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Solving Integral equations on Semi-Infinite Intervals via Rational third kind Chebyshev functions

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Abstract

In this paper, we employ the rational third kind Chebyshev functions on the interval $[0,\infty)$, to solve the linear integral equations of the second kind over infinite intervals. The properties of the rational third kind Chebyshev functions together with the Galerkin method are applied to reduce the integral equation to a system of linear algebraic equations. Using two numerical examples, we show that our estimates have a good degree of accuracy.

Keywords: Integral equation, Rational third kind Chebyshev functions, Semi-infinite interval, Galerkin method.

Introduction

In recent years, many different basic functions have been used to estimate the solution of integral equations, such as wavelets[1-3], orthonormal bases[4,5] and combination of Block-Pulse functions[6,7]. Besides many different method have been used to estimete the solution of mathematics equations, see[8,9].

In this paper we are going to use an efficient base that is rational third kind Chebyshev functions on $[0,\infty)$, which is called RTC functions.

Properties of RTC functions

In this section, we present some properties of RTC functions.

RTC functions: The third kind Chebyshev polynomials are orthogonal in the interval $[-1,1]$ with respect to the weight function

$$\rho(x)=\sqrt{\frac{1+x}{1-x}}$$

and we find that $V_n(x)$ satisfies the recurrence relation[10]

$$V_0(x)=1,\quad V_1(x)=2x-1,$$
$$V_n(x)=2xV_{n-1}(x)-V_{n-2}(x),\quad n\geq 2. \tag{1}$$

The RTC functions are defined by

$$R_n(x)=V_n\left(\frac{x-L}{x+L}\right),$$

thus RTC functions satisfy

$$R_0(x)=1,\quad R_1(x)=2\left(\frac{x-L}{x+L}\right)-1,$$
$$R_n(x)=2\left(\frac{x-L}{x+L}\right)R_{n-1}(x)-R_{n-2}(x),\quad n\geq 2. \tag{2}$$

Function approximation: Let $w(x)=\frac{2\sqrt{Lx}}{(x+L)^2}$ denotes a non-negative, integrable, realvalued function over the interval $I=[0,+\infty)$. We define

$$L_w^2(I)=\left\{y:I\to R\mid y \text{ is measurable and } \|y\|_w<\infty\right\}, \tag{3}$$

where

$$\|y\|_w=\left(\int_0^\infty |y(x)|^2 w(x)dx\right)^{\frac{1}{2}}, \tag{4}$$

is the norm induced by the scalar product

$$\langle y,z\rangle_w=\int_0^\infty y(x)z(x)w(x)dx. \tag{5}$$

Thus $\{R_n(x)\}_{n\geq 0}$ denote a system which are mutually orthogonal under Eq. (5), i.e.,

$$\int_0^\infty R_n(x)R_m(x)w(x)dx=\pi\delta_{nm}, \tag{6}$$

where δ_{nm} is the Kronecker delta function[11,12]. This system is complete in $L_w^2(I)$; as a result, any function $y\in L_w^2(I)$ can be expanded as follows:

$$y(x)=\sum_{k=0}^{\infty} a_k R_k(x), \tag{7}$$

with

$$a_k=\frac{1}{\pi}\langle y,R_k\rangle_w. \tag{8}$$

The a_k's are the expansion coefficients associated with the family $\{R_k(x)\}$. If the infinite series in Eq. (7) is truncated, then it can be written as

$$y(x)\approx\sum_{k=0}^{N} a_k R_k(x)=A^T R(x), \tag{9}$$

where

$$A=[a_0,a_1,\ldots,a_N]^T,$$

$$R(x)=[R_0(x),R_1(x),\ldots,R_N(x)]^T.$$

We can also approximate the function $k(x,t)$ in $L_w^2(I\times I)$ as follows

$$k(x,t)\approx k_M(x,t)=R^T(x)KR(t), \tag{10}$$

where K is an $M\times M$ matrix that

$$K_{ij}=\frac{1}{\pi^2}\langle R_i(x),\langle k(x,t),R_j(t)\rangle_w\rangle_w,\quad i,j=0,1,\ldots,M.$$

