3 = 3$. The parameters of the design are $v_1^* = 5$, $v_2^* = 4$, $b = 112$, $\mathbf{r}' = \{58\mathbf{1}'_5, 52\mathbf{1}'_4\}$, $\mathbf{k}' = \{3\mathbf{1}'_{24}, 4\mathbf{1}'_{40}, 4\mathbf{1}'_{20}, 6\mathbf{1}'_{10}, 7\mathbf{1}'_{18}\}$.

Theorem 3: Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters v_L, b_L, r_L, k_L, λ_L such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$, then

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{array}{cccccc} \mathbf{1}'_{v_3} \otimes N_2 & I_{v_2} \otimes \mathbf{1}'_{b_3} & \mathbf{1}'_{v_3} \otimes N_4 & I_{v_2} \otimes \mathbf{1}'_{b_5} & I_{v_2} & O_{v_2 \times v_3} \\ I_{v_3} \otimes \mathbf{1}'_{b_2} & \mathbf{1}'_{v_2} \otimes N_3 & I_{v_3} \otimes \mathbf{1}'_{b_4} & \mathbf{1}'_{v_2} \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{array}\right] \quad (3)$$

is the incidence matrix of a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + v_3b_2 + v_2b_3 + v_3b_4 + v_2b_5 + v_2 + v_3$, $\mathbf{r}' = \{(p_1r_1 + v_3r_2 + b_3 + v_3r_4 + b_5 + 1)\mathbf{1}'_{v_2}, (p_1r_1 + b_2 + v_2r_3 + b_4 + v_2r_5 + 1)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1b_1}, (k_2+1)\mathbf{1}'_{v_3b_2}, (k_3+1)\mathbf{1}'_{v_2b_3}, (k_4+1)\mathbf{1}'_{v_3b_4}, (k_5+1)\mathbf{1}'_{v_2b_5}, 1\mathbf{1}'_{v_2}, 1\mathbf{1}'_{v_3}\}$ having off-diagonal elements of its C matrix as

$$s_1 = \frac{p_1\lambda_1}{k_1} + \frac{v_3\lambda_2}{(k_2+1)} + \frac{v_3\lambda_4}{(k_4+1)},$$
$$s_0 = \frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)} + \frac{r_4}{(k_4+1)} + \frac{r_5}{(k_5+1)},$$
$$s_2 = \frac{p_1\lambda_1}{k_1} + \frac{v_2\lambda_3}{(k_3+1)} + \frac{v_2\lambda_5}{(k_5+1)},$$

and diagonal elements of C matrix as

$$a_1 = \frac{p_1r_1(k_1-1)}{k_1} + \frac{v_3r_2k_2}{(k_2+1)} + \frac{b_3k_3}{(k_3+1)} + \frac{v_3r_4k_4}{(k_4+1)} + \frac{b_5k_5}{(k_5+1)},$$
$$a_2 = \frac{p_1r_1(k_1-1)}{k_1} + \frac{b_2k_2}{(k_2+1)} + \frac{v_2r_3k_3}{(k_3+1)} + \frac{b_4k_4}{(k_4+1)} + \frac{v_2r_5k_5}{(k_5+1)}.$$

Example 8: Consider five BIB designs with parameters (11,11,5,5,2), (6,15,5,2,1), (5,5,4,4,3), (6,6,5,5,4) and (5,10,4,2,1) respectively. Then taking $p_1 = 1$, the design D with incidence matrix N as in (3) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 6$, $v_2^* = 5$, $b = 217$, $\mathbf{r}' = \{71\mathbf{1}'_6, 75\mathbf{1}'_5\}$, $\mathbf{k}' = \{5\mathbf{1}'_{11}, 3\mathbf{1}'_{75}, 5\mathbf{1}'_{30}, 6\mathbf{1}'_{30}, 3\mathbf{1}'_{60}, 1\mathbf{1}'_6, 1\mathbf{1}'_5\}$.

