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Memetic Heuristic Computation for Solving Nonlinear Singular Boundary Value Problems Arising in Physiology

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Abstract

We present a stochastic numerical method based on memetic heuristic computing for the approximate numerical solution of a class of nonlinear singular boundary value problems arising in physiology. The solution of the nonlinear problem is represented by the linear combination of some log sigmoid basis functions. A fitness function representing the mean square error consisting of unknown adaptable parameters (chromosome) is formulated. The minimization of the fitness function is carried out by employing Genetic algorithm (GA), Interior Point algorithm (IPA), and hybrid scheme combining GA and IPA (GA-IPA) for the optimal values of the chromosome. The efficiency of the presented method is testified by solving two examples from physiology. The results prove that the proposed heuristic method provides the approximate numerical solution comparable to some of the existing conventional numerical solutions.

Keywords*:* Nonlinear singular boundary value problem (BVP), Genetic algorithm (GA), Interior point algorithm (IPA), Evolutionary algorithms (EA's), finite difference method, B-spline method, Oxygen diffusion problem, nonlinear heat conduction model

Introduction

The singular boundary value problems (BVPs) are often encountered in many areas of engineering and science including the models of gas dynamics, thermal explosions, atomic physics, nuclear physics, and chemical reactions[1]. The exact analytical solution of most of the nonlinear singular BVPs either does not exist or it is difficult to be achieved, therefore many approximate analytical and numerical methods such as finite difference method, collocation method, B-spline functions, variational iteration method etc have been proposed to tackle these problems. A comprehensive survey of computational techniques used for singular BVPs given by Kumar M. and Singh N.[1].

The main aim of this paper is to present a stochastic method based on heuristic computing for the study of a class of nonlinear singular boundary value problems arising in physiology of the following form[2-5]

$$y''(x) + \frac{m}{x} y'(x) = f(x, y),\ 0 \le x \le 1 \quad (1)$$

with the following boundary conditions

$$y'(0) = 0 \quad (2)$$

$$\propto y(1) + \beta y'(1) = \gamma \quad (3)$$

Here α, β and γ are finite constants. The restrictions normally imposed on $f(x,y)$ are that it is continuous, $\frac{\partial f}{\partial y}$ exists and is continuous, and also nonnegative, $\forall\, 0 \le x \le 1$. The singular boundary value problem (1) – (3) with m = 0, 1, 2 appear in the study of many tumor growth problems[6-7] with linear $f(x,y)$ and with nonlinear $f(x,y)$ of the following form:

$$f(x,y) = \frac{ny}{y+k}\ , n > 0, k > 0 \quad (4)$$

and with m = 2 in the study of steady state oxygen diffusion in a spherical cell with Michaelis–Menten uptake kinetics[8-11].

A similar problem for m = 2 arise in the study of the distribution of heat sources in the human head[12-14] in which

$$f(x,y) = -ne^{-nky}\ , n > 0, k > 0 \quad (5)$$

Many authors have proposed several different approximate analytical and numerical techniques for the solution of singular boundary value problem (1) – (3)[2-5,15-16]. Ravi Kanth and Bhattacharya[3] applied cubic spline method, Khuri and Sayfy[5] employed combination of modified decomposition method and cubic B-spline collocation, Alipanah[4] applied a nonclassical pseudospectral method, Caglar et al.[2] used B-spline functions, Rashidinia et al.[15] applied non-polynomial cubic spline method for the approximate numerical solution of the problem (1) – (3). Although a rich variety of standard numerical methods have been proposed for the numerical treatment of BVPs but some may fail to converge when applied to singular BVPs[1,17].

