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Stress and Strain Analysis of Functionally Graded Annular Plate Subjected to Transverse Loading

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Abstract

In this paper, an exact analysis of stress and strain for a functionally graded circular plate under transverse loading and fix boundary conditions is studied. The Young's modules varies by power law and the loading changes by a polynomial function. These property and loading are most applicable. The influence of different functionally graded variation on the stress and displacement fields is studied through a numerical example. The exact solution shown that the graded material properties have significant effects on the mechanical behavior of the plate. The exact solution will help to prognosticate the distribution of the stress in the disk. By proper selection of the disk, the disk strength will be refine against loading. These parameters are called design parameters. Eventually, a numerical problem is solved and the impacts of design parameter are considered in order to find a proper annular disk for the specific loading.

Keywords: Functionally graded, annular disk, plate bending, stress, strain, FGM.

Introduction

Since the circular and sector plates combine light weight and a form efficiency with high load-carrying capacity, economy and technological effectiveness, they are extensively used in all field of engineering such as architectural structures, bridges, hydraulic structures, containers, airplane, missiles, ships and instruments.

Functionally graded materials are new modern materials which their properties change continuously[1]. These new materials are mostly constructed from metal and ceramic. One of the most prominent functions of such material is tolerating of high temperature. Investigations of static and dynamic response of FG material are the area of research over the last decade.

Sankar[2] presented elastic solution for functionally graded Euler-Bernoulli beam subjected to static transverse loads by assuming that Young's modulus of the beam varies exponentially along the thickness. Chakrabortyet al.[3] proposed a new beam finite element solution regarding the first-order shear deformation theory for studying the thermo elastic behavior of functionally graded beam. Chakraborty and Gopalakrishnan[4] studied the wave propagation behavior of FG beam under high frequency impulse of thermal or mechanical loading by using finite element method. Aydogdu and Taskin[5] studied the free vibration behavior of a simply supported FG beam based on Euler-Bernoulli beam theory. Zhong and Yu[6] presented an analytical solution of a cantilever FG beam with arbitrary graded variations of material property distribution based on two-dimensional elasticity theory. Ying et al.[7] obtained exact solutions for bending and free vibration of FG beams resting on a Winkler-Pasternak elastic foundation based on the two-dimensional elasticity theory by assuming that the beam is orthotropic at any point and the material properties change exponentially along the thickness direction. Kapuria et al.[8] presented a finite element model for static and free vibration responses of layered FG beams using an efficient third order zigzag theory for estimating the effective modulus of elasticity, and its experimental validation for two different FGM systems under various boundary conditions. Yang and Chen[9] studied the free vibration and elastic buckling of FG beams with open edge cracks by using Euler-Bernoulli beam theory. Li[10] proposed a new unified approach to investigate the static and the free vibration behavior of Euler-Bernoulli and Timoshenko beams. In a recent study, Amin Hadi et al studied an Euler-Bernoulli.[11] and Timoshenko[12] beam made of functionally graded material subjected to a transverse loading which Young's modulus of the beam vary by specific function.

In this study, ansolid Annular Platemade of exponentially FG.subjected to the uniform distribution transverse loading has been investigated by CLPT method.

Analysis

Consider a thin annular FGM disk, with internal radius "a" and external radius "b". For the following analysis, a cylindrical coordinates system (R, θ, z) is adopted with the origin at the center of disk. The geometry of the disk in relation with the coordinate axes is shown in figure 1. A three dimensional scheme of the annular disk subjected to an arbitrary transverse loading is shown in the figure 2. The Young modules is defined as follows;

$$E = E_o\left(\frac{r}{b}\right)^n \tag{1}$$

E_o iselastic modules at outer surface.