Product integration of the RTC functions: We also define the matrix P_a as follows

$$P_a=\int_0^a R(t)R^T(t)dt. \tag{11}$$

To illustrate the calculation P_a we choose $a=1$, we obtain

$$P_1=\begin{bmatrix} 1 & 1-4\ln2 & 9-12\ln2 & 17-24\ln2 & \frac{83}{3}-40\ln2 & \cdots \\ 1-4\ln2 & 9-8\ln2 & 9-16\ln2 & \frac{53}{3}-28\ln2 & \frac{91}{3}-44\ln2 & \cdots \\ 9-12\ln2 & 9-16\ln2 & \frac{59}{3}-24\ln2 & \frac{67}{3}-36\ln2 & \frac{559}{15}-52\ln2 & \cdots \\ 17-24\ln2 & \frac{59}{3}-28\ln2 & \frac{67}{3}-36\ln2 & \frac{559}{15}-48\ln2 & \frac{623}{15}-64\ln2 & \cdots \\ \frac{83}{3}-40\ln2 & \frac{91}{3}-44\ln2 & \frac{559}{15}-52\ln2 & \frac{623}{15}-64\ln2 & \frac{6217}{105}-80\ln2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Second kind integral equations over semi-infinite interval: In this phase, at first we consider the following second kind integral equation,

$$y(x)=f(x)+\int_0^a k(x,t)y(t)dt,\quad x\in I, \tag{12}$$

where $y,f\in L_w^2(I)$ and $k\in L_w^2(I\times I)$. Then we approximate f, y and k using (9) and (10) as follows

$$y(x)\approx Y^T R(x),$$

$$f(x)\approx F^T R(x),$$

$$k(x,t)\approx R^T(x)KR(t).$$

With substituting in (12) we have

$$\begin{aligned} R^T(x)Y &= R^T(x)F+\int_0^a R^T(x)KR(t)R^T(t)Ydt \\ &= R^T(x)F+R^T(x)K\left(\int_0^a R(t)R^T(t)dt\right)Y \\ &= R^T(x)(F+KP_aY), \end{aligned}$$

then one can conclude that

$$(I_{N+1}-KP_a)Y=F, \tag{13}$$

where I_{N+1} is the identity matrix. By solving this linear system of algebraic equations we can find the vector Y.

Numerical examples

With best of our knowledge this is the first time that the following examples are solved.

Example 1.: Consider the integral equation

$$y(x)=-\frac{1}{x+1}+\int_0^1\frac{4y(t)}{(x+1)(t+1)}dt,\quad x\in I, \tag{14}$$

with the exact solution $y(x)=\frac{1}{x+1}$.

In order to solve this example using the present method, we choose $L=1$ and $N=1$ therefore we have

$$F=\begin{bmatrix}-\frac{1}{4}\\ \frac{1}{4}\end{bmatrix},\qquad K=\frac{1}{4}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

So by solving the linear system $(I_2-KP_1)Y=F$ we obtain $Y=\begin{bmatrix}\frac{1}{4} & -\frac{1}{4}\end{bmatrix}^T$, thus

$$y(x)=Y^TR(x)=\frac{1}{4}R_0(x)-\frac{1}{4}R_1(x)=\frac{1}{x+1},$$

which is the exact solution.

Example 2.: Consider the integral equation

$$y(x)=e^{-x-2}+\int_0^1 2e^{-x-t}y(t)dt, \quad x\in I, \tag{15}$$

with the exact solution $y(x)=e^{-x}$. In Table 1, a comparison is made between the values of y obtained using the proposed method with $N=9$, 14 and the exact solution.

Table-1
Numerical results of $y(x)$ for Example 2

x	$y_9(x)$	$y_{14}(x)$	**Exact**
0	1.01004	1.00109	1.00000
1	0.36957	0.36791	0.36788
2	0.13467	0.13540	0.13534
3	0.05060	0.04972	0.04979
4	0.01924	0.01836	0.01832
5	0.00712	0.00681	0.00674
6	0.00206	0.00248	0.00248
7	0.00042	0.00085	0.00091
8	0.00050	0.00026	0.00034
9	0.00020	0.00006	0.00012
10	0.00009	0.00001	0.00005

Conclusion

The fundamental goal of this paper has been to construct an approximation to the solution of the second kind integral equations in a semi-infinite interval. In the above discussion, the Galerkin method with **RTC** functions, which have the property of orthogonality, were employed to achieve this goal. The contribution of this paper is that we do not reform the problem to a finite domain and with an small value of N accurate results are obtained. There is a good agreement between obtained results and exact values that demonstrates the validity of the present method for this type of problems and gives the method a wider applicability.

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