In recent years evolutionary computing based methods have attracted much attention. Several authors have used evolutionary computing based methods for the solution of linear and nonlinear ordinary differential equations (ODEs) and singular BVPs. To mention a few, Arqub et al.[17] employed continuous genetic algorithm (CGA) for the numerical solution of linear and nonlinear singular BVPs. Khan et al.[18] used genetic algorithm (GA) based differential equation neural (DEN) network for the solution of nonlinear singular system of polytrophic and isothermal sphere model. Malik et al.[19] employed a stochastic method based on hybrid approach of GA and interior point algorithm (IPA) for the solution of nonlinear ODE of Duffing van der pol oscillator model. Behrang et al.[20] used a particle swarm optimized (PSO) neural network (NN) method for the solution of nonlinear ODE of Darcian fluid of vertical cone implanted in porous media. The computational methods based on evolutionary computing have been proved to give the solution to nonlinear problems with greater accuracy besides some advantages over the standard numerical methods [17-20].

The motivation for this work is to introduce an alternate approximate stochastic numerical method for the solution of nonlinear singular boundary value problems arising in physiology. In this study genetic algorithm (GA), interior point algorithm (IPA) and hybrid scheme of GA and IPA called here as GA-IPA have been employed to obtain the numerical solution of nonlinear singular BVP (1) – (3). The efficiency and reliability of the proposed method are tested by solving two examples arising in physiology. To prove the validity of our results comparisons have been made with some numerical methods including B-spline method [2], cubic spline method [3], nonclassical pseudospectral method [4], and combined modified decomposition method and cubic B-spline collocation[5].

Methodology

The description of the proposed methodology is given and the learning procedural steps of heuristic hybrid schemes GA-IPA and GA-ASA used in this study are also presented.

Description of the Proposed Methodology: We may assume that the approximate solution $y(x)$ and its first and second derivatives, $y'(x),\ and\ y''(x)$ can be represented by a linear combination of some basis functions as follows.

$$y(x) = \sum_{i=1}^{m} a_i\, \varphi(b_i x + c_i) \qquad (6)$$

$$y'(x) = \sum_{i=1}^{m} a_i b_i\, \dot{\varphi}(b_i x + c_i) \qquad (7)$$

$$y''(x) = \sum_{i=1}^{m} a_i b_i^{\,2}\, \ddot{\varphi}(b_i x + c_i) \qquad (8)$$

where $\varphi(x)$ is taken as the log sigmoid function which is given by

$$\varphi(x) = \frac{1}{1+e^{-x}} \qquad (9)$$

$a_i\, , b_i\, ,\ and\ c_i$, are real valued unknown adaptable parameters (chromosome), and m is the number of basis functions.

To learn the optimal values of the chromosome a fitness function of the given problem is developed as follows:

$$\varepsilon_j = \varepsilon_1 + \varepsilon_2 \qquad (10)$$

where $\varepsilon_1$is represents the mean square error linked with the given differential equation, and ε_2 represents the mean square error linked with the boundary conditions, while j is the number of generations.

The fitness function given by (10) is a function of the unknown adaptable parameters (a_i, b_i, c_i) given in (6) to (8). The optimal values of the unknown adaptable parameters are achieved by the minimization of (10) using evolutionary algorithms. The optimal values of the adaptable parameters are acquired and consequently the approximate numerical solution $y(x)$ of the problem at hand is straight away obtained from (6).

Evolutionary Algorithms: Evolutionary algorithms (EAs) are population based stochastic search methods that mimic the process of natural evolution[21]. In past few decades evolutionary algorithms (EAs) have received a terrific attention and they have been extensively applied to a wide range of practical problems in engineering and science[21].

Genetic Algorithm (GA) is one of the most popular stochastic global search techniques in EAs which has been most widely used due to its simplicity and robustness in various optimization problems[22]. The GA begins with an initial population of individuals called chromosome. Each individual with a population represents one possible solution to the given problem. Each individual within a population is evaluated using a problem dependent fitness function. The algorithm evolves to global optimal solution over the successive generations using three primary operations selection, crossover, and evaluation[23].

One of the recent growing areas in evolutionary algorithms (EAs) is the memetic computing in which global and local search optimization methods are hybridized to achieve improved performance of the solution to the given problem[21]. The interior point algorithm (IPA) also called as barrier method is one of the widely used local search optimization methods. The IPA navigates through the interior feasible region until it reaches an optimal solution. The algorithm solves a sequence of barrier subproblems by applying either Newton step or conjugate gradient (CG) step at each iteration. A problem specific merit function is decreased by the algorithm[24].