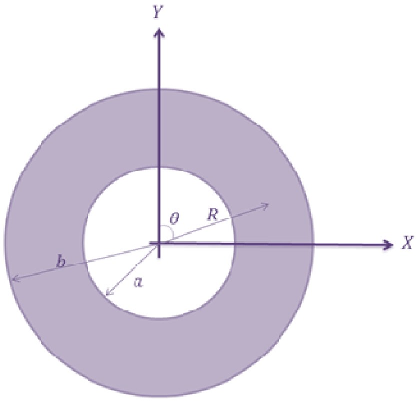

Figure-1
Annular disk with internal radius "a" and external radius"a"

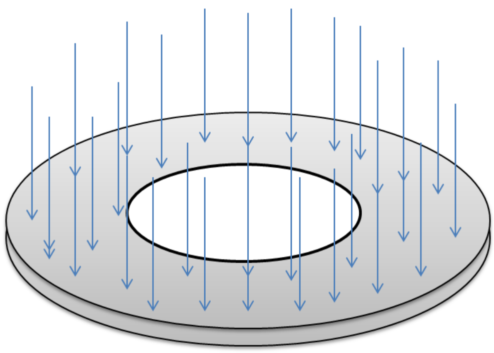

Figure-2
Three dimensional scheme of the annular disk subjected to an arbitrary transverse loading

In this section the analytical model and the applied theory in this study are briefy outlined. The FGM plates which are studied in this paper are thin so that the CLPT is applied in the analytical formulation. As a result of the CLPT assumption

$$u(r,z) = -z\frac{d\omega}{dr} \tag{2}$$

$$\omega = \omega(r) \tag{3}$$

Displacement in the r *and* z directions are denoted by u *and* ω, respectively. Three strain components can be expressed as:

$$\varepsilon_r = \frac{du}{dr} \tag{4}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{5}$$

$$\varepsilon_z = \frac{d\omega}{dz} \tag{6}$$

Where ε_r and ε_θ are normal strain components in r and θ direction, respectively.
For elastic materials, the relationships between the strains and stresses can be described by Hooke's law:

$$\sigma_r = \frac{-Ez}{(1+v)(1-2v)}\left[\frac{v}{r}\frac{d\omega}{dr} + (1-v)\frac{d^2\omega}{dr^2}\right] \tag{7}$$

$$\sigma_\theta = \frac{-Ez}{(1+v)(1-2v)}\left[\frac{(1-v)}{r}\frac{d\omega}{dr} + v\frac{d^2\omega}{dr^2}\right] \tag{8}$$

$$\sigma_z = -v(\sigma_r + \sigma_\theta) \tag{9}$$

The magnitude of the load across the radius increase by following relation

$$q = q_o\left(\frac{r}{b}\right)^m \tag{10}$$

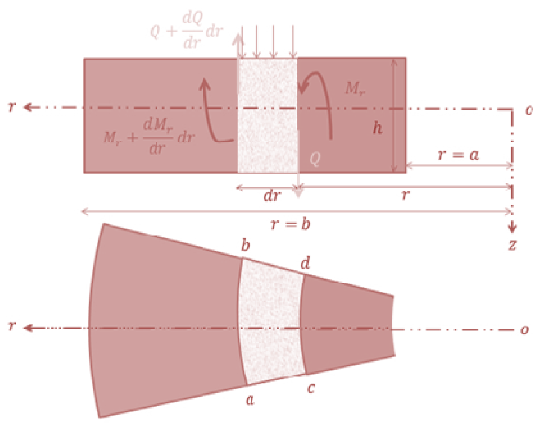

Figure-3
An infinitesimal cylindrical element of annular plate subjected to a transverse loading

The equilibrium equation for the shown element in cylindrical coordinates will be as follows;

$$M_r + \frac{dM_r}{dr}r - M_t + Qr = 0 \tag{11}$$

M_r and M_t are the moment around t and r axis defined as follow

$$M_r = -DE\left(\frac{d^2\omega}{dr^2} + \frac{v}{r}\frac{d\omega}{dr}\right) \tag{12}$$

$$M_t = -DE\left(\frac{1}{r}\frac{d\omega}{dr} + v\frac{d^2\omega}{dr^2}\right) \tag{13}$$

D is given by:

$$D = \frac{h^3}{12\left(1-v^2\right)} \tag{14}$$

φ defines as slope of the annular plate

$$\varphi = \frac{d\omega}{dr} \tag{15}$$

Substituting moments M_r and M_t in equilibrium equation will yield;

$$r\varphi'' + \left(1 + r\frac{E'}{E}\right)\varphi' + \left(\frac{vE'}{E} - \frac{1}{r}\right)\varphi = -\frac{Qr}{DE} \tag{16}$$

E' and φ' is the derivative of elastic modules and slope of the annular plate respect to r.

