Research Journal of Recent Sciences ISSN 2277-2502
Vol. **3(5),** 83-85, May **(2014)** *Res.J.Recent Sci.*

The Existence of General Solution of Non-linear partial differential equation in General case in complex space

N. Taghizadeh and S. Norozpour*
Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O Box 1914, Rasht, IRAN

Available online at: www.isca.in, www.isca.me
Received 23rd July 2013, revised 6th October 2013, accepted 20th November 2013

Abstract

In this paper, we discuss on the existence of general solution of non-linear partial differential equation $F\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}\right)=0$ *in complex space by using Fixed point theorem and contraction function.*

Keywords: Partial differential equation, complex space, fixed point theorem, Holomorphic function, contraction function

Introduction

Supposed $D \subseteq C$ and $S_D < \infty$, weakly and strongly singular operators T_D and Π_D are defined as follows:

$$T_D f(z) = \frac{-1}{\pi}\iint_D \frac{f(\xi)}{\xi - z} d\zeta d\eta,$$

$$\Pi_D f(z) = \frac{-1}{\pi}\iint_D \frac{f(\xi)}{(\xi - z)^2} d\zeta d\eta,$$

so that $\xi = \zeta + i\eta$, $z = x + iy$ and $\frac{\partial T_D f(z)}{\partial \bar{z}} = f(z)$ and $\frac{\partial T_D f(z)}{\partial z} = \Pi_D f(z)$ and if f is a bounded in D then $T_D f(z)$ is bounded and Holder Continuous[1].

An interest to the developing of this area is connected first of all with different types applications of generalized analytic functions. The most known constructions are those generalized analytic functions of Vekua type[2] defined as a solution to elliptic system of differential equations in complex domains containing so called (F, G)-derivatives. Different methods are developed for the study of the corresponding differential equations and the corresponding boundary value problems.

Now, we represent the most results in this direction.

In relation to this subject, mentioned topics have been studied as follows: i. Mamourian A, Esrafilian E, Taghizadeh.N. On the existence of general solution of first order elliptic systems by Fixed-point theorem[3]. ii. N.Taghizadeh. On the uniqueness of solution of first order non-linear complex elliptic systems of partial differential equations in Sobolev space[4]. iii. N.Taghizadeh, M.Akbari, the existence and uniqueness of solution of non-linear partial differential equations by Fixed-point theorem in sobolev space and their results[5].

Existence of general solution in complex space.

Firstly, we assume that $w \in C_\alpha(D)$ $(0 < \alpha < 1)$ is an arbitrary solution of

$$F(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}) = 0.$$

We define two functions as follows:

$$G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}\right) = \frac{\partial w}{\partial z} - T_D F\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}\right) \quad (2)$$

$$\varphi(z) = w(z) - T_D G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}\right). \quad (3)$$

By differentiating φ partially with respect to z and $\bar{z}$ we obtain that:

$$\frac{\partial \varphi}{\partial \bar{z}} = \frac{\partial w}{\partial \bar{z}} - G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}\right) = 0 \quad (4)$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial w}{\partial z} - \Pi_D G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}\right). \quad (5)$$

Since

$$G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}\right) \in C_\alpha(D) \quad (0 < \alpha < 1)$$

the following estimates hold:

$$\|\varphi\|_{\alpha,D} \le \|w\|_{\alpha,D} + \left\|T_D G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar{z}}\right)\right\|_{\alpha,D} \quad (6)$$

$$\left\|\frac{\partial\varphi}{\partial z}\right\|_{\alpha,D} \le \left\|\frac{\partial w}{\partial z}\right\|_{\alpha,D} + \left\|\Pi_D G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar z}\right)\right\|_{\alpha,D} . \quad (7)$$

It follows from the equation (4) and Wely's lemma[1] that φ is holomorphic in D and it belongs to the complex space according to (6), so, we deduce that, if w is a solution of (1) then it is in the form:

$$w(z) = \varphi(z) + T_D G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar z}\right) \quad (8)$$

where φ is holomorphic in D.