In this study we have hybridized stochastic global search method GA with the local search method IPA. The GA has been used as global optimizer while IPA has been employed for the local search refinement. The optimal chromosome acquired by the GA is given as a starting point for the IPA which performs the local refinement of the chromosome to improve the quality of the solution. The procedural steps of the hybrid scheme GA-IPA are given as follows while the parameter settings for the execution of these algorithms are given in table-1.

Algorithm 1: GA hybridized with IPA
Step 1: (Initialization of Population) A population of N chromosomes or individuals is generated using random number generator. In each population there are M numbers of genes which represent the number of unknown adaptable parameters. **Step 2:** (Evaluation of Fitness) Determine the fitness of each individual in the current population using a fitness function of the given problem and sorting the individuals on the basis of the fitness. **Step 3:** (Stopping Criteria) The algorithm ends if the fitness reaches a certain value or a certain number of cycles has reached. If the stopping criterion is fulfilled then go to step 6 for local search refinement, else continue and repeat steps 2 to 5. **Step 4:** (Reproduction) Crossover operation is used for populating a new generation. The new generation consists of the parents that are selected on the basis of their fitness which produces offspring (children) to act as parents for the next generation. **Step 5:** Mutation This operation is optional and it is carried if there is no improvement in the fitness in a generation. Mutation introduces intermittent changes in the genes to preserve the genetic diversity. **Step 6:** (Local Search Fine Tuning) The optimal chromosome acquired by the GA is provided to IPA as a starting point for fine tuning and improvement.

Table-1
Parameter Settings of Algorithms

GA		**IPA**	
Parameters	**Settings**	**Parameters**	**Settings**
Population size	240	Start point	Optimal chromosome from GA
Chromosome size	30	Maximum iterations	1000
Selection function	Stochastic uniform	Maximum function evaluations	150000
Mutation function	Adaptive feasible	Function tolerance	1e-18
Crossover function	Heuristic	Derivative type	Central differences
Hybridization	IPA	Hessian	BFGS
No. of generations	1000	Sub problem algorithm	ldl factorization
Function tolerance	1e-18	Initial barrier parameter	0.1
Bounds	-10, +10	Bounds	-10, +10

Results and Discussion

The proposed methodology is implemented on two problems arising in physiology. For the efficacy and viability of the proposed method, comparisons of the results are made with conventional approximate numerical methods such as B-spline method[2], cubic spline method[3], nonclassical pseudospectral method[4], and combined modified decomposition method and cubic B-spline collocation[5].

Example 1: We consider the special case of (1) – (4), the oxygen diffusion problem, with $m = 2, n = 0.76129, k = 0.03119, \propto = \gamma = 5, and\ \beta = 1$ as follows[2-5].

$$y''(x) + \frac{2}{x}y'(x) = \frac{0.76129y}{y+0.03119} \quad (11)$$

With the boundary conditions

$$y'(0) = 0 \quad (12)$$

$$5y(1) + y'(1) = 5 \quad (13)$$

The approximate numerical solution of (11) using the proposed method is obtained by developing its fitness function as follows

$$\varepsilon_1 = \frac{1}{11}\sum_i^{11}\left[y''(x_i) + \frac{2}{x_i}y'(x_i) - \frac{0.76129y}{y+0.03119}\right]^2 \quad (14)$$

$$\varepsilon_2 = \frac{1}{2}\{(y(0))^2 + ((5y(1) + y'(1)) - 5)^2\} \quad (15)$$

The number of basis functions is taken equal to 10. Therefore the fitness function ε_j is given as follows

$$\varepsilon_j = \frac{1}{11}\sum_{i=1}^{11}(y''(x_i) + \frac{2}{x_i}y'(x_i) - \frac{0.76129y}{y+0.03119})^2 + \frac{1}{2}\{(y(0))^2 + ((5y(1) + y'(1)) - 5)^2\}\Big|_j \quad (16)$$

where $y(x), y'(x)\ and\ y''(x)$ are given by (6) - (8) respectively.