Corollary 4: In theorem 3, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + v_3b_2 + v_2b_3 + v_3b_4 + v_2b_5$, $\mathbf{r}' = \{(p_1r_1 + v_3r_2 + b_3 + v_3r_4 + b_5)\mathbf{1}'_{v_2}, (p_1r_1 + b_2 + v_2r_3 + b_4 + v_2r_5)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1b_1}, (k_2+1)\mathbf{1}'_{v_3b_2}, (k_3+1)\mathbf{1}'_{v_2b_3}, (k_4+1)\mathbf{1}'_{v_3b_4}, (k_5+1)\mathbf{1}'_{v_2b_5}\}$.

Example 9: In example 8, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = 1$. The parameters of the design are $v_1^* = 6$, $v_2^* = 5$, $b = 206$, $\mathbf{r}' = \{70\mathbf{1}'_6, 74\mathbf{1}'_5\}$, $\mathbf{k}' = \{5\mathbf{1}'_{11}, 3\mathbf{1}'_{75}, 5\mathbf{1}'_{30}, 6\mathbf{1}'_{30}, 3\mathbf{1}'_{60}\}$.

Theorem 4: Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters v_L, b_L, r_L, k_L, λ_L such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$, then

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{array}{cccccc} \mathbf{1}'_{v_3} \otimes N_2 & I_{v_2} \otimes \mathbf{1}'_{b_3} & \mathbf{1}'_{p_2} \otimes N_4 & J_{v_2 \times p_3 b_5} & I_{v_2} & O_{v_2 \times v_3} \\ I_{v_3} \otimes \mathbf{1}'_{b_2} & \mathbf{1}'_{v_2} \otimes N_3 & O_{v_3 \times p_2 b_4} & \mathbf{1}'_{p_3} \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{array}\right] \quad (4)$$

is the incidence matrix of a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + v_3 b_2 + v_2 b_3 + p_2 b_4 + p_3 b_5 + v_2 + v_3$, $\mathbf{r}' = \{(p_1 r_1 + v_3 r_2 + b_3 + p_2 r_4 + p_3 b_5 + 1)\mathbf{1}'_{v_2}, (p_1 r_1 + b_2 + v_2 r_3 + p_3 r_5 + 1)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1 \mathbf{1}'_{p_1 b_1}, (k_2 + 1)\mathbf{1}'_{v_3 b_2}, (k_3 + 1)\mathbf{1}'_{v_2 b_3}, k_4 \mathbf{1}'_{p_2 b_4}, (k_5 + v_2)\mathbf{1}'_{p_3 b_5}, 1\mathbf{1}'_{v_2}, 1\mathbf{1}'_{v_3}\}$ having off-diagonal elements of its C matrix as

$$s_1 = \frac{p_1\lambda_1}{k_1} + \frac{v_3\lambda_2}{(k_2+1)} + \frac{p_2\lambda_4}{k_4} + \frac{p_3 b_5}{(k_5+v_2)},$$

$$s_0 = \frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)} + \frac{p_3 r_5}{(k_5+v_2)},$$

$$s_2 = \frac{p_1\lambda_1}{k_1} + \frac{v_2\lambda_3}{(k_3+1)} + \frac{p_3\lambda_5}{(k_5+v_2)},$$

and diagonal elements of C matrix as

$$a_1 = \frac{p_1 r_1(k_1-1)}{k_1} + \frac{v_3 r_2 k_2}{(k_2+1)} + \frac{b_3 k_3}{(k_3+1)} + \frac{p_2 r_4(k_4-1)}{k_4} + \frac{p_3 b_5(k_5+v_2-1)}{(k_5+v_2)},$$

$$a_2 = \frac{p_1 r_1(k_1-1)}{k_1} + \frac{b_2 k_2}{(k_2+1)} + \frac{v_2 r_3 k_3}{(k_3+1)} + \frac{p_3 r_5(k_5+v_2-1)}{(k_5+v_2)}.$$

Example 10: Consider five BIB designs with parameters (11,11,5,5,2), (7,7,3,3,1), (4,4,3,3,2), (7,7,4,4,2) and (4,6,3,2,1) respectively. Then taking $p_1 = p_2 = 1$ and $p_3 = 4$, the design D with incidence matrix N as in (4) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 7$, $v_2^* = 4$, $b = 109$, $\mathbf{r}' = \{50\mathbf{1}'_7, 46\mathbf{1}'_4\}$, $\mathbf{k}' = \{5\mathbf{1}'_{11}, 4\mathbf{1}'_{28}, 4\mathbf{1}'_{28}, 4\mathbf{1}'_7, 9\mathbf{1}'_{24}, 1\mathbf{1}'_7, 1\mathbf{1}'_4\}$.

