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Atherosclerotic Study of Non-Newtonian Steady Flow of Blood in Two Dimensions channel with Heat Transfer

Mirza A.A., Siddiqui A.M. and Haroon T.
Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Chak Shahzad, 44000, Islamabad, PAKISTAN

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Abstract

In this work, a steady two dimensional analytical solution is presented for non-isothermal, non-Newtonian fluid flowing through the channel having symmetric stenosis of cosine shape. The governing Navier-Stokes equations are reduced to compatibility and energy equations which are solved analytically with the help of regular perturbation method. The results obtained from the present analysis are presented analytically and graphically in terms of wall shear stress, separation and reattachment points, pressure gradient and temperature distribution on blood flow through a stenoised channel. It has been observed that the non-Newtonian nature of blood reduces the magnitude of the peak of flow over the stenoised region. Further, increase in second grade parameter (α)increases the temperature, pressure gradient, velocity distribution and wall shear stress while on the other hand the critical Re decreases. Its worth noting that the results presented in this article are compared with available results in literature and find good agreement.

Keywords: Non-Newtonian fluid, heat transfer, stenosis, wall shear stress, pressure distribution.

Introduction

It is well known that the deposit of cholesterol and proliferation of connective tissue may be responsible for the abnormal growth in lumen of artery. Its actual cause may not be known exactly but its effect on the cardiovascular system can easily be understood by studying the blood flow in its vicinity. One of the practical applications of blood flow through a membrane oxygenator is the flow with an irregular wall surface. Many authors have studied the behavior of blood in a constricted artery by considering different models of stenosis and assuming the blood to be Newtonian and non-Newtonian fluid. One of the earliest studies in this regard was conducted by Young[1]. He considered blood as a Newtonian fluid and suggested that the boundary irregularities can be an important factor in the development and progression of arterial diseases. Forrester and Young presented the analytical solution of Newtonian fluid for an axisymmetric, steady, incompressible flow and considered mild constriction for the flow of blood, both theoretically and experimentally in the converging and diverging tube[2]. Lee and Fung solved the flow model of the Newtonian fluid numerically through locally constricted tube for the low Reynolds number[3]. The constraints in their numerical procedure restricted the shape of the tube to be fixed and the Reynolds number to be moderate. Morgan and Young carried out the extension of Young[4,2]. They used an integral method and presented the approximate analytical solution of axisymmetric, steady state flow, which is applicable to both a mild and severe constriction. Haldar investigated the flow of blood through an axisymmetric cosine shape constricted artery and showed the solutions for velocity distribution, wall shearing stress and separation phenomena[5]. He indicated the presence of separation point, due to the occurrence of negative wall shear stresses at high Reynolds numbers. Chow et al. analyzed the steady laminar flow of incompressible Newtonian fluid for different physical parameters by considering the sinusoidal boundary[6]. It is observed that by increasing either Re or ε, the separation point would move down towards the throat in the divergent part of the channel with subsequent enlargement of the region of separation.

In addition to the Newtonian model many authors have studied the behavior of blood as non-Newtonian fluid. The non-Newtonian fluid may be considered as comparatively better model to represent the blood, due to its cells suspension property, even at a low shear rate. Further the Newtonian model is reasonable with regards the large channel assumption. However, for smaller channels the flow is expected to take on non-Newtonian character. Shukla et al. presented the analysis of blood by considering it as non-Newtonian fluid and studied the effect of constriction on the resistance to flow along with wall shear stress in an artery[7]. Mishra and Shit[8] considered the Herschel – Bulkley equation to represent the non-Newtonian characteristics of blood[8]. Haldar discussed the effect of shape of constriction on resistance of blood flow through an artery with mild local narrowing[9]. Cheng and Michel modeled the flow of blood as steady and pulsatile physiological flow[10]. Vahdati et al. designed a non linear ordinary differential equation for non-fatal disease in population and solved by Homotopy analysis method[11]. Thundil and Ramsai assumed the fluid to be air and presented numerical investigation using CFD[12]. Chauhan et al. studied the effect of turbulent flow over Ahmed's body by applying numerical technique[13].

It should be noted that all the above investigations are limited to flow patterns, pressure gradient, separation and reattachment points. However, the present work also investigates the effect of heat transfer in the channel. The solutions are presented graphically in terms of stream lines, wall shear stress, points of separation and reattachment, temperature distribution and pressure gradient. We assume the time independent flow of blood between two parallel plates, situated at the separation $2h_o$. To solve the highly non-linear equations, we apply the perturbation technique to find the analytical solution by taking δ as a small parameter.

Problem Formulation

It is assumed that the blood behaves like a homogeneous, incompressible, Non-Newtonian fluid of second grade with heat transfer. The governing equations for the present analysis are conservation of mass, momentum and energy equation. Consider the steady flow of blood through the channel of infinite length having stenosis of length $l_o/2$. The coordinate system is chosen in such a way that the channel lies in xy-plane and x-axis coincide with the center line in the direction of flow and y-axis perpendicular to x-axis.

Consider the boundary of the stenoised region of the form Haldar[5] as

$$h(\tilde{x}) = h_o - \frac{\lambda}{2}\left(1+\cos\left(\frac{4\pi\tilde{x}}{l_o}\right)\right), \qquad -\frac{l_o}{4} < \tilde{x} < \frac{l_o}{4}, \qquad (1)$$
$$= h_o, \qquad \text{otherwise,}$$

where $h(\tilde{x})$ is the variable width of channel, $2h_o$ the width of unobstructed channel and λ the maximum height of stenosis.

Assume that the blood behaves like non-Newtonian fluid and for steady, homogeneous, incompressible two dimensional flow of blood velocity field is taken as

$$\tilde{V} = (\ \tilde{u}(\tilde{x},\tilde{y}),\ \tilde{v}(\tilde{x},\tilde{y}),\ 0\ ) \qquad (2)$$