The fitness function given by (16) is minimized by applying GA, IPA and hybrid scheme GA-IPA for the learning of the unknown adaptable parameters. For the simulations Matlab 7.6.0 has been used in this work.

The parameter settings used for the implementation of the algorithms GA, IPA, and GA-IPA are given in table-1. The total number of unknown adaptable parameters (a_i, b_i , c_i) is chosen equal to 30. The values of these unknown adaptable parameters are restricted between -10 and + 10. This was noticed after several simulations that these bounds give us optimum results. The optimal values of the unknown adaptable parameters achieved by the algorithms GA, IPA, and GA-IPA with the prescribed settings are given in table 2. Consequently the approximate solution $y(x)$ of the oxygen diffusion problem (11) – (13) is obtained from (6) by using the values of unknown adaptable parameters from table-2. The approximate numerical results by our method are given in table-3. For comparison we present the numerical results of this problem by other methods given in table-4. It is evident from the results that the proposed method provides the approximate numerical solution of oxygen diffusion problem (11) – (13) comparable with other methods[2,3,5].

Table-2
Optimal values of unknown adaptable parameters for Example 1

Algorithm	i	a_i	b_i	c_i
GA	1	-2.50660510850856	0.91508858909537	-0.33605544286137
	2	0.17729566883183	-1.47048377808815	-0.92574105852736
	3	2.04672323692258	0.53113795072383	-0.33521014901773
	4	0.84424898190682	0.69622836279978	-0.84614541704560
	5	0.63708254632236	-1.82341669644584	-1.58714388985315
	6	-0.46739572429308	-0.15944966742429	-2.51570749968078
	7	0.81098933168428	1.09517679970993	0.54056579471413
	8	1.79308328430866	0.82521666943624	-2.09048029244534
	9	-0.29653584790633	-0.28613667436285	-0.83896324468903
	10	0.16469692668097	0.63590184335478	-1.85014932028984
IPA	1	-0.07211394360537	0.99227614225936	0.75279789489787
	2	1.62877525607231	0.50589883606511	-0.87608438218202
	3	0.41211348503627	0.77712989320100	0.18003088299471
	4	2.07289088469713	0.84761867382155	-2.01143269416064
	5	-0.60346823281644	0.64716488915703	1.09430299866458
	6	-0.63589374524471	-0.08915698494200	-1.84214623911952
	7	1.48811852796271	-0.97986507435511	-1.24373376963043
	8	-0.25675527387318	-0.66861398555971	0.25761260664088
	9	-0.25574323046808	0.77749438902812	-0.06193063668060
	10	0.49006769308929	-1.27379475016525	1.55041159187514
GA-IPA	1	-1.53974023488146	0.49376879357525	0.11917256911990
	2	0.19115641118917	-1.36844134866275	-1.09477599835617
	3	1.80408331908797	-0.00026854607897	0.11954584978765
	4	0.86505960272549	0.46896761812380	-0.98623255358284
	5	0.99353910366711	-1.12938015521738	-2.39355774550556
	6	-0.43923584338261	-0.14355683990310	-2.73915913522659
	7	0.17778647141545	1.09836625816913	0.46522397749471
	8	2.67860427393691	0.76942055786799	-2.71461456900907
	9	-0.11242948367301	-0.24061341285806	-0.84131638108855
	10	0.89533278054165	0.72982122908866	-1.99535741013724

Table-3
Numerical results of Example 1 by our method

x	GA	IPA	GA-IPA
0.1	0.828452624272893	0.828469664625725	0.828478525695687
0.2	0.829688567485952	0.829701986546119	0.829704436800511
0.3	0.833358349299386	0.833371699714595	0.833373276372227
0.4	0.839475507646817	0.839487071466120	0.839488470974904
0.5	0.848042203584424	0.848051399810071	0.848051833709369
0.6	0.859056888715474	0.859065523180713	0.859064620762517
0.7	0.872519055695347	0.872529890249512	0.872528299630194
0.8	0.888431321322146	0.888445998568066	0.888445001060461
0.9	0.906799304830408	0.906817042359152	0.906817649393439
0.1	0.927629857152453	0.927647660867015	0.927649784982863