Corollary 5: In theorem 4, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + v_3 b_2 + v_2 b_3 + p_2 b_4 + p_3 b_5$, $\mathbf{r}' = \{(p_1 r_1 + v_3 r_2 + b_3 + p_2 r_4 + p_3 b_5)\mathbf{1}'_{v_2}, (p_1 r_1 + b_2 + v_2 r_3 + p_3 r_5)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1 \mathbf{1}'_{p_1 b_1}, (k_2 + 1)\mathbf{1}'_{v_3 b_2}, (k_3 + 1)\mathbf{1}'_{v_2 b_3}, k_4 \mathbf{1}'_{p_2 b_4}, (k_5 + v_2)\mathbf{1}'_{p_3 b_5}\}$.

Example 11: In example 10, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = p_2 = 1$ and $p_3 = 4$. The parameters of the design are $v_1^* = 7$, $v_2^* = 4$, $b = 98$, $\mathbf{r}' = \{49\mathbf{1}'_7, 45\mathbf{1}'_4\}$, $\mathbf{k}' = \{5\mathbf{1}'_{11}, 4\mathbf{1}'_{28}, 4\mathbf{1}'_{28}, 4\mathbf{1}'_7, 9\mathbf{1}'_{24}\}$.

Theorem 5: Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters v_L, b_L, r_L, k_L, λ_L such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$, then

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{array}{cccccc} \mathbf{1}'_{v_3} \otimes N_2 & I_{v_2} \otimes \mathbf{1}'_{b_3} & \mathbf{1}'_{p_2} \otimes N_4 & I_{v_2} \otimes \mathbf{1}'_{b_5} & I_{v_2} & O_{v_2 \times v_3} \\ I_{v_3} \otimes \mathbf{1}'_{b_2} & \mathbf{1}'_{v_2} \otimes N_3 & O_{v_3 \times p_2 b_4} & \mathbf{1}'_{v_2} \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{array}\right] \quad (5)$$

is the incidence matrix of a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + v_3 b_2 + v_2 b_3 + p_2 b_4 + v_2 b_5 + v_2 + v_3$, $\mathbf{r}' = \{(p_1 r_1 + v_3 r_2 + b_3 + p_2 r_4 + b_5 + 1)\mathbf{1}'_{v_2}, (p_1 r_1 + b_2 + v_2 r_3 + v_2 r_5 + 1)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1 \mathbf{1}'_{p_1 b_1}, (k_2 + 1)\mathbf{1}'_{v_3 b_2}, (k_3 + 1)\mathbf{1}'_{v_2 b_3}, k_4 \mathbf{1}'_{p_2 b_4}, (k_5 + 1)\mathbf{1}'_{v_2 b_5}, 1\mathbf{1}'_{v_2}, 1\mathbf{1}'_{v_3}\}$ having off-diagonal elements of its C matrix as

$$s_1 = \frac{p_1\lambda_1}{k_1} + \frac{v_3\lambda_2}{(k_2+1)} + \frac{p_2\lambda_4}{k_4},$$

$$s_0 = \frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)} + \frac{r_5}{(k_5+1)},$$

$$s_2 = \frac{p_1\lambda_1}{k_1} + \frac{v_2\lambda_3}{(k_3+1)} + \frac{v_2\lambda_5}{(k_5+1)},$$

and diagonal elements of C matrix as

$$a_1 = \frac{p_1 r_1(k_1-1)}{k_1} + \frac{v_3 r_2 k_2}{(k_2+1)} + \frac{b_3 k_3}{(k_3+1)} + \frac{p_2 r_4(k_4-1)}{k_4} + \frac{b_5 k_5}{(k_5+1)},$$

$$a_2 = \frac{p_1 r_1(k_1-1)}{k_1} + \frac{b_2 k_2}{(k_2+1)} + \frac{v_2 r_3 k_3}{(k_3+1)} + \frac{v_2 r_5 k_5}{(k_5+1)}.$$