Introducing the dimensionless quantities of the form

$$x=\frac{\tilde{x}}{l_o},\ y=\frac{\tilde{y}}{h_o},\ u=\frac{\tilde{u}}{u_o},\ v=\frac{\tilde{v}}{u_o},\ q=\frac{h_o^2}{\mu u_o l_o}\tilde{q},\ \theta=\frac{\tilde{T}-T_o}{T_1-T_o} \qquad (3)$$

where u_o is the characteristic velocity. and T_1, T_o are temperatures on the boundary of stenosis and fluid respectively. Dimensionless form of the boundary profile becomes

$$f = 1-\frac{\varepsilon}{2}(1+\cos 4\pi x), \quad -\frac{1}{4} < x < \frac{1}{4}, \qquad (4)$$
$$=1, \qquad \text{otherwise,}$$

where $f = h(\tilde{x})/h_o$ and $\varepsilon = \lambda/h_o$ is dimensionless height of stenosis. Introducing the stream functions of the form

$$u=\frac{\partial\psi}{\partial y},\quad v=-\delta\frac{\partial\psi}{\partial x} \qquad (5)$$

which satisfy the continuity equation identically and momentum equations reduces to component form by making use of (2), (3) and (5) along with the energy equation in dimensionless variables as follows

$$\frac{\partial q}{\partial x} - \text{Re}\,\delta\frac{\partial\psi}{\partial x}\nabla^2\Psi = \nabla^2\left(\frac{\partial\Psi}{\partial y}\right) - \alpha\delta\frac{\partial\Psi}{\partial x}\nabla^4\Psi, \qquad (6)$$

$$\frac{\partial q}{\partial y} - \text{Re}\,\delta\frac{\partial\psi}{\partial y}\nabla^2\Psi = -\delta^2\nabla^2\left(\frac{\partial\Psi}{\partial x}\right) - \alpha\delta\frac{\partial\Psi}{\partial y}\nabla^4\Psi \qquad (7)$$

$$Pe\delta\frac{\partial(\psi,\theta)}{\partial(y,x)} = \nabla^2\theta + Br\left(1+\frac{\alpha}{2}\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\right) \left(4\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)^2\right) \qquad (8)$$

where q is modified pressure in terms of stream function given by

$$q = p+\frac{1}{2}\text{Re}\,\delta\left(\left(\frac{\partial\psi}{\partial y}\right)^2+\delta^2\left(\frac{\partial\psi}{\partial x}\right)^2\right) - \alpha\delta\left\{\frac{\partial\psi}{\partial y}\nabla^2\left(\frac{\partial\psi}{\partial y}\right)+\delta^2\frac{\partial\psi}{\partial x}\nabla^2\left(\frac{\partial\psi}{\partial x}\right)\right\} \qquad (9)$$
$$-\delta(3\alpha+2\beta)\left\{4\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2+\frac{1}{2}\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)^2\right\}.$$

Eliminating modified pressure from (6) and (7), we get compatibility equation in terms of stream function of the form[14]

$$\text{Re}\,\delta\frac{\partial(\psi,\nabla^2\psi)}{\partial(y,x)} = \nabla^4\psi + \alpha\delta\frac{\partial(\psi,\nabla^4\psi)}{\partial(y,x)} \qquad (10)$$

Boundary conditions in terms of stream functions are

$$\frac{\partial\psi}{\partial y}=0,\quad \psi=-\frac{1}{2},\quad \theta=1 \quad at \quad y=f, \quad and$$
$$\frac{\partial^2\psi}{\partial y^2}=0,\quad \psi=0,\quad \frac{\partial\theta}{\partial y}=0 \quad at \quad y=0. \qquad (11)$$

where $\tilde{\nabla}^2 = \delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

$$\alpha=\frac{\alpha_1 u_o}{\mu h_o},\quad \beta=\frac{\alpha_2 u_o}{\mu h_o},\quad \delta=\frac{h_o}{l_o},\quad \text{Re}=\frac{u_o h_o}{\nu},$$

and

$$\text{Br}=\frac{u_o^2\mu}{k(T_1-T_o)},\quad \text{Pe}=\frac{\rho u_o h_o c_p}{k}.$$

It should be noted that for $\alpha = 0$, the above model reduces to viscous case and reduced compatibility equation (10) for $\alpha = 0$ has been discussed by[1-6].

Solution

The resulting compatibility and energy equations are non-linear and exact solution is very difficult to find, for the analytical solution we apply perturbation technique in these equations by considering δ as a small parameter, which is requirement of concerned method, as follows

$$\psi = \psi_o(x,y) + \delta\psi_1(x,y) + \delta^2\psi_2(x,y) + \cdots$$
$$\theta = \theta_o + \delta\theta_1 + \delta^2\theta_2 + \cdots \tag{12}$$

Zeroth order problem and its solution: Zeroth order system is obtained by substituting (12) in (8), (10) and (11), then equating the coefficients of δ^0, we obtain

$$\frac{\partial^4\psi_o}{\partial y^4} = 0, \tag{13}$$

$$\frac{\partial^2\theta_o}{\partial y^2} = -Br\left(\frac{\partial^2\psi_o}{\partial y^2}\right)^2 \tag{14}$$

Subject to the boundary conditions

$$\frac{\partial\psi_o}{\partial y} = 0,\ \psi_o = -\frac{1}{2},\ \theta_o = 1 \quad at \quad y = f, \quad and$$
$$\frac{\partial^2\psi_o}{\partial y^2} = 0,\ \psi_o = 0,\ \frac{\partial\theta_o}{\partial y} = 0 \quad at \quad y = 0. \tag{15}$$

The solution of (13) is obtained by integrating successively along with the boundary conditions on stream function in (15) as follows

$$\psi_o = \frac{\eta}{4}\left(\eta^2 - 3\right), \qquad \eta = \frac{y}{f} \tag{16}$$

which is similar to zeroth order viscous solution and free from second grade parameter. The solution of (14) by making use of (16) and subject to boundary conditions on temperature in (15) becomes

$$\theta_o = 1 - \frac{3Br}{16f^2}\left(\eta^4 - 1\right). \tag{17}$$

It is observed that zeroth order solution of energy equation is independent of second grade parameter and depends upon ratio of heat production by viscous dissipation to heat transport by conduction.

First order problem and its solution: The first order system is obtained by comparing the coefficients of δ, we obtain

$$\frac{\partial^4\psi_1}{\partial y^4} = \mathrm{Re}\frac{\partial\left(\psi_o, \frac{\partial^2\psi_o}{\partial y^2}\right)}{\partial(y,x)} - \alpha\frac{\partial\left(\psi_o, \frac{\partial^4\psi_o}{\partial y^4}\right)}{\partial(y,x)}, \tag{18}$$

$$\frac{\partial^2\theta_1}{\partial y^2} = Pe\left(\frac{\partial\psi_o}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi_o}{\partial x}\frac{\partial}{\partial y}\right)\theta_o - 2$$
$$Br\left(\frac{\partial^2\psi_o}{\partial y^2}\frac{\partial^2\psi_1}{\partial y^2} + \frac{\alpha}{4}\left(\frac{\partial\psi_o}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi_o}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi_o}{\partial y^2}\right)^2\right), \tag{19}$$

along with the boundary conditions

$$\frac{\partial\psi_1}{\partial y} = 0,\ \psi_1 = 0,\ \theta_1 = 0 \quad at \quad y = f, \quad and$$
$$\frac{\partial^2\psi_1}{\partial y^2} = 0,\ \psi_1 = 0,\ \frac{\partial\theta_1}{\partial y} = 0 \quad at \quad y = 0. \tag{20}$$

The solution of equation (18) is obtained by integrating and making use of (16) subject to corresponding boundary conditions of the form

$$\psi_1 = -\frac{3\mathrm{Re}\,f'\eta}{1120}\left(\eta^6 - 7\eta^4 + 11\eta^2 - 5\right), \tag{21}$$

which is first order viscous solution and independent of second grade parameter. The solution of (19) is achieved by using expressions for ψ_o, θ_o, ψ_1 and integrating twice along with the boundary conditions on temperature giving

$$\theta_1 = \frac{3Br\left(\eta^2 - 1\right)f'}{8960f^4}\left[\begin{array}{l} f^2(2\mathrm{Re}(9\eta^6 - 47\eta^4 + 19\eta^2 + 19) + \\ Pe(15\eta^6 - 13\eta^4 - 83\eta^2 + 337)) + 168\alpha(2\eta^4 - 3\eta^2 - 3) \end{array}\right] \tag{22}$$

It is observed that by setting $\alpha = 0$, the first order solution reduces to first order viscous solution for energy equation. The solution of first order temperature distribution depends upon ratio of convection to conduction.