Theorem 1.2: supposed $D \subseteq \mathbb{C}$ and $S_D < \infty$. A function $w \in C_\alpha(D)$ $(0 < \alpha < 1)$ is a solution of the partial differential equation (1) if and only if for a function $\varphi \in C_\alpha(D)$ and holomorphic in D, (w,h) satisfied the system below:

$$\begin{cases} w(z) = \varphi(z) + T_D G\left(z,w,h,\frac{\partial w}{\partial \bar z}\right) \\ h(z) = \varphi'(z) + \Pi_D G\left(z,w,h,\frac{\partial w}{\partial \bar z}\right) \end{cases} \quad (9)$$

Proof: Firstly, supposed w is a solution of (1), as it is proved, w is in form (8) where φ is holomorphic in $C_\alpha(D)$ and G is in form (2).

By differentiating (8) partially with respect to z we obtain that:

$$\frac{\partial w}{\partial z} = \frac{\partial \varphi}{\partial z} + \Pi_D G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar z}\right)$$

we denote: $\frac{\partial w}{\partial z} = h$

then $h(z) = \varphi'(z) + \Pi_D G\left(z,w,h,\frac{\partial w}{\partial \bar z}\right)$

so (w,h) is a solution of the system (9).

Now, we suppose (w,h) is a solution of the following system. By differentiating the first equation in this system partially with the respect to z and $\bar z$ we obtain:

$$\begin{cases} \frac{\partial w}{\partial \bar z} = G\left(z,w,h,\frac{\partial w}{\partial \bar z}\right) \\ \frac{\partial w}{\partial z} = \varphi'(z) + \Pi_D G\left(z,w,h,\frac{\partial w}{\partial \bar z}\right) = h(z) \end{cases} \quad (10)$$

on substitution $\frac{\partial w}{\partial z} = h$ in (10) we obtain the following result.

Remark: We denote the set of all pairs (w,h) for which both w and h belong to the space $C_\alpha(D)$. The norm in this space shall be defined as follows:

$$\|(w,h)\|_{\alpha,D} = Max\,(\|w\|_{\alpha,D}, \|h\|_{\alpha,D})$$

thus making $C_\alpha(D)$ a Banach space.

Theorem 2.2: If

i. The domain D has a finite area.

ii. As a function of the variables $z \in D \subseteq \mathbb{C}$, $G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar z}\right)$ is a continuous of its variables.

iii. The function G satisfy in the Lipschitz condition of the form:

$$\left|G\left(z,w,h,\frac{\partial w}{\partial \bar z}\right) - G\left(z,\tilde w,\tilde h,\frac{\partial \tilde w}{\partial \bar z}\right)\right| \le L_1|w-\tilde w| + L_2|h-\tilde h|$$

almost everywhere in D, whereas constant L_2 is strictly less than 1 and L_1 is arbitrary positive number.

There exist $w,h \in \Omega_p(D) = \{(w,h) \mid w,h \in C_\alpha(D)\}$ such that $G\left(z,w,\frac{\partial w}{\partial z},\frac{\partial w}{\partial \bar z}\right) \in \Omega_p(D)$.

Then differential equation (1) has unique solution in the complex space.

Proof: We denote $\Omega_p(D) = \{(w,h) \mid w,h \in C_\alpha(D)\}$

and we define the norm as follows

$$\|(w,h)\|_{\alpha,D} = Max\,(\|w\|_{\alpha,D}, \|h\|_{\alpha,D})$$

then the set $\Omega_p(D)$ is a Banach space.

For a pair $(w,h) \in \Omega_p(D)$ we define an operator L as follows:

$$L : \Omega_p(D) \to \Omega_p(D)$$
$$L(w,h) = (W,H)$$

where

$$\begin{cases} W(z) = \varphi(z) + T_D G\left(z, w, h, \dfrac{\partial w}{\partial \bar{z}}\right) \\ H(z) = \varphi'(z) + \Pi_D G\left(z, w, h, \dfrac{\partial w}{\partial \bar{z}}\right) \end{cases} \quad (12)$$

where φ is holomorphic in D and it belongs to $C_\alpha(D)$.