Example 12: Consider five BIB designs with parameters (9,12,4,3,1), (5,10,4,2,1), (4,4,3,3,2), (5,5,4,4,3) and (4,6,3,2,1) respectively. Then taking $p_1 = 1$ and $p_2 = 2$, the design D with incidence matrix N as in (5) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 5$, $v_2^* = 4$, $b = 121$, $\mathbf{r}' = \{39\mathbf{1}'_5, 45\mathbf{1}'_4\}$, $\mathbf{k}' = \{3\mathbf{1}'_{12}, 3\mathbf{1}'_{40}, 4\mathbf{1}'_{20}, 4\mathbf{1}'_{10}, 3\mathbf{1}'_{30}, 1\mathbf{1}'_5, 1\mathbf{1}'_4\}$.

Corollary 6: In theorem 5, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + v_3 b_2 + v_2 b_3 + p_2 b_4 + v_2 b_5$, $\mathbf{r}' = \{(p_1 r_1 + v_3 r_2 + b_3 + p_2 r_4 + b_5)\mathbf{1}'_{v_2}, (p_1 r_1 + b_2 + v_2 r_3 + v_2 r_5)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1 \mathbf{1}'_{p_1 b_1}, (k_2 + 1)\mathbf{1}'_{v_3 b_2}, (k_3 + 1)\mathbf{1}'_{v_2 b_3}, k_4 \mathbf{1}'_{p_2 b_4}, (k_5 + 1)\mathbf{1}'_{v_2 b_5}\}$.

Example 13: In example 12, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = 1$ and $p_2 = 2$. The parameters of the design are $v_1^* = 5$, $v_2^* = 4$, $b = 112$, $\mathbf{r}' = \{38\mathbf{1}'_5, 44\mathbf{1}'_4\}$, $\mathbf{k}' = \{3\mathbf{1}'_{12}, 3\mathbf{1}'_{40}, 4\mathbf{1}'_{20}, 4\mathbf{1}'_{10}, 3\mathbf{1}'_{30}\}$.

Theorem 6: Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters $v_L, b_L, r_L, k_L, \lambda_L$ such that $v_2 = v_4, v_3 = v_5$ and $v_1 = v_2 + v_3$, then

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{matrix} \mathbf{1}'_{v_3} \otimes N_2 & I_{v_2} \otimes \mathbf{1}'_{b_3} & \mathbf{1}'_{p_2} \otimes N_4 & I_{v_2} \otimes \mathbf{1}'_{b_5} & I_{v_2} & O_{v_2 \times v_3} \\ I_{v_3} \otimes \mathbf{1}'_{b_2} & \mathbf{1}'_{v_2} \otimes N_3 & J_{v_3 \times p_2 b_4} & \mathbf{1}'_{v_2} \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{matrix}\right] \quad (6)$$

is the incidence matrix of a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + v_3b_2 + v_2b_3 + p_2b_4 + v_2b_5 + v_2 + v_3$, $\mathbf{r}' = \{(p_1r_1 + v_3r_2 + b_3 + p_2r_4 + b_5 + 1)\mathbf{1}'_{v_2}, (p_1r_1 + b_2 + v_2r_3 + p_2b_4 + v_2r_5 + 1)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1b_1}, (k_2+1)\mathbf{1}'_{v_3b_2}, (k_3+1)\mathbf{1}'_{v_2b_3}, (k_4+v_3)\mathbf{1}'_{p_2b_4}, (k_5+1)\mathbf{1}'_{v_2b_5}, 1\mathbf{1}'_{v_2}, 1\mathbf{1}'_{v_3}\}$ having off-diagonal elements of its C matrix as