Second order problem and its solution: The second order system of equations is obtained by equating the coefficients of δ^2 as follows

$$\frac{\partial^4\psi_2}{\partial y^4} = \mathrm{Re}\left[\frac{\partial\left(\psi_o, \frac{\partial^2\psi_1}{\partial y^2}\right)}{\partial(y,x)} + \frac{\partial\left(\psi_1, \frac{\partial^2\psi_o}{\partial y^2}\right)}{\partial(y,x)}\right] - \alpha$$
$$\left[\frac{\partial\left(\psi_o, \frac{\partial^4\psi_1}{\partial y^4}\right)}{\partial(y,x)} + \frac{\partial\left(\psi_1, \frac{\partial^4\psi_o}{\partial y^4}\right)}{\partial(y,x)}\right] - 2\frac{\partial^4\psi_o}{\partial x^2\partial y^2}, \tag{23}$$

$$\frac{\partial^2\theta_2}{\partial y^2} = Pe\left(\frac{\partial(\psi_o,\theta_1)}{\partial(y,x)} + \frac{\partial(\psi_1,\theta_o)}{\partial(y,x)}\right) - \frac{\partial^2\theta_o}{\partial x^2} - Br \left[4\left(\frac{\partial^2\psi_o}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\psi_1}{\partial y^2}\right)^2 + 2\frac{\partial^2\psi_o}{\partial y^2}\frac{\partial^2\psi_2}{\partial y^2} - 2\frac{\partial^2\psi_o}{\partial y^2}\frac{\partial^2\psi_o}{\partial x^2} + \frac{\alpha}{2}\left\{\left(\frac{\partial\psi_o}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi_o}{\partial x}\frac{\partial}{\partial y}\right)\left(2\frac{\partial^2\psi_o}{\partial y^2}\frac{\partial^2\psi_1}{\partial y^2}\right) + \left(\frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi_o}{\partial y^2}\right)^2\right\}\right], \quad (24)$$

subject to boundary conditions

$$\frac{\partial\psi_2}{\partial y} = 0, \quad \psi_2 = 0, \quad \theta_2 = 0 \quad at \quad y = f, \quad and$$
$$\frac{\partial^2\psi_2}{\partial y^2} = 0, \quad \psi_2 = 0, \quad \frac{\partial\theta_2}{\partial y} = 0 \quad at \quad y = 0. \quad (25)$$

The solution of equation (23) is obtained by making use of ψ_o, ψ_1 and integrating successively four times along with corresponding boundary conditions in (25) giving

$$\psi_2 = \frac{\eta(\eta^2-1)^2}{3449600 f^2}\left[385\left(ff'' - 4f'^2\right)\left(\mathrm{Re}\,\alpha\left(5\eta^4 - 26\eta^2 + 69\right) + 672 f^2\right) + f^2\,\mathrm{Re}^2\left(f'^2\left(98\eta^6 - 959\eta^4 + 2472\eta^2 - 2875\right) - ff''\left(35\eta^6 - 315\eta^4 + 853\eta^2 - 1213\right)\right)\right].. \quad (26)$$

It is observed that by setting $\alpha = 0$, equation (26) reduces to second order viscous solution. To find the second order temperature using the expression for ψ_o, $\psi_1, \psi_2, \theta_o, \theta_1$ in (24) and integrating twice along with the corresponding boundary conditions on temperature, we obtain

$$\theta_2 = \frac{(\eta^2-1)}{55193600 f^4}\Big[-2f'^2\{924\alpha\,\mathrm{Re}(36\eta^8 - 279\eta^6 - 771\eta^4 + 379\eta^2 + 379) - f^2\{4Pe\,\mathrm{Re}(840\eta^{10} - 6860\eta^8 + 11455\eta^6 + 3139\eta^4 - 26891\eta^2 + 56269) + 2\{-517440(13\eta^4 + 9\eta^2 - 36) + \mathrm{Re}^2(2303\eta^{10} - 21721\eta^8 + 63122\eta^6 - 68086\eta^4 + 17183\eta^2 + 17183)\}\}\} + ff''\{29568\alpha\,\mathrm{Re}(-18\eta^4 + 7\eta^2 + 7) + f^2\{2Pe\,\mathrm{Re}(525\eta^{10} - 5173\eta^8 + 14132\eta^6 - 7120\eta^4 - 29065\eta^2 + 102605) + 8\{-258720(7\eta^4 - 4\eta^2 - 9) + \mathrm{Re}^2(175\eta^{10} - 1673\eta^8 + 5158\eta^6 - 7778\eta^4 + 2059\eta^2 + 2059)\}\}\} + 6468Pe\alpha(8\eta^8 - 37\eta^6 + 23\eta^4 + 113\eta^2 - 427) + 7Pe^2f^2(225\eta^{10} - 721\eta^8 - 2206\eta^6 + 30134\eta^4 - 94771\eta^2 + 238859)\Big]. \quad (27)$$

The second order viscous solution for the energy equation can be recovered by setting $\alpha = 0$ in equation (27). The second order temperature depends upon ratio of conduction to convection and dissipation to conduction. Now velocity components u, v and the temperature θ can be recovered from the above solutions.

Pressure Distribution: In this section, our aim is to find the pressure for which we have to perturb modified pressure as

$$q = q_o + \delta q_1 + \delta^2 q_2 + \cdots. \quad (28)$$

Using expression (28) in (6) - (7) and equating the coefficients of like powers of δ we obtain the various systems. To find the solution of these systems, we apply

$$q = \int_0^x \frac{\partial q}{\partial x}dx + \int_0^y \frac{\partial q}{\partial y}dy. \quad (29)$$

Further to find the pressure within the channel, substitute the perturbed pressure as

$$p = p_o + \delta p_1 + \delta^2 p_2 + \cdots. \quad (30)$$

in equation (9), we get the following perturbed system for pressure.