Example 2: Consider the special case of (1), (2), (3) and (5), the non-linear heat conduction model of the human head, with $m = 2, n = 1,\ k = 1$ as follows[2-5].

$$y''(x) + \frac{2}{x}y'(x) = -e^{-y} \quad (17)$$

with the following two cases of boundary conditions

Case 1)

$$y'(0) = 0 \quad (18)$$

$$y(1) + y'(1) = 0 \quad (19)$$

Here $\propto = \beta = 1, and\ \gamma = 0$

Case 2)

$$y'(0) = 0 \quad (20)$$

$$0.1y(1) + y'(1) = 0 \quad (21)$$

Here $\propto = 0.1,\ \beta = 1, and\ \gamma = 0$

To find the approximate solution of example 2 its fitness function is developed for each case as follows.

$$\varepsilon_j = \frac{1}{11}\sum_{i=1}^{11}(y''(x_i) + \frac{2}{x_i}y'(x_i) + e^{-y})^2 + \frac{1}{2}\{(y(0))^2 + ((y(1) + y'(1)))^2\}\Big|_j \quad (22)$$

$$\varepsilon_j = \frac{1}{11}\sum_{i=1}^{11}(y''(x_i) + \frac{2}{x_i}y'(x_i) + e^{-y})^2 + \frac{1}{2}\{(y(0))^2 + ((0.1y(1) + y'(1)))^2\}\Big|_j \quad (23)$$

The fitness functions given by (22) and (23) are minimized by applying GA, IPA and hybrid scheme GA-IPA for the learning of the unknown adaptable parameters.

The algorithms are executed according to the prescribed settings in table-1. The optimal values of unknown adaptable parameters (a_i, b_i , c_i) for this problem are presented in table-5 and table-6 for case 1 and case 2 respectively. The approximate numerical results by our method are presented in table-7. For comparison we also present the numerical results of this problem by other methods given in[2-4] in table-8. From the comparison of the results the efficiency of our method is evident. Further the results reveal the improved performance of the hybrid approach of GA-IPA.

Conclusion

A new stochastic method based on memetic heuristic computing has been employed for the approximate numerical solution of a class of nonlinear singular boundary value problems arising in physiology. From the comparison of the results made with some conventional numerical methods it can be concluded that the proposed heuristic approach is effective and viable to broad range of nonlinear singular boundary value problems. It is observed that the hybridization of GA with local search algorithm such as IPA is very effective and handy. Furthermore the presented method can give the approximate numerical solution of the given problem conveniently and on the continuous grid of time once the optimal values of the unknown adaptable parameters has been accomplished.

Table- 4
Comparison of numerical results of Example 1 between our method and other methods[2,3,5]

x	Our Method GA-IPA	B-Spline[2]	MDM-cubic B-spline[5]	Cubic B-spline[3]
0.1	0.828478525695687	0.82848327295802	0.82848329481355	0.82848327300049
0.2	0.829704436800511	0.82970607521884	0.82970609688790	0.82970607526174
0.3	0.833373276372227	0.83337471691089	0.83337473804308	0.83337471695366
0.4	0.839488470974904	0.83948989814383	0.83948991833986	0.83948989818632
0.5	0.848051833709369	0.84805277036165	0.84805278876051	0.84805277040321
0.6	0.859064620762517	0.85906491397434	0.85906492753032	0.85906491401483
0.7	0.872528299630194	0.87252830841853	0.87252831569855	0.87252830845759
0.8	0.888445001060461	0.88844529589927	0.88844529949702	0.88844529593717
0.9	0.906817649393439	0.90681854026297	0.90681854179965	0.90681854029886
0.1	0.927649784982863	0.92765098252660	0.92765098305256	0.92765098256074