$$s_1 = \frac{p_1\lambda_1}{k_1} + \frac{v_3\lambda_2}{(k_2+1)} + \frac{p_2\lambda_4}{(k_4+v_3)},$$
$$s_0 = \frac{p_1\lambda_1}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)} + \frac{p_2r_4}{(k_4+v_3)} + \frac{r_5}{(k_5+1)},$$
$$s_2 = \frac{p_1\lambda_1}{k_1} + \frac{v_2\lambda_3}{(k_3+1)} + \frac{p_2b_4}{(k_4+v_3)} + \frac{v_2\lambda_5}{(k_5+1)},$$

and diagonal elements of C matrix as

$$a_1 = \frac{p_1r_1(k_1-1)}{k_1} + \frac{v_3r_2k_2}{(k_2+1)} + \frac{b_3k_3}{(k_3+1)} + \frac{p_2r_4(k_4+v_3-1)}{(k_4+v_3)} + \frac{b_5k_5}{(k_5+1)},$$
$$a_2 = \frac{p_1r_1(k_1-1)}{k_1} + \frac{b_2k_2}{(k_2+1)} + \frac{v_2r_3k_3}{(k_3+1)} + \frac{p_2b_4(k_4+v_3-1)}{(k_4+v_3)} + \frac{v_2r_5k_5}{(k_5+1)}.$$

Example 14: Consider five BIB designs with parameters (11,11,5,5,2), (6,15,5,2,1), (5,5,4,4,3), (6,6,5,5,4) and (5,10,6,3,3) respectively. Then taking $p_1 = 1$ and $p_2 = 3$, the design D with incidence matrix N as in (6) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 6$, $v_2^* = 5$, $b = 205$, $\mathbf{r}' = \{61\mathbf{1}'_6, 99\mathbf{1}'_5\}$, $\mathbf{k}' = \{5\mathbf{1}'_{11}, 3\mathbf{1}'_{75}, 5\mathbf{1}'_{30}, 10\mathbf{1}'_{18}, 4\mathbf{1}'_{60}, 1\mathbf{1}'_6, 1\mathbf{1}'_5\}$.

Corollary 7: In theorem 6, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + v_3b_2 + v_2b_3 + p_2b_4 + v_2b_5$, $\mathbf{r}' = \{(p_1r_1 + v_3r_2 + b_3 + p_2r_4 + b_5)\mathbf{1}'_{v_2}, (p_1r_1 + b_2 + v_2r_3 + p_2b_4 + v_2r_5)\mathbf{1}'_{v_3}\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1b_1}, (k_2+1)\mathbf{1}'_{v_3b_2}, (k_3+1)\mathbf{1}'_{v_2b_3}, (k_4+v_3)\mathbf{1}'_{p_2b_4}, (k_5+1)\mathbf{1}'_{v_2b_5}\}$.

Example 15: In example 14, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = 1$ and $p_2 = 3$. The parameters of the design are $v_1^* = 6$, $v_2^* = 5$, $b = 194$, $\mathbf{r}' = \{60\mathbf{1}'_6, 98\mathbf{1}'_5\}$, $\mathbf{k}' = \{5\mathbf{1}'_{11}, 3\mathbf{1}'_{75}, 5\mathbf{1}'_{30}, 10\mathbf{1}'_{18}, 4\mathbf{1}'_{60}\}$.

Methods of Construction of BBPBUB Designs Using GD and BIB Designs

In this section, we describe some methods of construction of BBPBUB designs making use of the incidence matrices of BIB designs and GD designs, etc.