Zeroth order pressure: Equating the coefficients of δ^0 on both sides of (6)-(7) for modified pressure as follows

$$\frac{\partial q_o}{\partial x}=\frac{\partial^3\psi_o}{\partial y^3}, \tag{31}$$

$$\frac{\partial q_o}{\partial y}=0, \tag{32}$$

and equation (9) gives the zeroth order pressure as

$$p_o=q_o. \tag{33}$$

Solution of equations (31)-(32) by using (29) obtained of the form

$$q_o=-\frac{3}{32\pi(\varepsilon-1)^2}\left[\frac{1}{\sqrt{\varepsilon-1}}\left(3\varepsilon^2+8(1-\varepsilon)\right)\tanh^{-1}\left(\frac{\tan(2\pi x)}{\sqrt{\varepsilon-1}}\right)\right. \\ \left.+\frac{f'}{8\pi f^2}\{16(\varepsilon-1)-3\varepsilon^2-3\varepsilon(\varepsilon-2)\cos(4\pi x)\}\right], \tag{34}$$

which is viscous pressure and independent of second grade parameter (α) .

First order system and solution: Comparing the coefficients of δ on both sides of (6)-(7), the resulting equations are obtained as

$$\frac{\partial q_1}{\partial x}=\frac{\partial^3\psi_1}{\partial y^3}+\mathrm{Re}\left\{\frac{\partial^2\psi_o}{\partial y^2}-\alpha\frac{\partial^4\psi_o}{\partial y^4}\right\}\frac{\partial\psi_o}{\partial x}, \tag{35}$$

$$\frac{\partial q_1}{\partial y}=\mathrm{Re}\left\{\frac{\partial^2\psi_o}{\partial y^2}-\alpha\frac{\partial^4\psi_o}{\partial y^4}\right\}\frac{\partial\psi_o}{\partial y}, \tag{36}$$

and first order pressure becomes

$$p_1=q_1-\frac{1}{2}\mathrm{Re}\left(\frac{\partial\psi_o}{\partial y}\right)^2+\alpha\frac{\partial\psi_o}{\partial y}\frac{\partial^3\psi_o}{\partial y^3}+\frac{1}{2}(3\alpha+2\beta)\left(\frac{\partial^2\psi_o}{\partial y^2}\right)^2. \tag{37}$$

The solution of first order modified pressure is obtained by using ψ_o, ψ_1 in (35)-(36) along with (29) as

$$q_1=\frac{1}{1120}\left\{\frac{9\,\mathrm{Re}}{f^2}\left(35\eta^4-70\eta^2+11\right)-\frac{1}{(\varepsilon-1)^6}\left(35\eta^4f^4-70\eta^2f^2(\varepsilon-1)^2+11(\varepsilon-1)^4\right)\right\}. \tag{38}$$

The first order pressure is given by (37) as follows

$$p_1=-\frac{9}{1120}\left\{\begin{array}{l}\frac{\mathrm{Re}}{(\varepsilon-1)^6}\left(35\eta^4f^4-70\eta^2f^2(\varepsilon-1)^2+11(\varepsilon-1)^4\right)\\+\frac{35\eta^2}{f^6}\left(\mathrm{Re}\,\eta^2+8(2\alpha+\beta)\right)-\frac{70}{f^4}\left(\mathrm{Re}\,\eta^2+2\alpha\right)-\frac{24\,\mathrm{Re}}{f^2}\end{array}\right\}. \tag{39}$$

It is observed that second grade parameters do not appear in the first order modified pressure but it is present in first order pressure.

Second order system and solution: Equating the coefficients of like power of δ^2 on both sides of (6)-(7), the system is obtained as

$$\frac{\partial q_2}{\partial x}=\frac{\partial^3\psi_2}{\partial y^3}+\frac{\partial^3\psi_o}{\partial x^2\partial y}+\mathrm{Re}\left(\frac{\partial\psi_o}{\partial x}\frac{\partial^2\psi_1}{\partial y^2}+\frac{\partial\psi_1}{\partial x}\frac{\partial^2\psi_o}{\partial y^2}\right)-\alpha\left(\frac{\partial\psi_o}{\partial x}\frac{\partial^4\psi_1}{\partial y^4}+\frac{\partial\psi_1}{\partial x}\frac{\partial^4\psi_o}{\partial y^4}\right), \tag{40}$$

$$\frac{\partial q_2}{\partial y}=\mathrm{Re}\left(\frac{\partial\psi_o}{\partial y}\frac{\partial^2\psi_1}{\partial y^2}+\frac{\partial\psi_1}{\partial y}\frac{\partial^2\psi_o}{\partial y^2}\right)-\frac{\partial^3\psi_o}{\partial y^2\partial x}-\alpha\left(\frac{\partial\psi_o}{\partial y}\frac{\partial^4\psi_1}{\partial y^4}+\frac{\partial\psi_1}{\partial y}\frac{\partial^4\psi_o}{\partial y^4}\right), \tag{41}$$

and equation for second order pressure is

$$p_2 = q_2 - \mathrm{Re}\frac{\partial\psi_o}{\partial y}\frac{\partial\psi_1}{\partial y} + \alpha\left(\frac{\partial\psi_o}{\partial y}\frac{\partial^3\psi_1}{\partial y^3} + \frac{\partial\psi_1}{\partial y}\frac{\partial^3\psi_o}{\partial y^3}\right) + (3\alpha+2\beta)\frac{\partial^2\psi_o}{\partial y^2}\frac{\partial^2\psi_1}{\partial y^2}. \tag{42}$$

The solution of second order modified pressure is obtained in the integral form by using (40) and (41) in (29) as follows