Table-5
Optimal values of unknown adaptable parameters for Example 2 (case 1)

Algorithm	i	a_i	b_i	c_i
GA	1	-1.12327280832697	1.79337358890822	-2.12384215660137
	2	-3.13016151577742	0.48246650235393	2.51699877215490
	3	1.74878803122207	2.75572659648116	1.85404984767479
	4	-0.03417756354181	-4.61621535090682	-1.24312959239023
	5	0.63471534729992	1.34659593859644	-3.82209476460761
	6	1.03408046054757	-0.37147929989994	-4.86585454010577
	7	1.30217421422479	2.02457674265681	-1.03005029454391
	8	3.24730557810278	-1.00304097829012	0.87149921419898
	9	-0.89150588449611	-2.39165685712962	2.89627769669125
	10	0.50300653166396	-4.10376028109355	-1.80937890497580
IPA	1	-0.61931827960276	0.03764901495447	0.23010801612501
	2	-3.01177636040799	1.55216320744863	-5.16182814257681
	3	-0.10089552779581	-0.26471954438667	0.60701964297834
	4	0.76474640113586	1.46167782485829	2.32104707708574
	5	-1.11540262529679	0.09158469244436	-0.56557936714867
	6	0.81044580545661	0.17090193738545	0.49894009460589
	7	0.67686234290422	-1.18560136979938	1.90772346556965
	8	-0.10902894807365	0.14193266645683	-1.33401789753674
	9	0.19200734756989	0.07103979288489	-1.23438937389536
	10	-0.98326631172991	-0.07203518706278	0.49724449595999
GA-IPA	1	-1.29708564868413	0.92063744633280	-1.66937165003394
	2	-2.10629280869242	0.38352606317426	1.30381370631817
	3	0.94929872432130	1.97236355046111	1.34759549757890
	4	0.28771439230222	-2.25743735024180	-0.88314661227434
	5	-0.27774137234434	0.07539618227323	-2.26909407711221
	6	0.64095446448981	-0.25610944231123	-2.34092803646557
	7	0.38747862918347	1.95597421169347	-0.73689821167995
	8	2.40387272881544	-0.63919068204860	1.30264691093135
	9	-0.58901649702233	-1.87639524476186	2.34665849245932
	10	-0.35483789751106	-1.81581112121196	-0.71970019442402

Table- 6
Optimal values of unknown adaptable parameters for Example 2 (case 2)

Algorithm	i	a_i	b_i	c_i
GA	1	-1.87089533724844	0.19805182198055	-1.47076115192967
	2	0.28591866237701	1.50059916841293	2.06516676439330
	3	0.60346379591220	-0.92118968672311	1.95147412538976
	4	0.51056733311811	0.39374765721594	1.75547383725058
	5	-2.09294185919529	0.27543547663242	3.21347642069440
	6	0.21653564572270	-1.19614570660237	0.31213941496941
	7	-0.14615364007079	-1.07658104562748	-0.44268102804453
	8	1.14093740819197	-0.39994685475271	0.42248527094357
	9	1.59074662803655	0.70165842499501	-0.66686315532486
	10	1.05653984555036	-0.77922937616325	2.77998579625194
IPA	1	-0.34685840784782	0.44041567369138	-1.63864125165239
	2	0.41057001130553	0.48973162867117	-1.68085692923495
	3	1.15925527246105	0.87169329522954	2.30122824487012
	4	0.05260608357958	0.03910483996635	-0.56029472284878
	5	0.13507500850458	-0.44758394758680	-1.51019747595450
	6	-2.21352276451025	0.78244768188113	-3.12781087832936
	7	-0.33039873415728	-0.43828966876423	-1.72693497049137
	8	-0.11110520437373	0.22650346318431	-1.65347105005859
	9	0.43919487378095	-0.33290803954683	-1.46127701056223
	10	0.15925465514660	-0.13728727048227	1.10749799561891
GA-IPA	1	-1.86849789853038	0.32676800947685	-1.45876494913674
	2	0.35531781157450	1.44662664213565	2.17569422120668
	3	0.54075565077487	-0.52516463275994	1.98525942672710
	4	0.50521816879522	0.43571042779363	1.74071986885599
	5	-2.10086491465531	0.21050319325445	3.20079008467813
	6	0.18400738873472	-0.98951798178992	0.42898978894560
	7	-0.14926470197882	-1.07585985668589	-0.42209937593746
	8	1.12479914543136	-0.38013543822792	0.43657499037072
	9	1.56256335893424	0.63389417891404	-0.68899153524522
	10	1.09814291356268	-0.82137435059103	2.81289596083360

