# Heteroclinic Bifurcation and Chaotic Analysis in Rotational-Translational Motion of a Kelvin-Type Gyrostat Satellite 

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#### Abstract

The different methodologies for the study of nonlinear asymmetric Kelvin-type gyrostat satellite consisting of the heteroclinic bifurcation and chaos are investigated in this work. The dynamical model of the gyrostat satellite involves the attitude orientation along with the translational motion in the circular orbit. The mathematical model of the Kelvin-type gyrostat satellite is first derived using the Hamiltonian approach in the Roto-Translatory motion under the gravity gradient perturbations. Since the model of the system is too complex, the coupled equations of motion are reduced using the modified Deprit canonical transformation by the Serret-Andoyer variables in the spin-orbit dynamics. The simulation results demonstrate the heteroclinic bifurcation route to chaos in the Roto-Translatory motion of the gyrostat satellite due to the effects of the orbital motion and the gravity gradient perturbation on the attitude dynamics. According to the numerical solutions, the intersection of the stable and unstable manifolds in the heteroclinic orbits around the saddle point lead to the occurrence of the heteroclinic bifurcation and chaotic responses in the perturbed system. Chaos behaviour in the system is also analyzed using the phase portrait trajectories, Poincare' section, and the time history responses. Moreover, the Lyapunov exponent criterion verifies numerically the existence of chaos in the Roto-Translatory motion of the system.


Keywords: Gyrostat satellite, Roto-Translatory motion, Chaos, Heteroclinic Bifurcation, Lyapunov exponent.

## Introduction

Recently, many research works have been focused on the chaotic dynamics of the satellites only in attitude motion. These investigations study the occurrence of chaos and verified it in the Hamiltonian satellite system numerically and mathematically under the perturbation influences such as gravity gradient torques. The effects of the orbital motion on the rotational dynamics, however, are not considered at present ${ }^{1-3}$.

In a Gyrostat Satellite (GS), the orientation of the satellite platform can be stabilized because of the rotation of gyros in its structure. Since the spin of rotors is assumed constant, the gyrostat satellite is called Kelvin-type. Therefore, the GSs are used in the highly technical missions because of the stabilizing reaction moments due to the rotation of gyros. On the other hand, the gravity gradient perturbation torque leads to the appearance of the chaotic behaviour in the Hamiltonian dynamics of the GS. As a result, modelling and prediction of chaos in the GS seems to be necessary in order to preserve its proper performance ${ }^{4-7}$.

Eliminating the effects of the orbital motion in rotational dynamics of the GS increases the uncertainties between the model and actual system. Adding the translational motion to rotational dynamics in the GS augments the complexity of the model in the nonlinear studies. Hence, one needs to apply the
reduction method on such a complex system. The order of the Rotational-Translational system can be reduced using the Deprit canonical transformation by the Serret-Andoyer variables ${ }^{8}$.

After demonstrating chaos in the responses of the system, the chaotic behaviour must be verified by the use of some procedures such as Lyapunov exponent criterion, the Poincare' section analysis, and the phase space trajectories. Then, in the modelling process, the effective perturbation torques which creates chaos in the system is identified for the rejection of chaos in the behaviour of the system ${ }^{9,10}$.

The chaotic dynamics of the Roto-translatory motion of a triaxial GS is considered in this work. The governing equations of motion are first derived using the Hamiltonian approach under the gravity gradient torques. The order of the Hamiltonian is then reduced using the modified Deprit canonical transformation because of the complexity in the coupled equations of the Rotational-Translational dynamics. In order to reduce the Hamiltonian, the new Serret-Andoyer variables are defined for both attitude and orbital states. In this modelling process, the orbital motion along with the gravity gradient torques affects the rotational system in the form of perturbation terms. With neglecting the effects of these perturbations, the regular responses are illustrated based on the simulation results. Increasing the influences of the perturbation parameter on the system leads to the validation of the intersection of stable and
unstable manifolds in the heteroclinic orbits. According to the numerical solution for the Hamiltonian GS system under the perturbation torques, the Poincare section, phase portrait trajectories, and the time series responses demonstrate the heteroclinic bifurcation and chaos. In addition, the Lyapunov exponent method verifies the heteroclinic bifurcation route to chaos from the quasi-periodic behaviours.