$$\begin{aligned}
q_2 = {} & \frac{1}{f^4}\Big\{ \frac{9}{16}\mathrm{Re}\,\pi\eta^2\alpha\varepsilon\sin(4\pi x)\left(\eta^4-3\eta^2+3\right) - \frac{9}{13860}\pi\eta^2\varepsilon f^2\sin(4\pi x)\left(\mathrm{Re}^2\left(7\eta^6-42\eta^4+62\eta^2-38\right)\right)\Big\} \\
& + \frac{1}{3449600}\int_0^x \frac{1}{f^5}\Big[\, 2f'^4\big\{ f^2\{ +\mathrm{Re}^2\left(71148\,\eta^{10}-467775\,\eta^8+879648\,\eta^6-658350\,\eta^4+147996\,\eta^2-2875\right) \\
& -1034880\left(75\eta^4-36\eta^2+1\right)\}-27720\,\mathrm{Re}\,\alpha\left(165\eta^8-630\eta^6+980\eta^4-410\eta^2+23\right)\big\}+f'^2\{12f^2(\\
& -1034880\left(5\eta^2-1\right)+\mathrm{Re}^2\left(40425\,\eta^8-194040\,\eta^6+235620\,\eta^4-87780\,\eta^2+4111\right))+6930\,\mathrm{Re}\,\alpha ff''(825\eta^8 \\
& -3528\eta^6-6370\eta^4-3280\eta^2+253)-18480\,\mathrm{Re}\,\alpha\left(525\eta^6-1260\eta^4+945\eta^2-82\right)+f^3f''\{1034880 \\
& \left(150\eta^4-99\eta^2+5\right)+\mathrm{Re}^2(-101640\,\eta^{10}+717255\,\eta^8-1545852\,\eta^6+142650\,\eta^4-429012\,\eta^2+14375)\} \\
& +2f^2f'f'''\{-2310\,\mathrm{Re}\,\alpha\left(105\eta^8-504\eta^6+1050\eta^4-656\eta^2+69\right)+f^2\{-1034880\left(10\eta^4-9\eta^2+1\right) \\
& +\mathrm{Re}^2(4928\eta^{10}-38115\,\eta^8+95172\,\eta^6-108570\,\eta^4+44340\,\eta^2-2875)\}\}+f\{-6f''\{-1540\,\mathrm{Re}\,\alpha \\
& \left(105\eta^6-315\eta^4+315\eta^2-41\right)+3f^2\{-172480\left(5\eta^2-1\right)+\mathrm{Re}^2(2695\eta^8-16170\,\eta^6+26180\,\eta^4-14630\,\eta^2 \\
& +1093)\}\}+2ff''^2\{-1155\,\mathrm{Re}\,\alpha\left(135\eta^8-672\eta^6+1470\eta^4-984\eta^2+115\right)+f^2\{-517440\{15\eta^4 \\
& -15\eta^2+2)+\mathrm{Re}^2(3003\eta^{10}-24255\,\eta^8+63294\,\eta^6-76230\,\eta^4+34503\,\eta^2-2875)\}\}-f^2f^{(4)}\left(\eta^2-1\right) \\
& \{-1155\,\mathrm{Re}\,\alpha\left(15\eta^6-69\eta^4+141\eta^2-23\right)+f^2\{-258720\left(5\eta^2-1\right)+\mathrm{Re}^2(385\eta^8-3080\eta^6+7546\eta^4 \\
& -8624\eta^2+1213)\}\}\}\}\}\,\Big]dx,
\end{aligned} \tag{43}$$

and (42) gives the second order pressure as

$$\begin{aligned}
p_2 = {} & \frac{9}{4480f^2}\{2\pi\eta^2\varepsilon\sin(4\pi x)\left(\mathrm{Re}^2\left(7\eta^6-42\eta^4+68\eta^2-38\right)-1120\right)-\mathrm{Re}^2 f'\left(\eta^2-1\right)^2(7\eta^4-28\eta^2 \\
& +5)\}+\frac{9\,\mathrm{Re}}{1120f^4}\{70\pi\eta^2\alpha\varepsilon\sin(4\pi x)(\eta^4-3\eta^2+3)+f'\{\alpha(-119\eta^6+385\eta^4-237\eta^2+19)-2\eta^2\beta \\
& (21\eta^4-70\eta^2+33)\}\} \\
& + \frac{1}{3449600}\int_0^x \frac{1}{f^5}\Big[\, 2f'^4\big\{ f^2\{ +\mathrm{Re}^2\left(71148\,\eta^{10}-467775\,\eta^8+879648\,\eta^6-658350\,\eta^4+147996\,\eta^2-2875\right) \\
& -1034880\left(75\eta^4-36\eta^2+1\right)\}-27720\,\mathrm{Re}\,\alpha\left(165\eta^8-630\eta^6+980\eta^4-410\eta^2+23\right)\big\}+f'^2\{12f^2(\\
& -1034880\left(5\eta^2-1\right)+\mathrm{Re}^2\left(40425\,\eta^8-194040\,\eta^6+235620\,\eta^4-87780\,\eta^2+4111\right))+6930\,\mathrm{Re}\,\alpha ff''(825\eta^8 \\
& -3528\eta^6-6370\eta^4-3280\eta^2+253)-18480\,\mathrm{Re}\,\alpha\left(525\eta^6-1260\eta^4+945\eta^2-82\right)+f^3f''\{1034880 \\
& \left(150\eta^4-99\eta^2+5\right)+\mathrm{Re}^2(-101640\,\eta^{10}+717255\,\eta^8-1545852\,\eta^6+142650\,\eta^4-429012\,\eta^2+14375)\} \\
& +2f^2f'f'''\{-2310\,\mathrm{Re}\,\alpha\left(105\eta^8-504\eta^6+1050\eta^4-656\eta^2+69\right)+f^2\{-1034880\left(10\eta^4-9\eta^2+1\right) \\
& +\mathrm{Re}^2(4928\eta^{10}-38115\,\eta^8+95172\,\eta^6-108570\,\eta^4+44340\,\eta^2-2875)\}\}+f\{-6f''\{-1540\,\mathrm{Re}\,\alpha \\
& \left(105\eta^6-315\eta^4+315\eta^2-41\right)+3f^2\{-172480\left(5\eta^2-1\right)+\mathrm{Re}^2(2695\eta^8-16170\,\eta^6+26180\,\eta^4-14630\,\eta^2 \\
& +1093)\}\}+2ff''^2\{-1155\,\mathrm{Re}\,\alpha\left(135\eta^8-672\eta^6+1470\eta^4-984\eta^2+115\right)+f^2\{-517440\{15\eta^4 \\
& -15\eta^2+2)+\mathrm{Re}^2(3003\eta^{10}-24255\,\eta^8+63294\,\eta^6-76230\,\eta^4+34503\,\eta^2-2875)\}\}-f^2f^{(4)}\left(\eta^2-1\right) \\
& \{-1155\,\mathrm{Re}\,\alpha\left(15\eta^6-69\eta^4+141\eta^2-23\right)+f^2\{-258720\left(5\eta^2-1\right)+\mathrm{Re}^2(385\eta^8-3080\eta^6+7546\eta^4 \\
& -8624\eta^2+1213)\}\}\}\}\}\,\Big]dx.
\end{aligned} \tag{44}$$

One can find the pressure *p* by substituting the expressions for p_o, p_1 *and* p_2 in equation (30). Viscous pressure could be obtained by setting the second grade parameters $\alpha = \beta = 0$.