Table-7
Approximate numerical results of Example 2 by our method

		Case 1			Case 2	
x	GA	IPA	GA-IPA	GA	IPA	GA-IPA
0	0.36750779524127	0.36751756893963	0.36751588119744	1.14739969626101	1.14708216909984	1.14705825598891
0.1	0.36634775250347	0.36636250990421	0.36636087185142	1.14686249070124	1.14655429348007	1.14652697105963
0.2	0.36287853251650	0.36289418240385	0.36289254188700	1.14527162584015	1.14496541447850	1.14493764576553
0.3	0.35708175027041	0.35709751797256	0.35709539123525	1.14262052227733	1.14231326091689	1.14228564777570
0.4	0.34893160136317	0.34894827065180	0.34894643470649	1.13890281674537	1.13859337456431	1.13856554492182
0.5	0.33839540280592	0.33841203611810	0.33841087423934	1.13411113876322	1.13379895322401	1.13377039454498
0.6	0.32542707641218	0.32544356877601	0.32544235082695	1.12823621270162	1.12792075914365	1.12789123007345
0.7	0.30996803115962	0.30998620980780	0.30998410658520	1.12126630822933	1.12094709209643	1.12091672617820
0.8	0.29195059379233	0.29197124251226	0.29196858564536	1.11318705019611	1.11286382666188	1.11283302893868
0.9	0.27129618732011	0.27131700746154	0.27131500993096	1.10398158351661	1.10365451347012	1.10362374199432
1.0	0.24790830410069	0.24792764186634	0.24792648975714	1.09363107069048	1.09330054441016	1.09327006266004

Table-8
Comparison of numerical results of Example 1 between our method and other methods[2-4]

	Case 1			Case 2		
x	Our Method GA-IPA	Cubic spline[3]	Nonclassical Pseudospectral[4]	Our Method GA-IPA	Cubic spline[3]	B-spline[2]
0	0.36751588119744	0.36751798060679	0.3675168151	1.14705825598891	1.14704108351547	1.14703993670271
0.1	0.36636087185142	0.36636349221841	0.3663623292	1.14652697105963	1.14651170579035	1.14651055946170
0.2	0.36289254188700	0.36289522191628	0.3628940661	1.14493764576553	1.14492256347331	1.14492141825538
0.3	0.35709539123525	0.35709868923803	0.3570975457	1.14228564777570	1.14227062156003	1.14226947822689
0.4	0.34894643470649	0.34894954627856	0.3489484206	1.13856554492182	1.13855080147511	1.13854966085306
0.5	0.33841087423934	0.33841325026918	0.3384121487	1.13377039454498	1.13375594993740	1.13375481292594
0.6	0.32544235082695	0.32544459259504	0.3254435224	1.12789123007345	1.12787679502618	1.12787566262296
0.7	0.30998410658520	0.30998707059095	0.3099860402	1.12091672617820	1.12090188873858	1.12090076206338
0.8	0.29196858564536	0.29197208361443	0.2919711030	1.11283302893868	1.11281753529231	1.11281641561478
0.9	0.27131500993096	0.27131792896572	0.2713170101	1.10362374199432	1.10360770422896	1.10360659299888
1.0	0.24792648975714	0.24792856599196	0.2479277233	1.09327006266004	1.09325392715304	1.09325282603337

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