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The Bandwidth of Maximal Planar Graphs

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Abstract

Bandwidth labelling is one of the interesting labellings of graphs. The bandwidth of a simple graph is the minimum of all possible maximum differences of adjacent labelled vertices. We consider bandwidth calculation for the maximal planar graphs. We characterize bandwidth of graphs via power of paths and show embedding of some planar graphs in power of path graphs. We also show alternatively that all graphs having bandwidth at most 3 are planar and prove that P_n^3 is maximal planar graph.

Keywords: Labelling of graphs, bandwidth of graphs, planar graphs, bipartite graphs, and maximal planar graphs.

Introduction

Graph labelling is one of the well worked areas of graph theory. Simple graphs have a variety of labellings like graceful labellings, cordial, elegant, magic labellings and many more. The bandwidth labelling is one of the interesting labellings of simple graphs. However, the problem to determine the bandwidth of a general graph is NP-complete[1]. Since then, this kind of labelling has perhaps attracted the most attention in the literature. Bandwidth labelling of graphs has a wide range of engineering applications like data security, mobile telecommunication systems, cryptography, various coding theory problems, communication networks and many more[2]. Suppose that G is a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For undefined terminology of graph theory we refer the reader to the book of Jonathan L. Gross and Jay Yellen[3]. A labelling f is a bijection $f : V \to X_n$ where $|V| = n$ and $X_n = \{1, 2, \cdots, n\}$. Let $F = \{ f : V \to X_n, f \text{ a bijection}\}$. We define the bandwidth of a labelling f of G as $BW_f(G) = \max_{uv \in E} |f(u) - f(v)|$. The bandwidth of G is given by $BW(G) = \min_{f \in F} \{\max_{uv \in E} |f(u) - f(v)|\}$. We say that f is a bandwidth labelling of G if $BW_f(G) = BW(G)$. Bandwidth of some basic graphs can be found in references[4,5] and bandwidth of direct product of graphs has been discussed in references[6-8].

Characterizing Bandwidth via Powers of the Path Graph

We start with the following definition:

Definition 1.
Let G and H be graphs. An injection $\alpha : V(G) \to V(H)$ is called an embedding if α preserves adjacency; that is α satisfies

$$uv \in E(G) \to \alpha(u)\alpha(v) \in E(H) \qquad (2.1)$$

in which case we say α embeds G in H and write $G \mapsto H$.

Note that there is no requirement that α preserves non-adjacency; i.e. it may happen that $u, v \in V(G)$ with $uv \notin E(G)$ but $\alpha(u)\alpha(v) \in E(H)$. Note also that the composition of embeddings is an embedding.

Let $m, n \in \overset{+}{Z} = \{1, 2, 3, \cdots\}$. The m^{th} power of a graph G with $|G| = n$ on n vertices, denoted by G^m, is the graph having vertex-set $V(G^m) = V(G)$ and edge-set $E(G^m) = \{uv : d_G(u, v) \le m\}$, where $d_G(u, v)$ is the length of a path from *u* to *v* in G of minimum value.

Let $G = P_n$ in (b) and $m, n \in \omega = \overset{+}{Z}$. Then $P_n^{\,m}$ is the graph with $V(P_n^{\,m}) = \{1, 2, 3, \cdots, n\}$ and $ij \in E(P_n^{\,m})$ if $|i - j| \le m$. ω^m :is the *m*-uniform graph on $V(\omega^m) = \{1, 2, 3, \cdots\}$ and $ij \in E(\omega^m)$ if $|i - j| \le m$;

ζ^m: is the *m*-uniform graph over the integers; $V(\zeta^m) = Z$ and $ij \in E(\zeta^m)$, with $ij \in E(\zeta^m)$ if $|i - j| \le m$.

For example for a path graph $G = P_n$, $P_n^{\,1} = P_n$ is the path P_n on n vertices, while $P_n^{\,m} = K_n$, the complete graph on n vertices, for all $m \geq n-1$.

Proposition[3] 2. *Let f be a bandwidth labelling of a graph G. Then every 1 in the corresponding adjacency matrix, M_f lies in the band containing the $BW_f(G)$ diagonals above and the $BW_f(G)$ diagonals below the leading diagonal.*

It can be observed from the above definition that for the natural labelling $f\,(v_i) = i$, the adjacency matrix M_f of $P_n^{\,m}$ with vertex sequence $(v_1, v_2, \cdots, v_n)$, has all 1's on its first m diagonals that are closer to the leading diagonal, for $1 \leq m \leq n-1$ and zero everywhere else.