Integration of both sides can be done if Q is presented by a function of r. It will be advantageous to present the right hand side as form of intensity loading. For this purpose we multiply both sides of the equation by $2\pi r$. Then,

$$Q\left(2\pi r\right) = \int_0^r q\left(2\pi r\right)dr \tag{17}$$

The following equation will be given as result of integration of above equation;

$$\frac{1}{r}\left(\frac{d}{dr}\left(E\left(r\omega''' + \left(1 + r\frac{E'}{E}\right)\omega'' + \left(\frac{vE'}{E} - \frac{1}{r}\right)\omega'\right)\right)\right) = -\frac{q}{D} \quad 18)$$

Deflection of the Annular disk will be obtained by solving the above equation.

$$\omega = J_1\left(\frac{r}{b}\right)^{m-n} r^3 + C_1J_2r^{1-\frac{n+T}{2}} + C_2J_3r^{1-\frac{n-T}{2}} + C_3J_4r^{2-n} + C_4 \tag{19}$$

The slope of each point will be given by following relation;

$$\varphi = C_3J_4\left(2-n\right)r^{1-n} + \frac{J_1}{b}(m-n)\left(\frac{r}{b}\right)^{m-n-1} r^3 + 3J_1r^2\left(\frac{r}{b}\right)^{m-n} + C_1J_2r^{-\frac{n+T}{2}}\left(1 - \frac{n+T}{2}\right)$$
$$+C_2J_3r^{-\frac{n-T}{2}}\left(1 - \frac{n-T}{2}\right) \tag{20}$$

Which the parameter T , J_i are defined as bellow;

$$T = \left(4 + n^2 - 4nv\right)^{0.5} \tag{21}$$

$$J_1 = -\frac{48q_o\left(1+m\right)\left(1-v^2\right)}{h^3E_o\left(2+m\right)\left(3+m-n\right)\left(\left(4+2m-n\right)^2 - T^2\right)} \tag{22}$$

$$J_2 = \frac{2}{2-n-T} \tag{23}$$

$$J_3 = \frac{2}{2-n+T} \tag{24}$$

$$J_4 = \frac{1}{2-n} \tag{25}$$

If the annular plate disk is welded at inner and outer radius, we have following boundary condition;

$$\begin{cases} \omega(a) = 0 \\ \left.\dfrac{d\omega}{dr}\right|_{r=a} = 0 \\ \omega(b) = 0 \\ \left.\dfrac{d\omega}{dr}\right|_{r=b} = 0 \end{cases} \tag{26}$$

The unknown constants are given;

$$C_1 = -\frac{f_1f_3 - f_2f_4}{f_3f_5 - f_4f_6} \tag{27}$$

$$C_2 = f_7 + \frac{f_6\left(f_1f_3 - f_4f_6\right)}{f_3\left(f_3f_5 - f_4f_6\right)} \tag{28}$$

$$C_3 = f_8 + \frac{f_9\left(f_1f_3 - f_2f_4\right)}{f_{10}\left(f_3f_5 - f_4f_6\right)} \tag{29}$$

$$C_4 = \frac{f_{11}}{f_{12}} \tag{30}$$

Where

$$f_1 = J_1 J_4 b^{1-n}\left[(m-n)\left(1-\left(\frac{a}{b}\right)^{2-n}\right)-(2-n)\left(1-\left(\frac{a}{b}\right)^{m-n}\right)\right] \tag{31}$$

$$f2 = \frac{J_1 J_4}{a} b^{2-n}\left[(m-n)\left(\frac{a}{b}\right)^{m-n}\left(1-\left(\frac{a}{b}\right)^{2-n}\right)-(2-n)\left(\frac{a}{b}\right)^{2-n}\left(1-\left(\frac{a}{b}\right)^{m-n}\right)\right] \tag{32}$$