Theorem 7: Let N_1 be the incidence matrix of a GD design with parameters $v_1 = 2n$, b_1, r_1, k_1, $m = 2$, n, λ_1^*, λ_2^* and N_L (L = 2,3) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters $v_L = n$, b_L, r_L, k_L, λ_L and the n treatments in the BIB designs are the treatments in any of the two groups of the (2,n) association scheme, then

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{matrix} \mathbf{1}'_n \otimes N_2 & I_n \otimes \mathbf{1}'_{b_3} & I_n & O_{n\times n} \\ I_n \otimes \mathbf{1}'_{b_2} & \mathbf{1}'_n \otimes N_3 & O_{n\times n} & I_n \end{matrix}\right] \quad (7)$$

is the incidence matrix of a BBPB design D with unequal block sizes with parameters $v_1^* = n$, $v_2^* = n$, $b = p_1b_1 + nb_2 + nb_3 + 2n$, $\mathbf{r}' = \{(p_1r_1 + nr_2 + b_3 + 1)\mathbf{1}'_n, (p_1r_1 + b_2 + nr_3 + 1)\mathbf{1}'_n\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1b_1}, (k_2+1)\mathbf{1}'_{nb_2}, (k_3+1)\mathbf{1}'_{nb_3}, 1\mathbf{1}'_n, 1\mathbf{1}'_n\}$ having off-diagonal elements of its C matrix as

$$s_1 = \frac{p_1\lambda_1^*}{k_1} + \frac{n\lambda_2}{(k_2+1)},$$
$$s_0 = \frac{p_1\lambda_2^*}{k_1} + \frac{r_2}{(k_2+1)} + \frac{r_3}{(k_3+1)},$$
$$s_2 = \frac{p_1\lambda_1^*}{k_1} + \frac{n\lambda_3}{(k_3+1)},$$

and diagonal elements of C matrix as

$$a_1 = \frac{p_1r_1(k_1-1)}{k_1} + \frac{nr_2k_2}{(k_2+1)} + \frac{b_3k_3}{(k_3+1)},$$
$$a_2 = \frac{p_1r_1(k_1-1)}{k_1} + \frac{b_2k_2}{(k_2+1)} + \frac{nr_3k_3}{(k_3+1)}.$$

Example 16: Consider a GD design D_1 SR9 (Clatworthy[10]) with parameters $v_1 = 8$, $b_1 = 16$, $r_1 = 4$, $k_1 = 2$, $m = 2$, $n = 4$, $\lambda_1^* = 0$, $\lambda_2^* = 1$ and two BIB designs with parameters D_2(4,4,3,3,2) and D_3(4,6,3,2,1) respectively. Then taking $p_1 = 1$, the design D with incidence matrix N as in (7) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 4$ (1,3,5,7), $v_2^* = 4$ (2,4,6,8), $b = 64$, $\mathbf{r}' = \{23\mathbf{1}'_4, 21\mathbf{1}'_4\}$, $\mathbf{k}' = \{2\mathbf{1}'_{16}, 4\mathbf{1}'_{16}, 3\mathbf{1}'_{24}, 1\mathbf{1}'_4, 1\mathbf{1}'_4\}$.

Corollary 8: In theorem 7, if we remove last 2n blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = n$, $v_2^* = n$, $b = p_1b_1 + nb_2 + nb_3$, $\mathbf{r}' = \{(p_1r_1 + nr_2 + b_3)\mathbf{1}'_n, (p_1r_1 + b_2 + nr_3)\mathbf{1}'_n\}$, $\mathbf{k}' = \{k_1\mathbf{1}'_{p_1b_1}, (k_2+1)\mathbf{1}'_{nb_2}, (k_3+1)\mathbf{1}'_{nb_3}\}$.

Example 17: In example 16, if we remove last 2n blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = 1$. The parameters of the design are $v_1^* = 4$ (1,3,5,7), $v_2^* = 4$ (2,4,6,8), $b = 56$, $\mathbf{r}' = \{22\mathbf{1}'_4, 20\mathbf{1}'_4\}$, $\mathbf{k}' = \{2\mathbf{1}'_{16}, 4\mathbf{1}'_{16}, 3\mathbf{1}'_{24}\}$.