Using this observation and proposition 2 we have the following:

Proposition[3] 3. *For the path graph P_n, $BW(P_n^{\,m}) = m$ for $1 \leq m \leq n-1$.*

Proposition[3] 4. *Let G be a graph on n vertices. Then $BW(G) \leq m$ if and only if G is a sub graph of $P_n^{\,m}$.*

From these propositions immediately follows:

Corollary[3] 5. *Let G be a graph on n vertices. Then $BW(G) = m$ if and only if m is the smallest integer such that G is a sub graph of $P_n^{\,m}$.*

The next result relates a graph G on n vertices of at most bandwidth m to the power of path graphs as defined in Definition 1.

Lemma 6. *The following are equivalent for a graph G on n vertices: i. $BW(G) \leq m$; ii. $G \mapsto P^m$; iii. $G \mapsto \omega^m$; iv. $G \mapsto \zeta^m$;*

Proof. Let G be a graph on n vertices. Then **(i)⇒(ii).** Assume that $BW(G) \leq m$, so $\exists$ a bijection $f: V(G) \to X_n$ (we may write this as $f: V \to X_n$) such that $\max_{uv \in E} |f(u) - f(v)| = k \leq m$. Since $X_n = V(P_n^m)$ we have $f: V(G) \to V(P_n^m)$. Moreover, let $uv \in E(G)$. Then $|f(u) - f(v)| \leq m \Rightarrow f(u)f(v) \in E(P_n^m)$. Hence $f: G \mapsto P_n^m$.

(ii)⇒(iii). Suppose $\alpha: G \mapsto P_n^m$. Let $\eta: V(P_n^m) = X_n \to V(\omega^n) = W$ be the natural mapping defined by $\eta(i) = i$. Hence η is an injection and if $i, j \in X_n$ with $ij \in E(P_n^m)$ then,

$$|i-j| \leq m \Rightarrow |\eta(i) - \eta(j)| \leq m \Rightarrow \eta(i)\eta(j) = ij \in E(\omega^m)$$

Hence $\eta: P_n^m \mapsto \omega^m$ and so $\alpha\eta: G \mapsto \omega^m$, as required.

(iii) ⇒ (iv). Let $\alpha: G \mapsto \omega^m$ and similar to (iii) by introducing the natural embedding $\eta: V(\omega^m) \mapsto V(\zeta^m)$ by $\eta(i) = i$. We then have $\alpha\eta: G \mapsto \zeta^m$.

(iv)⇒(i). Let $\alpha: G \mapsto \zeta^m$, and let us write $\alpha(G) = \{i_1 < i_2 < \cdots < i_n\}$ say, observe that $i_{j+t} - i_j \geq t$, for $1 \leq j \leq j+t \leq n$. Define $f: V(G) \to X_n$ by $f(u) = j$, where $\alpha(u) = i_j$. Since α is injection, then so is f. Now take any $uv \in E(G)$, whence $\alpha(u)\alpha(v) \in E(\zeta^m) \Rightarrow |\alpha(u) - \alpha(v)| \leq m$.

Let $\alpha(u) = i_j$, $\alpha(v) = i_k$ say. Then

$$|f(u) - f(v)| = |j-k| \leq |i_j - i_k| = |\alpha(u) - \alpha(v)| \leq m.$$

It follows that $BW(G) \leq m$, which completes the proof.

Planarity and Bandwidth of Graphs

Here we will characterize the planarity of graphs via their bandwidth by applying Lemma 6 to a graph G such that $BW(G) \leq k$, for some k. The following corollaries immediately follow from Lemma 6.

Corollary 7. *$BW(G) = k$, if and only if k is the least positive integer such that $G \mapsto \zeta^k$.*

$k = 1$, ζ^1:

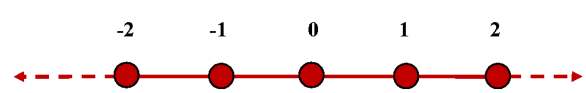

Figure 1

Corollary 8: *$BW(G) = 1$, if and only if the components of G are paths.*

$k = 2, \zeta^2$:

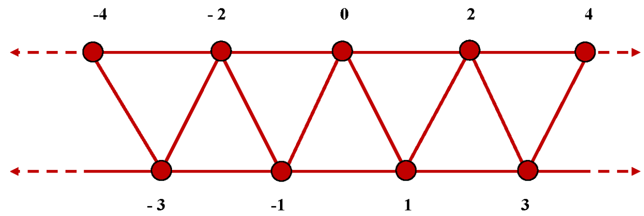

Figure 2

Again ζ^2 is an (infinite) planar graph as can be seen in figure-2 and $BW(G) \le 2$ if and only if G is a subgraph of ζ^2, so in particular G is also planar.