$$f_3 = \frac{J_3 J_4}{a} b^{3-\frac{3n-T}{2}}\left[\left(1-\frac{n-T}{2}\right)\left(\frac{a}{b}\right)^{1-\frac{n-T}{2}}\left(1-\left(\frac{a}{b}\right)^{2-n}\right)-(2-n)\left(\frac{a}{b}\right)^{2-n}\left(1-\left(\frac{a}{b}\right)^{1-\frac{n-T}{2}}\right)\right] \tag{33}$$

$$f_4 = \frac{J_3 J_4}{b} b^{3-\frac{3n-T}{2}}\left[\left(1-\frac{n-T}{2}\right)\left(1-\left(\frac{a}{b}\right)^{2-n}\right)-(2-n)\left(1-\left(\frac{a}{b}\right)^{1-\frac{n-T}{2}}\right)\right] \tag{34}$$

$$f_5 = \frac{J_2 J_4}{b} b^{3-\frac{3n+T}{2}}\left[\left(1-\frac{n+T}{2}\right)\left(1-\left(\frac{a}{b}\right)^{2-n}\right)-(2-n)\left(1-\left(\frac{a}{b}\right)^{1-\frac{n+T}{2}}\right)\right] \tag{35}$$

$$f_6 = \frac{J_2 J_4}{a} b^{3-\frac{3n+T}{2}}\left[\left(1-\frac{n-T}{2}\right)\left(\frac{a}{b}\right)^{1-\frac{n+T}{2}}\left(1-\left(\frac{a}{b}\right)^{2-n}\right)-(2-n)\left(\frac{a}{b}\right)^{2-n}\left(1-\left(\frac{a}{b}\right)^{1-\frac{n+T}{2}}\right)\right] \tag{36}$$

$$f_7 = -\frac{J_1}{J_3}\frac{(m-n)\left(\frac{a}{b}\right)^{m-n}\left(1-\left(\frac{a}{b}\right)^{2-n}\right)-(2-n)\left(\frac{a}{b}\right)^{2-n}\left(1-\left(\frac{a}{b}\right)^{m-n}\right)}{\left(1-\frac{n-T}{2}\right)\left(\frac{a}{b}\right)^{1-\frac{n-T}{2}}\left(1-\left(\frac{a}{b}\right)^{2-n}\right)-(2-n)\left(\frac{a}{b}\right)^{2-n}\left(1-\left(\frac{a}{b}\right)^{1-\frac{n-T}{2}}\right)} \tag{37}$$

$$f_8 = \frac{J_1 b^n\left\{2\left(\frac{a}{b}\right)^{m-\frac{n}{2}}(n-m)-\left(\frac{a}{b}\right)^{1+\frac{T}{2}}\left[(n-2m+2+T)\left(\frac{a}{b}\right)^{m-n}+(n-2-T)\right]\right\}}{J_4\left((n-2+T)\left(\frac{a}{b}\right)^{3-n+\frac{T}{2}}-2(n-2)a^{-\frac{n}{2}}b^{\frac{T}{2}}\left(\frac{a}{b}\right)^2+(n-2-T)\left(\frac{a}{b}\right)^{1+\frac{T}{2}}\right)} \tag{38}$$

$$f_9 = a^{\frac{n}{2}} b^{n+1} J_2\left[(2-n+T)\left(\frac{a}{b}\right)^{\frac{n+T}{2}}-(2-n-T)\left(\frac{a}{b}\right)^{\frac{n-T}{2}}-2T\left(\frac{a}{b}\right)\right] \tag{39}$$

$$f_{10} = J_4 b^{2+n+\frac{T}{2}}\left[(2-n+T)\left(\frac{a}{b}\right)^{n+\frac{T}{2}}+(2-n-T)\left(\frac{a}{b}\right)^{2+\frac{T}{2}}-(4-2n)\left(\frac{a}{b}\right)^{1+\frac{n}{2}}\right] \tag{40}$$