Wall shear stress

Wall shear stress for the second grade fluid in dimensionless form is obtained from the component of Cauchy shear stress as follows

$$\tau_w = \left\{1 + \alpha\left(\delta u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right)\right\}\left(\frac{\partial u}{\partial y} + \delta \frac{\partial v}{\partial x}\right) + 2\alpha\Omega \frac{\partial v}{\partial y}. \tag{45}$$

Wall shear stress up to second order in δ is obtained by making use of velocity components defined in equation (5) as follows

$$\tau_w = \frac{3}{f^2}\left[\frac{1}{2} + \frac{\delta f'}{70 f^2}\left(2\,\mathrm{Re}\, f^2 - 105\alpha\right) - \delta^2 \left\{ \frac{\mathrm{Re}\,\alpha}{70 f^2}\left(16 f'^2 - ff''\right) + \frac{\mathrm{Re}^2}{80850}\left(79 f'^2 - 40 ff''\right) + \frac{1}{10}\left(13 f'^2 - 2 ff''\right) \right\} \right]. \tag{46}$$

The points of separation and reattachment are given by setting $\tau_w = 0$, the resulting equation in terms of *Re* becomes

$$\mathrm{Re} = \frac{7}{2\delta f^2\left(40 f f'' - 79 f'^2\right)}\Big[165\left(\alpha\delta\left(16 f'^2 - f f''\right) - 2 f^2 f'\right) \pm \sqrt{165}\sqrt{\begin{matrix}\left\{ 165\left(2 f^2 f' + \alpha\delta\left(f f'' - 16 f'^2\right)\right)^2 \right. \\ \left. - 4 f^2\left(40 f f'' - 79 f'^2\right)\left(2\delta^2 f^2 f'' - 15\alpha\delta f' + 4 f^2\left(5 - 13\delta^2 f'^2\right)\right) \right\}\end{matrix}}\ \Big] \tag{47}$$

From the expression (47), we have to find the critical Reynolds number at which the separation and reattachment points occur.

Graphical Discussion

In this section solutions are presented graphically for wall shear stress, zero wall shear stress, temperature distribution and pressure gradient. Solutions are analyzed numerically through graphs for second grade parameters (α, β), height of stenosis (ε), Reynolds number (Re), Brinkman number (Br) and Peclet number (Pe). The geometry of the proposed model for the study of the stenoised channel is depicted in figure 1. The radii of obstructed and unobstructed regions are $h(x)$ and h_o.

The distribution of wall shear stress for the various values of Re is given in figure 2 for fixed $\varepsilon = 0.2, \delta = 0.1, \alpha = 0.04$. An increase in Re, wall shear stress increases near the throat of stenoised region and becomes adverse in the converging and diverging section of the channel. The negative shearing in converging and diverging sections of channel indicates that there is point of separation in the upstream region and reattachment point in the downstream region of the channel. It is observed that wall shear stress holds for both small and large Re. It is also observed that the magnitude of adverse wall shear stress in the diverging part is smaller than in the converging part. In figure 3 effect of second grade parameter $\alpha = 0, 0.04, 0.08$ is shown on wall shear stress, τ_w, other parameters are chosen to be $\varepsilon = 0.7, \delta = 1/7$ and Re=38. It is observed that for $\alpha = 0$, the present result corresponds to viscous fluid. As the second grade parameter increases wall shear stress increases near the throat and becomes negative in converging and diverging sections due to separation and reattachment points. It is noted that the effect of Re and second grade parameter on wall shear stress have same adverse behavior. In figure 4 effect of ε on wall shear stress is presented. The straight line indicates that there is no stenosis and the flow is Poiseuille flow. By the increase in ε wall shear stress increases near the throat and becomes negative in the converging and diverging sections of the channel, which is the prediction for the points of separation and reattachment. The separation point was considered to be the point nearest the throat where adverse flow along the wall of channel is observed. The point farthest down stream from the throat where back flow occurs is defined as reattachment point. Figure 5 presents the distribution for the point of separation in converging section of the channel for different ε along with fixed δ and α. The separation point lies to the right of minimum point; actually the purpose for zero wall shear stress is to find the critical Reynolds number where separation occurs. It is observed that the critical Re decreases as the ε increases. The theory that the critical Reynolds number decreases with the increase in height of stenosis is verified. Figure 6 predicts the separation point for different α in the converging region for fixed δ and ε. It is observed that with the increase in α critical Re decreases and this behavior has observed earlier. It is also observed that the critical Re have same behavior for negative values of α. In figure 7 zero wall shear stress is plotted for ε having fixed α and δ in diverging section of the channel. The aim of investigation is to determine the critical value of Re at which reattachment occurred in the diverging region of channel. As the critical Re reached the reattachment occurs in the diverging region of channel and separation point occur in the upstream region of channel. It is observed form figure 7 that as ε increases critical Re decreases. In figure 8 zero wall shear stress is presented for various values of α along with fixed ε and δ. It is observed that critical Re decreases as α increases in the diverging region of the channel. It is noted that the reattachment point lies to the left of minimum point and shows similar behavior as in figure 7. Now numerical results are carried out to study the behavior of the temperature distribution graphically for α, ε, Br and Pe. In figure 9 behavior of Newtonian ($\alpha = 0$) and non-Newtonian ($\alpha \neq 0$) effects are observed over the distribution of temperature. Increase in α temperature increases over the stenoised region along with the fixed values of remaining parameters and becomes negative in converging and diverging regions. It is noted that the maximum value of temperature occurs at the middle of the stenoised region. The adverse temperature in these regions causes back flow as observed earlier in wall shear stress. In figure 10 effect of Br is shown over the distribution of temperature for fixed Pe and Re. It is observed that with the increase in Br temperature increases over the stenosis and decrease due to back flow in the converging and diverging section of the channel. Figure 11 presents the effects of Peclet number, Pe, on temperature by keeping other parameters fixed. It is observed that with the increase in Pe temperature increases which firmly ensures that the whole region is dominated by convection. The magnitude of the adverse temperature in the diverging region is smaller as compared to that in the converging region. The adverse temperature in these sections causes back flow. Figure 12 depict the pressure distribution for various values of ε in the converging as well as in the diverging sections of channel for fixed Re, δ and α. It is observed that increase in ε increases the pressure gradient over the stenoised region. The magnitude of adverse pressure gradient is greater than in the converging part as compared with the diverging part. Straight line presents the pressure gradient in the absence of stenosis, which is known as Poiseuille flow. Figure 13 shows the effect of Re over the pressure gradient for fixed ε, δ and second grade parameters (α, β). It is observed that with the increase in Re pressure gradient increases and becomes maximum at the throat. The adverse pressure gradient in the converging and diverging part indicates flow separation and reattachment from wall also confirm the results for the velocity field. The magnitude of adverse pressure gradient is higher in the converging part as compared with the diverging part. Figure 14 describes the distribution of pressure gradient for Newtonian and non-Newtonian behavior. It is observed that with the increase in α pressure gradient increases over stenosis and becomes negative in the converging and diverging sections of channel. It is also noted that the pressure in the non-Newtonian fluid is higher than that in the Newtonian fluids.