$k = 3, \zeta^3$:

For $k = 3$, ζ^k is a planar graph. The next figure shows this in two stages. The first graph in figure 3 shows its subgraph ω^3. The triangle 012 contains the subgraph $\zeta^3 - \omega^3$ as illustrated in the second figure-4.

If we take the subgraph of ω^3 on the vertex set X_n we get a plane version of P_n^3 .Each vertex of P_n^3 has degree 6 except for $1(3), 2(4), 3(5); \ n(3), n-1(4), n-2(5)$, where the bracketed numbers are the respective degrees. The degree sum of P_n^3 is evidently $6(n-6)+2(3+4+5)=6n-36+24=6n-12$. Hence $|E(P_n^3)| = 3n-6$ (the count here assumes that $n \ge 6$, but the expression is also valid for all $n \ge 3$, as is seen by inspecting cases). As is well known, for any planar graph on n vertices, $|E| \le 3n-6$, and so each P_n^3 is a maximal (with respect to edges) planar graph. This is equivalent to each face being a triangle.

Each subgraph of ω^3 on any 5 (consecutive) vertices, has a maximum of 9 edges and is planar as each subgraph $K_5 - \{e\}$ is planar for the deletion of any edge e of K_5. Similarly a subgraph for ζ^3 on any 6 consecutive vertices has a maximum of 8 edges of the bipartite subgraph $K_{3,3}$ of ω^3. Hence ω^3 does not contain any subdivision of $K_{3,3}$ or K_5 and hence is a planar. In conclusion.

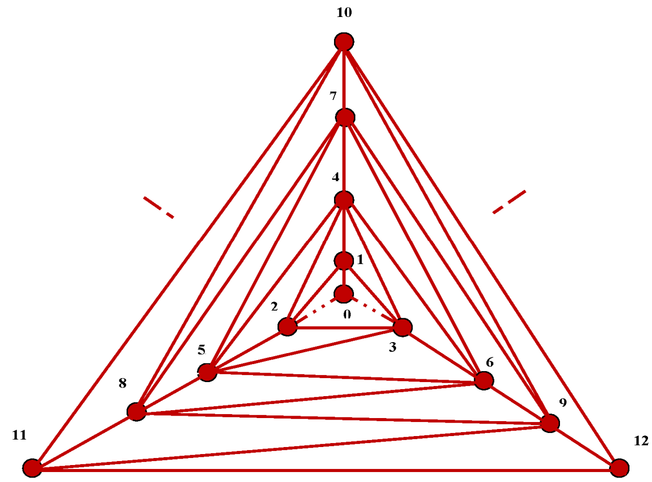

Figure 3

Proposition 9. ζ^k is planar if and only if $k \le 3$.

Proof.⇒ Suppose that $k \ge 4$. Then ζ^k contains K_5 as a subgraph (as any 5 consecutive labelled vertices will be pairwise adjacent to each other) which is non planar and so nor is ζ^k hence for $k \ge 4$, ζ^k is non-planar.

Conversely ζ^3 is planar by the above discussion and since $\zeta^k \mapsto \zeta^3$ if $k \le 3$, the result follows.

Corollary 10. For any graph $G, BW(G) \le 3$ implies G is planar.

Proof. If $BW(G) \le 3$ then $G \overset{Lamma\ 6}{\mapsto} \zeta^3$, which is planar. This is stated in the contrapositiveform byJ. Chvatalova[4] as:

Theorem 11. If G is non-planar, then $BW(G) \le 4$.

Although Theorem 11 is contained in reference[4], but it gives no explanation or proper answerto the question as to why P_n^3 is planar. It does draw a simple picture.