$$\begin{aligned} f_{11} = b^{2+n+T} J_1 &\left\{4(m-2)T\left(\frac{a}{b}\right)^{1+\frac{n+T}{2}}\left[1+\left(\frac{a}{b}\right)^{m-n}\right]\right. \\ &+(2-2m+n-T)(n+T-2)\left[\left(\frac{a}{b}\right)^{m}+\left(\frac{a}{b}\right)^{2+T}\right] \\ &\left.-(n-T-2)(2-2m+n+T)\left[\left(\frac{a}{b}\right)^{T+m}+\left(\frac{a}{b}\right)^{2}\right]\right\} \end{aligned} \tag{41}$$

$$\begin{aligned} f_{12} = b^{2+n+T} &\left\{\left[\left(\frac{a}{b}\right)^{2+T}+\left(\frac{a}{b}\right)^{n}\right](2-n-T)^2-8(n-2)T\left(\frac{a}{b}\right)^{1+\frac{n+T}{2}}\right. \\ &\left.-\left[\left(\frac{a}{b}\right)^{n+T}+\left(\frac{a}{b}\right)^{2}\right](2-n+T)^2\right\} \end{aligned} \tag{42}$$

The radial displacement will be calculated as follows;

$$u_r = -z\left[C_3J_4(2-n)r^{1-n} + \frac{J_1}{b}(m-n)\left(\frac{r}{b}\right)^{m-n-1} r^3 + 3J_1r^2\left(\frac{r}{b}\right)^{m-n} + C_1J_2r^{-\frac{n+T}{2}}\left(1-\frac{n+T}{2}\right) + C_2J_3r^{-\frac{n-T}{2}}\left(1-\frac{n-T}{2}\right)\right] \quad (43)$$

And the strain components will be given by following relations;

$$\varepsilon_r = -\frac{z}{4}\left[4C_3J_4\left(2-3n+n^2\right)r^{-n} - 4J_1(m-n)(1+m-n)r^{-2}\left(\frac{r}{b}\right)^{m-n} + C_2J_3r^{-1-\frac{n-T}{2}}(n-T-2)(n-T) + C_1J_2r^{-1-\frac{n+T}{2}}(n+T-2)(n+T)\right] \quad (44)$$

$$\varepsilon_\theta = z\left[C_3J_4(n-2)r^{-n} + J_1(m-n)r^{-2}\left(\frac{r}{b}\right)^{m-n} + \frac{1}{2}C_2J_3r^{-1-\frac{n-T}{2}}(n-T-2) + \frac{1}{2}C_1J_2r^{-1-\frac{n+T}{2}}(n+T-2)\right] \quad (45)$$

$$\varepsilon_z = 0 \quad (46)$$

Component of the stress will be determined as follows;

$$\sigma_r = \frac{E_oz\left(\frac{r}{b}\right)^n}{(1-2v)(1+v)}\left\{-\frac{1-v}{4}\begin{bmatrix}4C_3J_4\left(2-3n+n^2\right)r^{-n} - 4J_1(m-n)(1-m+n)r^{-2}\left(\frac{r}{b}\right)^{m-n} \\ +C_2J_3r^{-1-\frac{n-T}{2}}(n-T-2)(n-T) + C_1J_2r^{-1-\frac{n+T}{2}}(n+T-2)(n+T)\end{bmatrix} + v\begin{bmatrix}C_3J_4(-2+n)r^{-n} - J_1(m-n)r^{-2}\left(\frac{r}{b}\right)^{m-n} \\ -\frac{1}{2}C_2J_3r^{-\frac{n-T}{2}}(2-n+T) + \frac{1}{2}C_1J_2r^{-\frac{n+T}{2}}(-2+n+T)\end{bmatrix}\right\} \quad (47)$$