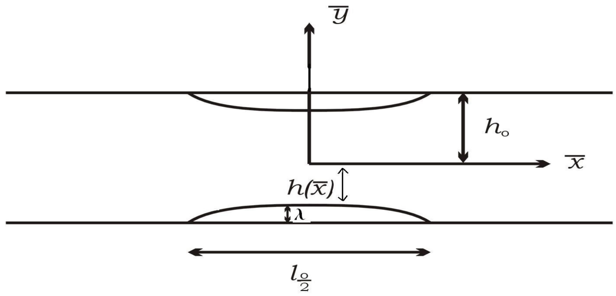

Figure-1
Geometry of the problem

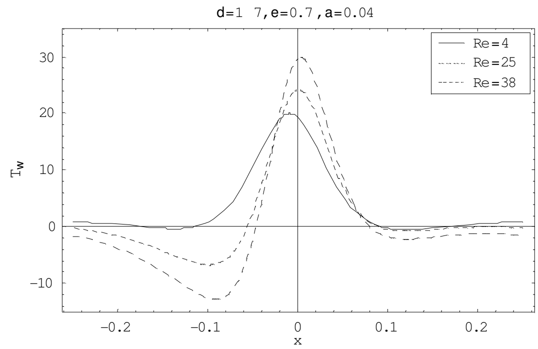

Figure-2
Effect of Re on wall shear stress

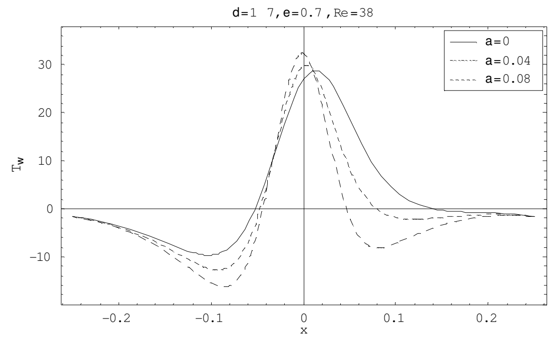

Figure-3
Effect of α on wall shear stress

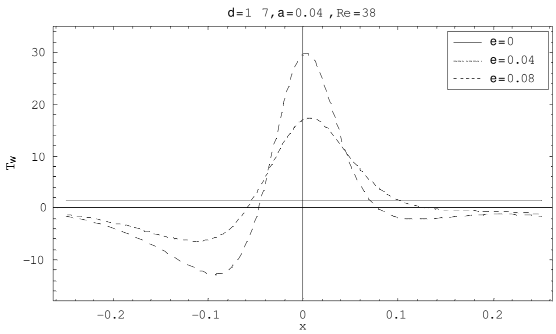

Figure-4
Effect of ε on wall shear stress

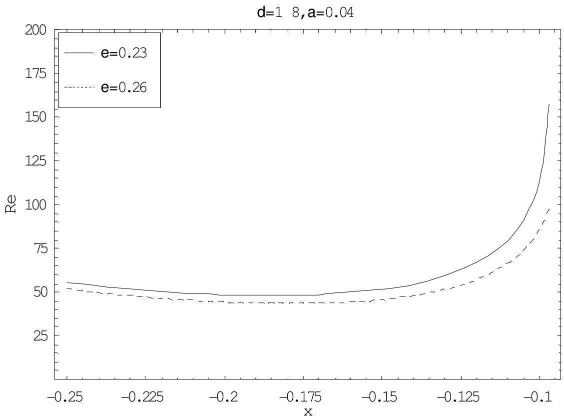

Figure-5
Separation point for ε in converging region

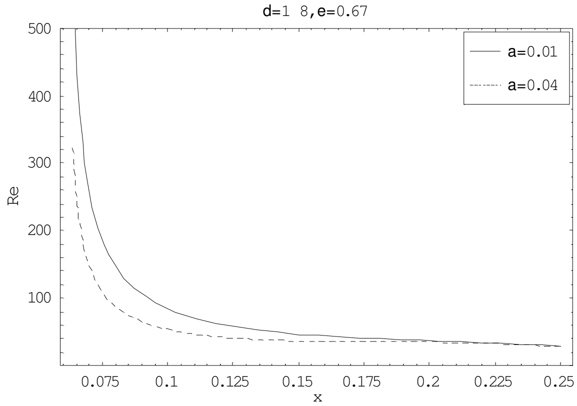

Figure-8
Reattachment point for α in the diverging region

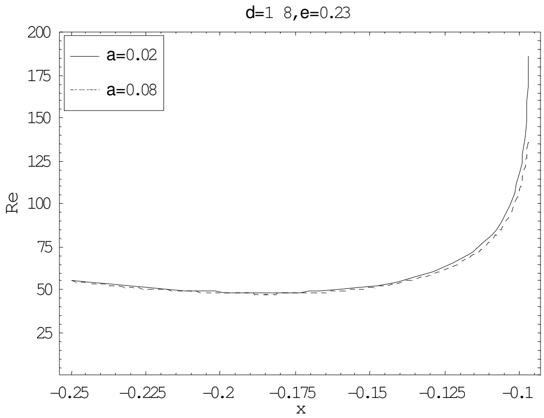

Figure-6
Separation point for α in the converging region

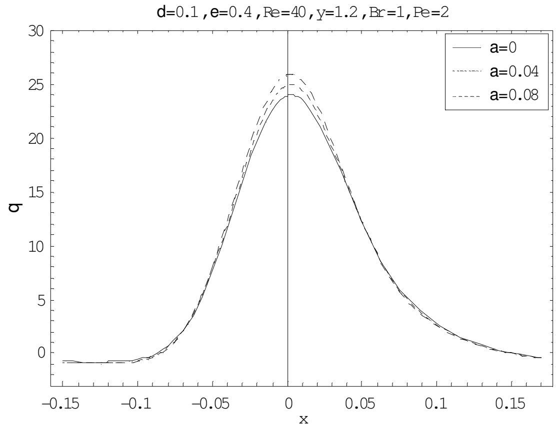

Figure-9
Temperature distribution for α

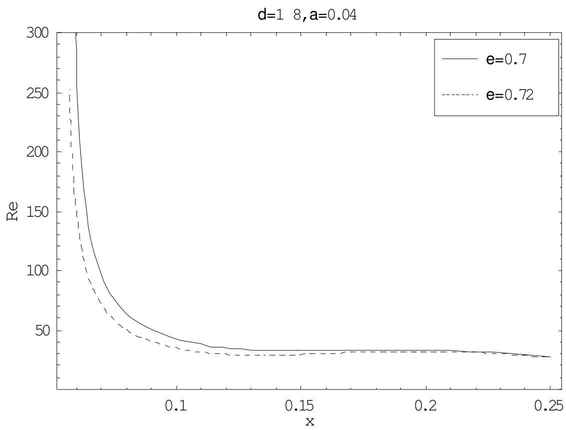

Figure-7
Reattachment point for ε in the diverging region

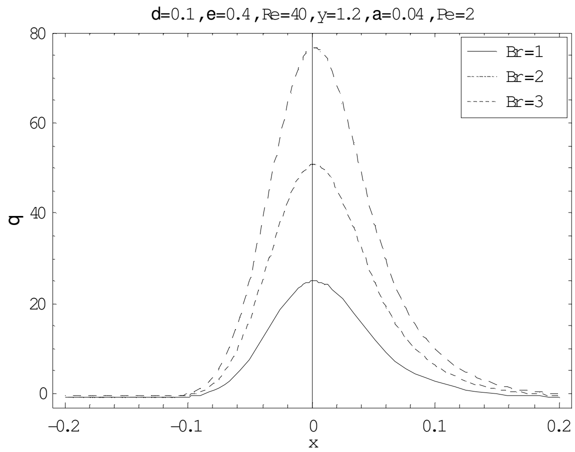

Figure-10
Temperature distribution for Br.