Theorem 8: Let N_1 be the incidence matrix of a GD design with parameters $v_1 = 2n$, b_1, r_1, k_1, $m = 2$, n, λ_1^*, λ_2^* and N_2 be the $v_L \times b_L$ incidence matrix of a BIB design with parameters $v_2 = n, b_2, r_2, k_2, \lambda_2$ and the n treatments in the BIB design are the treatments in any of the two groups of the (2,n) association scheme, then

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{matrix} \mathbf{1}'_{p_2} \otimes N_2 & J_{n\times n} & I_n & O_{n\times n} \\ J_{n\times p_2 b_2} & J_{n\times n} & O_{n\times n} & I_n \end{matrix}\right] \quad (8)$$

is the incidence matrix of a BBPB design D with unequal block sizes with parameters $v_1^* = n$, $v_2^* = n$, $b = p_1 b_1 + p_2 b_2 + 3n$, $\mathbf{r}' = \{(p_1 r_1 + p_2 r_2 + n + 1)\mathbf{1}'_n, (p_1 r_1 + p_2 b_2 + n + 1)\mathbf{1}'_n\}$, $\mathbf{k}' = \{k_1 \mathbf{1}'_{p_1 b_1}, (k_2 + n)\mathbf{1}'_{p_2 b_2}, 2n\mathbf{1}'_n, 1\mathbf{1}'_n, 1\mathbf{1}'_n\}$ having off-diagonal elements of its C matrix as

$$s_1 = \frac{p_1 \lambda_1^*}{k_1} + \frac{p_2 \lambda_2}{(k_2 + n)} + \frac{1}{2},$$
$$s_0 = \frac{p_1 \lambda_2^*}{k_1} + \frac{p_2 r_2}{(k_2 + n)} + \frac{1}{2},$$
$$s_2 = \frac{p_1 \lambda_1^*}{k_1} + \frac{p_2 b_2}{(k_2 + n)} + \frac{1}{2},$$

and diagonal elements of C matrix as

$$a_1 = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + n - 1)}{(k_2 + n)} + \frac{(2n - 1)}{2},$$
$$a_2 = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 b_2 (k_2 + n - 1)}{(k_2 + n)} + \frac{(2n - 1)}{2}.$$

Example 18: Consider a GD design D_1 R94 (Clatworthy[10]) with parameters $v_1 = 6$, $b_1 = 6$, $r_1 = 4$, $k_1 = 4$, $m = 2$, $n = 3$, $\lambda_1^* = 3$, $\lambda_2^* = 2$ and a BIB design with parameters D_2(3,3,2,2,1). Then taking $p_1 = 2$ and $p_2 = 4$, the design D with incidence matrix N as in (8) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 3$ (1,3,5), $v_2^* = 3$ (2,4,6), $b = 33$, $\mathbf{r}' = \{20\mathbf{1}'_3, 24\mathbf{1}'_3\}$, $\mathbf{k}' = \{4\mathbf{1}'_{12}, 5\mathbf{1}'_{12}, 6\mathbf{1}'_3, 1\mathbf{1}'_3, 1\mathbf{1}'_3\}$.

Corollary 9: In theorem 8, if we remove last 2n blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = n$, $v_2^* = n$, $b = p_1 b_1 + p_2 b_2 + n$, $\mathbf{r}' = \{(p_1 r_1 + p_2 r_2 + n)\mathbf{1}'_n, (p_1 r_1 + p_2 b_2 + n)\mathbf{1}'_n\}$, $\mathbf{k}' = \{k_1 \mathbf{1}'_{p_1 b_1}, (k_2 + n)\mathbf{1}'_{p_2 b_2}, 2n\mathbf{1}'_n\}$.

Example 19: In example 18, if we remove last 2n blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = 2$ and $p_2 = 4$. The parameters of the design are $v_1^* = 3$ (1,3,5), $v_2^* = 3$ (2,4,6), $b = 27$, $\mathbf{r}' = \{19\mathbf{1}'_3, 23\mathbf{1}'_3\}$, $\mathbf{k}' = \{4\mathbf{1}'_{12}, 5\mathbf{1}'_{12}, 6\mathbf{1}'_3\}$.

Conclusion

A number of balanced bipartite block designs for comparing a set of test treatments to a set of control treatments generated by the new methods of construction given here. The methods are flexible enough to incorporate number of incidence matrices of BIB and other designs. The designs so constructed are found to have applications in agricultural and industrial experiments. Thus, the methods presented here will be of both statistical and combinatorial usefulness.

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