$$\sigma_\theta = \frac{E_oz\left(\frac{r}{b}\right)^n}{(1-2v)(1+v)}\left\{(1-v)\begin{bmatrix}C_3J_4(n-2)r^{2-n} - J_1(m-n)r^{-2}\left(\frac{r}{b}\right)^{m-n} \\ -\frac{1}{2}C2J3r^{-1-\frac{n-T}{2}}(2-n+T) + \frac{1}{2}C1J2r^{-1-\frac{n+T}{2})}(n+T-2)\end{bmatrix} - \frac{v}{4}\begin{bmatrix}4C_3J_4(n-2)(n-1)r^{-n} - 4J_1(m-n)r^{-2}\left(\frac{r}{b}\right)^{m-n}(1-m+n) \\ +C_2J_3r^{-1-\frac{n-T}{2}}(-2+n-T)(n-T) + C_1J_2r^{-1-\frac{n+T}{2}}(-2+n+T)(n+T)\end{bmatrix}\right\} \quad (48)$$

$$\sigma_z = -\frac{E_ovzr^{-2-n-\frac{T}{2}}\left(\frac{r}{b}\right)^n}{4(1-2v)(1+v)}\left\{4J_1(m-n)^2r^{n+\frac{T}{2}}\left(\frac{r}{b}\right)^{m-n} + r\begin{bmatrix}4C_3J_4(2-n)^2r^{1+\frac{T}{2}} \\ +C_2J_3r^{\frac{n}{2}+T}(2-n+T)^2 \\ +C_1J_2r^{\frac{n}{2}}(2-n-T)^2\end{bmatrix}\right\} \quad (49)$$

Results and Discussion

In the following, the obtained solution will be employed to analyze the effect of material inhomogeneity on the elastic field in the a FGM annular disk with an inner radius $a = 0.5m$, an outer radius $b = 0.9m$ and with the thickness $h = 0.2m$, with material property $E_o = 70GPa$ and is subjected to transverse loading $q = -10^4 \times \left(r/b\right)^{10}$. It is assumed that the Poisson's ratio υ has a constant value of 0.3. For different values of n dimensionless ω along the r direction is plotted in Fig. 4. According to this figure, at the same position, dimensionless ω Increases as n Increases.

Figure 5 is shown the displacement in the r direction of the FGM annular disk versus to r. in these plots displacement is increasing as the parameter n is increasing. Figure 6 display the normal stress of the FGM annular disk for different values of parameter n.

Conclusion

This paper investigates a annular disk made of functionally graded material with power function variation of properties. Then presented exact solution packages for stresses, displacements and other results that taken of stress. To show the effect of inhomogeneity on the stress distributions, different values were considered for material inhomogeneity parameter *n*. The presented results show that the material inhomogeneity has a significant influence on the mechanical behaviors of the solid Annular Platemade of functionally graded materials.