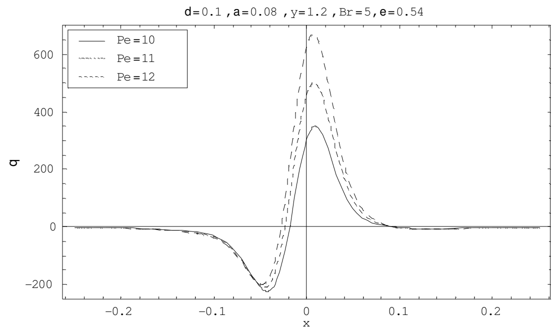

Figure-11
Temperature distribution for Pe.

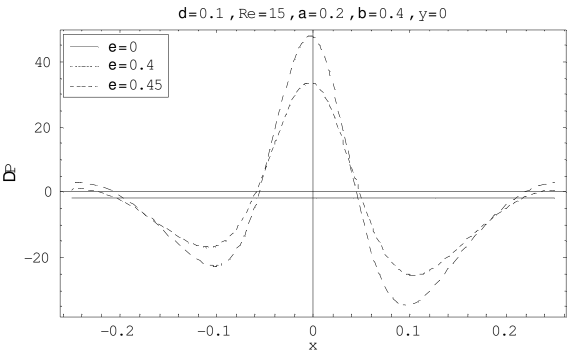

Figure-12
Pressure distribution for ε

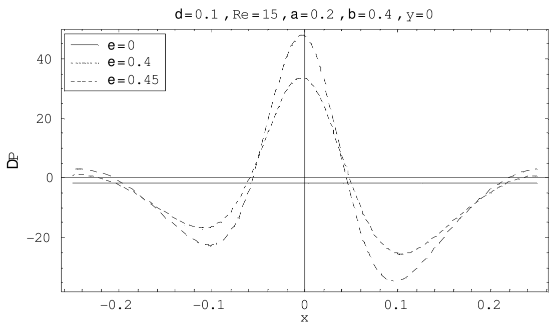

Figure-13
Pressure distribution for Re

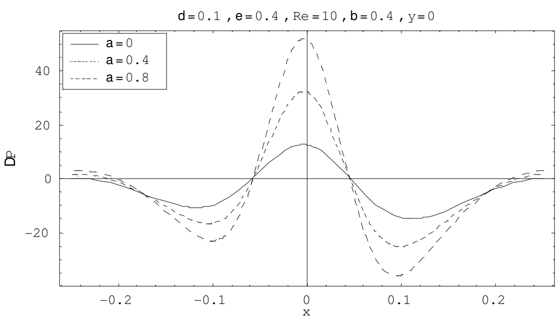

Figure-14
Pressure distribution for α

Conclusion

In the present article, consideration has been given to second grade steady state flow of blood through the channel of infinite length with heat transfer having stenosis of length $l_o/2$. The highly non-linear equations are solved with the help of regular perturbation method. The results thus obtained are discussed graphically in terms of wall shear stress, pressure gradient, separation and reattachment point and temperature distribution. It is noted that by setting $\alpha = 0$, the present model reduces to viscous case[1-6]. Furthermore, the general pattern of streamlines is same as discussed in literature[4-5]. Wall shear stress, separation and reattachment points are similar with available literature[2, 3]. From the present investigation the following conclusions are made: i. Increase in Re increases wall shear stress and pressure gradient. ii. Increase in ε increases wall shears stress, pressure gradient and temperature. iii. Critical Re decreases with an increase in ε. iv. Increase in α leads to increase in temperature, pressure gradient and wall shear stress. v. Temperature increases with an increase in Br and Pe. vi. The critical Re decreases by increasing α in the converging and diverging region.

References

1. Young D.F, Effect of a time-dependent stenosis on flow through a tube, *J. Engng Ind., Trans. Am. Soc. Mech. Engrs.*, **90,** 248-254 (**1968**)
2. Forrester J.H. and Young D.F., Flow through a converging-diverging tube and its implications in occlusive vascular disease, *J. Biomech.*, **3**, 297-316(**1970**)
3. Lee J.S. and Fung Y.C., Flow in locally constricted tubes at low Reynolds number, *J. Appl. Mech.*, **37**, 9-16 (**1970**)
4. Morgan B.E. and Young D.F., An integral method for the analysis of flow in arterial stenoses., *J. Math. Bio.* **36**, 39-53(**1974**)
5. Haldar K., Analysis of separation of blood flow in constricted arteries, *Archives of Mechanics*, **43(1)**, 107-113 (**1991**)
6. Chow J.C.F., Soda K. and Dean C., On laminar flow in wavy channel, Developments in Mechanics, **6**, proceedings of the 12th Midwestern Mechanics Conference
7. Shukla J.B., Parihar R.S., Rao B.R.P., Effects of stenosis on Non-Newtonian flow of the blood in an artery, *Bull. of Math. Bio.* **42,** 283-294 (**1980**)
8. Mishra J.C. and Shit G.C., Blood flow through arteries in a pathological state, a theoretical study, *Int. J. of Eng. Sci.*, **44**, 662-671 (**2006**)
9. Haldar K., Effect of the shape of stenosis on the resistance to blood flow through an artery, *Bull. of Math. Bio.* **47(4)**, 545-550 (**1985**)

10. Cheng Tu and Michel Deville, Pulsatile flow of non-Newtonian fluids through arterial stenoses, *J. Biomech.* **29**(**7**), 899-908 **(1996)**

11. Vahdati S., Tavassoli Kajani M. and Ghasemi M., Application to Homotopy Analysis Method to SIR Epidemic Model, *Res. J. Recent Sci.,* **2**(**1**), 91-96 **(2013)**

12. Thundil Karuppa Raj R. and Ramsai R., Numerical study of fluid flow and effect of inlet pipe angle in catalytic converter using CFD, *Res. J. Recent Sci.,* **1**(**7**), 39-44 **(2012)**

13. Chauhan Rajsinh B. and Thundil Karuppa Raj R., Numerical investigation of external flow around the Ahmed reference body using computational fluid dynamics, *Res. J. Recent Sci.,* **1**(**9**), 1-5 **(2012)**

14. Siddiqui A.M. and Kaloni P.N., Certain inverse solutions of a non-Newtonian fluid, *Int. J. Non-Linear Mech*., **21(6)**, 459-473 **(1986)**