## Dynamical Model

The mathematical model of a Kelvin-type Gyrostat Satellite is obtained in an equatorial orbit under the gravity gradient torque. The asymmetric triaxial GS consists of a main platform with three rotors as momentum wheels. In the modelling process, the attitude rotational dynamics of the GS along with the translational motion called Roto-Translatory motion is analyzed. The following assumptions are introduced for the derivation of the GS dynamical model in the coupled Roto-Translatory motion. i. The platform and the rotors are assumed as rigid bodies. ii. The rotors are located along the principal axes of the GS. Therefore, the total inertia tensor of the satellite and its three wheels is diagonal. iii. Since the GS is considered as the Kelvin-type, the angular velocities of three rotors with respect to the main platform are nonzero constants. iv. The mass center of the total GS moves on a circular orbit in the unperturbed case. The radius vector from the mass center of the earth to the satellite mass center is $R_{O}$ and its angular velocity is $\omega_{o}=\dot{\mu}_{O}$ around the earth, with the $\mu_{o}$ being similar in concept to the definition of eccentric anomaly (figure 1).

In order to derive the mathematical model of the system, the following orthogonal coordinate systems are considered in accordance with figure 1. i. The inertia reference frame $I\left\{I_{1}, I_{2}, I_{3}\right\}$ - The origin of frame is located at the mass center of the earth, and $I_{1}$ axis passes through the Vernal Equinox. ii. The orbital coordinate $O\left\{O_{1}, O_{2}, O_{3}\right\}$ - with its unit vectors as $(\hat{I}, \hat{J}, \hat{K})$ and its origin located at the mass center of the total GS, the $O_{3}$ axis in the direction of the nadir, the $O_{2}$ axis in the direction of the orbital velocity vector, and the $O_{1}$ axis perpendicular to the orbital plane in the direction of the positive angular momentum of the spacecraft orbital motion. iii. The body coordinate $B\left\{B_{1}, B_{2}, B_{3}\right\}$, fixed to the GS in its center of mass point with unit vectors as $(\hat{i}, \hat{j}, \hat{k})$. Their axes coincide with the spacecraft principal axes. Also, it is assumed that the rotation axes of each rotor are aligned with the axes of the body frame.

The mathematical model of the Kelvin-type GS consists of the kinetic and kinematic equations. The kinetic model of the system is derived using the Hamiltonian approach under the gravity gradient perturbation. In the Hamiltonian- based modelling, the Lagrangian is first calculated as $L=T-U_{g}$, where $T$ is the kinetic energy of the Roto-Translatory motion for the GS. The kinetic energy is obtained in the perturbation form due to the fact that the terms corresponding to $\omega_{o}$ and $\omega_{o}^{2}$ are comparatively small. The result is obtained as ${ }^{11,12}$ :


Figure-1
Roto-Translatory motion of the gyrostat satellite along with its coordinate systems

$$
\begin{align*}
T & =\frac{1}{2} m_{s} R_{O}^{2} \omega_{O}^{2}+\frac{1}{2} m_{s} \dot{R}_{O}^{2}+\frac{1}{2} I_{W}\left(\Omega_{x}^{2}+\Omega_{y}^{2}+\Omega_{z}^{2}\right)+\frac{1}{2}\left[\begin{array}{lll}
\omega_{x} & \omega_{y} & \omega_{z}
\end{array}\right]\left[\begin{array}{ll}
\tilde{I}
\end{array}\right]\left[\begin{array}{lll}
\omega_{x} & \omega_{y} & \omega_{z}
\end{array}\right]^{T} \\
& +I_{W}\left(\omega_{x} \Omega_{\mathrm{x}}+\omega_{y} \Omega_{y}+\omega_{z} \Omega_{z}\right)+\omega_{o}\left\{\frac{1}{2}\left[\begin{array}{lll}
C_{\theta} C_{\psi} & -C_{\theta} S_{\psi} & S_{\theta}
\end{array}\right][\tilde{I}]\left[\begin{array}{lll}
\omega_{x} & \omega_{y} & \omega_{z}
\end{array}\right]^{T}\right. \\
& \left.+I_{W}\left[\begin{array}{llll}
C_{\theta} C_{\psi} & -C_{\theta} S_{\psi} & S_{\theta}
\end{array}\right]\left[\begin{array}{lll}
\Omega_{\mathrm{x}} & \Omega_{\mathrm{y}} & \Omega_{\mathrm{z}}
\end{array}\right]^{T}\right\}+\omega_{O}^{2}\left\{\frac{1}{2}\left[\begin{array}{lll}
C_{\theta} C_{\psi} & -C_{\theta} S_{\psi} & S_{\theta}
\end{array}\right][\tilde{I}]\left[\begin{array}{lll}
C_{\theta} C_{\psi} & -C_{\theta} S_{\psi} & S_{\theta}
\end{array}\right]^{T}\right\} \tag{1}
\end{align*}
$$