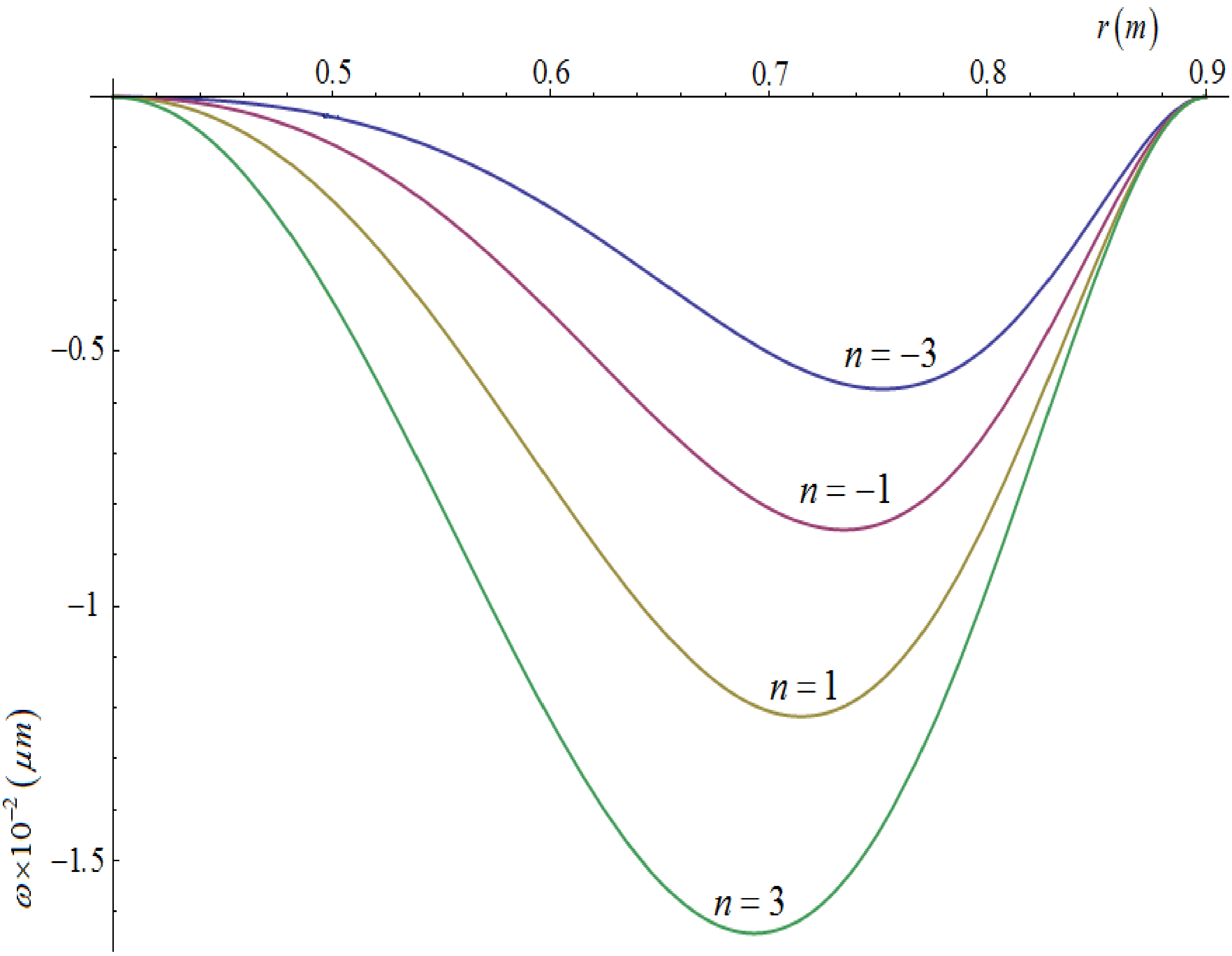

Figure-4
Distribution of displacement in z direction of the plate versus r

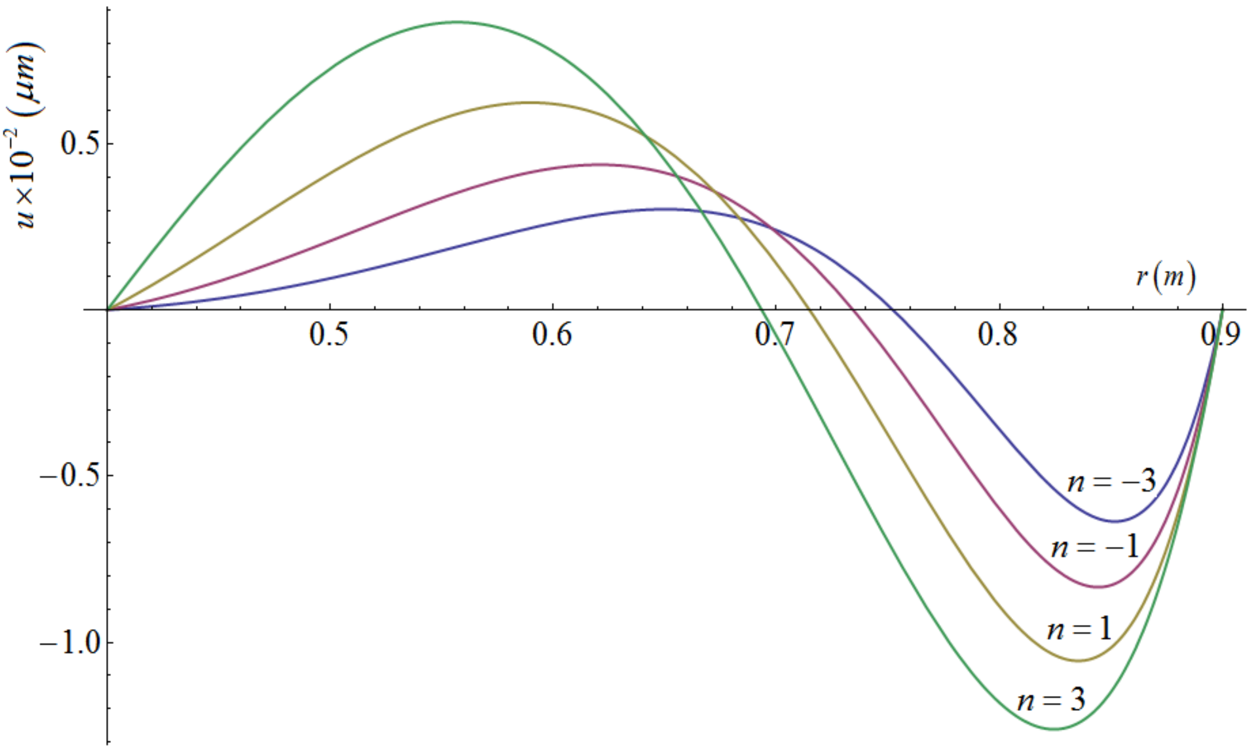

Figure-5
Distribution of displacement in r direction of the plate versus r