We will prove L satisfy in Lipschitz condition. Supposed $(w,h),(\tilde{w},\tilde{h}) \in \Omega_p(D)$ so:

$$L(w,h) = (W,H)$$

where

$$\begin{cases} W(z) = \varphi(z) + T_D G\left(z, w, h, \dfrac{\partial w}{\partial \bar{z}}\right) \\ H(z) = \varphi'(z) + \Pi_D G\left(z, w, h, \dfrac{\partial w}{\partial \bar{z}}\right) \end{cases}$$

and

$$L(\tilde{w},\tilde{h}) = (\tilde{W},\tilde{H})$$

where

$$\begin{cases} \tilde{W}(z) = \varphi(z) + T_D G\left(z, \tilde{w}, \tilde{h}, \dfrac{\partial \tilde{w}}{\partial \bar{z}}\right) \\ \tilde{H}(z) = \varphi'(z) + \Pi_D G\left(z, \tilde{w}, \tilde{h}, \dfrac{\partial \tilde{w}}{\partial \bar{z}}\right) \end{cases} . \quad (13)$$

Then

$$\left\| L(w,h) - L(\tilde{w},\tilde{h}) \right\| = \left\| (W,H) - (\tilde{W},\tilde{H}) \right\|$$
$$= \left\| (W - \tilde{W}),(H - \tilde{H}) \right\|$$
$$= Max\left(\left\| W - \tilde{W} \right\|, \left\| H - \tilde{H} \right\| \right) .$$

According to the systems (12) and (13):

$$\left\| W - \tilde{W} \right\| = \left\| T_D G\left(z, w, h, \frac{\partial w}{\partial \bar{z}}\right) - T_D G\left(z, \tilde{w}, \tilde{h}, \frac{\partial \tilde{w}}{\partial \bar{z}}\right) \right\|$$
$$= \left\| T_D\left(G\left(z, w, h, \frac{\partial w}{\partial \bar{z}}\right) - G\left(z, \tilde{w}, \tilde{h}, \frac{\partial \tilde{w}}{\partial \bar{z}}\right)\right) \right\|$$
$$\le A(D)\left(L_1 \left\| w - \tilde{w} \right\| + L_2 \left\| h - \tilde{h} \right\|\right)$$
$$\le A(D)(L_1 + L_2)\left(\left\| (w,h) - (\tilde{w},\tilde{h}) \right\|\right)$$

and similarly:

$$\left\| H - \tilde{H} \right\| \le B(D)(L_1 + L_2)\left(\left\| (w,h) - (\tilde{w},\tilde{h}) \right\|\right).$$

This mean that:

$$\left\| (W,H) - (\tilde{W},\tilde{H}) \right\| \le (L_1 + L_2)\max(A(D),B(D)) \left\| (w,h) - (\tilde{w},\tilde{h}) \right\|.$$

We denote $K = (L_1 + L_2)\max(A(D),B(D))$

then $\left\| (W,H) - (\tilde{W},\tilde{H}) \right\| \le K \left\| (w,h) - (\tilde{w},\tilde{h}) \right\|$.

Now, if $0 \le K \le 1$ then the operator L is contraction function in $\Omega_p(D)$ and such as, there exists a unique fixed element (w,h) of the operator L which is also a solution of (9). The corresponding w is then by theorem (1.2) a general solution of differential equation (1).

Conclusion

In this paper, we discuss on the existence of general solution of Non-linear partial differential equation in general case in complex space. We deduce that the proposed method can be extended in other spaces.

References

1. Vekua I.N., Elliptic first order system of partial differential equations boundary value problems, Matem, Sbornik, **31**, 217-314 **(1952)**
2. Vekua I.N., Generalized analytic functions, 2nd ed, Nauka, Moscow, (in Rus-sian) **(1988)**
3. Mamourian A., Esrafilian E. and Taghizadeh N., On the existence of general solution of first order elliptic systems by Fixed-point theorem, *Non-linear Analysis theory and Application*, **30(1),** 5351-5356 **(1997)**
4. Taghizadeh N., On the uniqueness of solution of first order non-linear complex elliptic systems of partial differential equations in Sobolev space, *Honam Mathematical*, **27(2),** 205-209 **(2005)**
5. Taghizadeh N. and Akbari M., the existence and uniqueness of solution of non-linear partial differential equations by Fixed-point theorem in sobolev space and their results, *International journal of Applied Mathematics*, **22(6),** 879-885 **(2009)**