An Alternative Proof for the Planarity of $G = P_n^3$

The subdivision of some edge e with endpoints $\{a,b\}$ yields a graph containing one new vertex c, and with an edge set replacing e by two new edges, $\{a,c\}$ and $\{c,b\}$. A subdivision of a graph G is a graph resulting from the subdivision of edges in G. We also recall the Kuratowski's theorem that states that a finite graph G is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

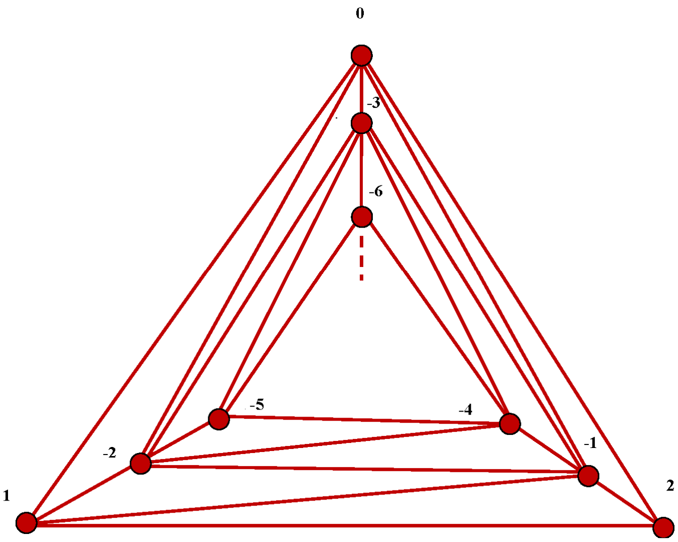

Figure 4

Proposition 12. P_n^3 *is planar.*

Proof. We use the Kuratowski's Theorem to prove that the graph $G = P_n^3$ is planar. For the K_5 subdivision of G it is enough to show that there is no set S of 5 vertices of G that are pairwise connected by a set of vertex disjoint paths.

Let $S = \{a, b, c, d, e\}$ be any set of 5 vertices of G and without loss assume that $a < b < c < d < e$. Any path from a to any other vertex of S must pass through one of the vertices $a+1$, $a+2$ and $a+3$. It follows that if we take any set of 4 paths from a to the other members of S, two of them pass through the same vertex $a+i$ $(1 \le i \le 3)$ and so there is no disjoint set of paths from a to all the other members of S, as required.

If on the other hand the $K_{3,3}$ subdivision were present in G then there would exist 6 vertices $a < b < c$ and $d < e < f$ and a set P of 9 vertex disjoint paths from each of the members of the first set to each of the members of the second set. Without loss we may assume that a is the least of the six integers involved (but we cannot assume that $b < d$ for instance). Any path from a to d must pass through at least one of the vertices $a+1$, $a+2$, $a+3$, and at least one of each set of three consecutive vertices from this point onwards, until d is reached. Hence, if we consider the (vertex disjoint) paths in P from a to each of d, e and f we see that they collectively include all of the vertices from a up to and including $a+3m, m \in Z$ where d lies in the set $I = \{a+3m-2, a+3m-1, a+3m\}$.

It follows that neither b nor c lie in the interval from a up to and including $a+3m$, so that $d < b < c$. The paths in P from b and from c to d each contain (distinct) vertices in the interval
$J = \{a+3m+1, a+3m+2, a+3m+3\}$.

Now d lies in I and the two other vertices in I are on the paths in P from a to e and to f respectively. Hence e lies in I for otherwise, since $e < f$, two vertices of J lie on paths in P from a to e and to f, whence there exists a vertex v in J that lies on a path in P from b or from c to d and also on a path from a to e or from a to f, contradicting the assumption that all the paths of P are vertex disjoint. Hence we do indeed have that both d and e lie in I.

But then there exist two vertices in J on paths in P from b and from c to e. There must then be some vertex u in J that is both on a path in P from b or from c to e and on a path from b or c to d, which again contradicts vertex disjointness of the set of paths. Hence $K_{3,3}$ is not a subdivision of G either and therefore G is planar by Kuratowski's Theorem.

Conclusion

The bandwidth labelling is one of the most interesting labellings of finite simple graphs that has enormous applications in various fields including coding theory, telecommunication and VLSI designs and many more. This article describes the maximal planar graphs and shows that if the bandwidth of a graph is at most 3 then it is necessarily a planar graph.We have proved that graphs having bandwidth at most 3 can be embedded in P_n^3 and hence P_n^3 is the maximal planar graph. However, this is not sufficient that is, planar graphs could have bandwidth more than 3, e.g. the bandwidth of the planar graph $P_m \times P_n$ is given as[6] $BW(P_m \times P_n) = \min\{m, n\}$,is more than 3 unless $\min\{m, n\} \le 3$. On the other hand the bipartite graph, $K_{3,3}$is evidently non planar having bandwidth 4.

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