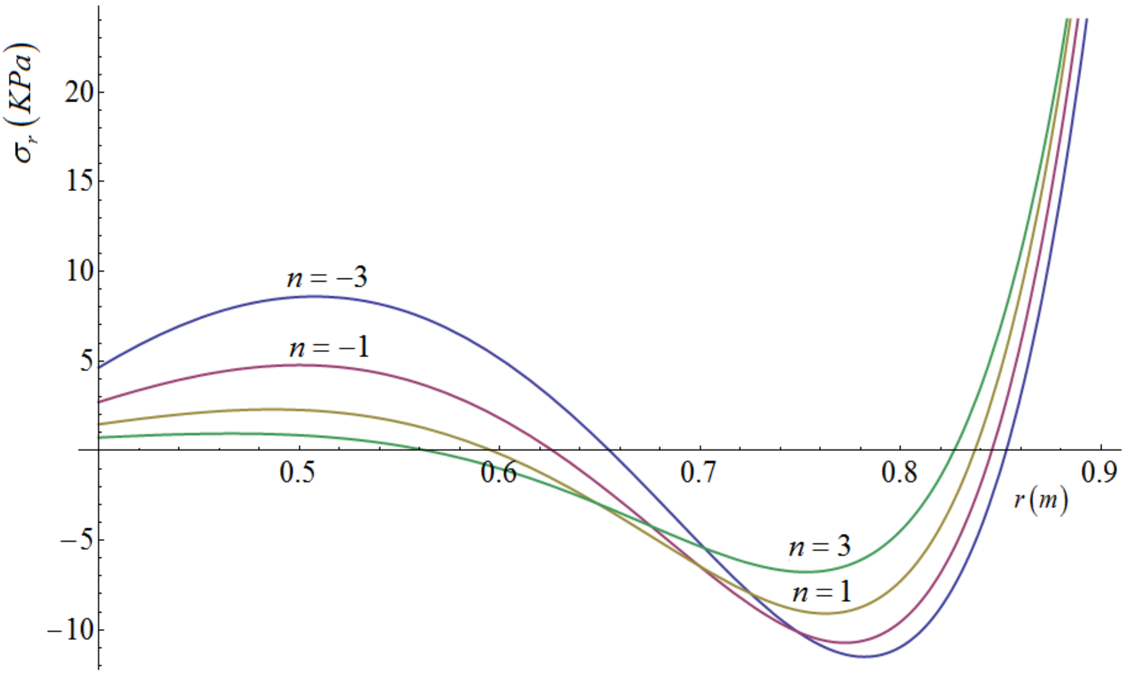

Figure-6
Distribution of non-dimensional stress in the r **direction versus** r

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