, where
$\tilde{I}=\left[\begin{array}{ccc}I_{x} & m_{g}^{2} r^{2} / m_{s} & m_{g}^{2} r^{2} / m_{s} \\ m_{g}^{2} r^{2} / m_{s} & I_{y} & m_{g}^{2} r^{2} / m_{s} \\ m_{g}^{2} r^{2} / m_{s} & m_{g}^{2} r^{2} / m_{s} & I_{z}\end{array}\right]$
, and $m_{s}$ is the total mass of the GS as $m_{s}=m_{B}+3 m_{g}$, where $m_{B}$ is the mass of the main platform and $m_{g}$ is the mass of each rotor, $\Omega_{x}, \Omega_{y}$, and $\Omega_{z}$ are the angular velocities of the wheels with respect to the platform, $I_{x}, I_{y}$, and $I_{z}$ are the inertia moment of the main platform relative to its mass center, $I_{W}$ is the inertia moment of each rotor relative to its mass center, $r$ is the distance between the mass center of each rotors and the main body, and $C_{(.)} \triangleq \cos (),. S_{(.)} \triangleq \sin ($.$) . Moreover, \omega_{x}, \omega_{y}$, and $\omega_{z}$ are the angular velocities of the main platform expressed in the body frame, and $\varphi, \theta, \psi$ are the Euler angles describing the attitude orientation of the body coordinate relative to the orbital coordinates, expressed in the attitude kinematic equations of motion as follows ${ }^{13}$.
$\omega_{x}=\dot{\varphi} \cos \psi \cos \theta+\dot{\theta} \sin \psi$
$\omega_{y}=-\dot{\varphi} \sin \psi \cos \theta+\dot{\theta} \cos \psi$
$\omega_{z}=\dot{\psi}+\dot{\varphi} \sin \theta$
Also, the gravitational potential energy of the GS neglecting the gravity gradient perturbations is given as $U_{g}=-G M m_{s} / R_{o}$, where $G$ is the universal gravity constant and $M$ is the mass of the earth. After derivation of the Lagrangian, the generalized momentum is obtained as $P_{q_{i}}=\partial L / \partial \dot{q}_{i}$, where $q_{i}$ 's are the generalized coordinates involving the attitude coordinates $\varphi, \theta, \psi$ and orbital coordinates $\mu_{o}, R_{O}$. Using the Legendre transformation, the Hamiltonian of the system is then simplified as ${ }^{14}$.

$$
\begin{equation*}
H=T_{0}+\left(\omega_{o}\right) T_{1}+\left(\omega_{O}^{2}\right) T_{2}+U_{g} \tag{6}
\end{equation*}
$$

Choosing the $\xi_{1}, \xi_{2}, \xi_{3}$ body coordinate system with its axes coinciding with the GS principal axes, the matrices in equation (2) become diagonal. For constant values of the angular velocities of the momentum wheels, the Hamiltonian can be put into the simplest form as

$$
\begin{align*}
& H_{n}=\frac{P_{\Psi}^{2}}{2 I_{3}}+\frac{P_{R_{o}}^{2}}{2 m_{s}}+P_{\Theta}^{2}\left(\frac{C_{\Psi}^{2}}{2 I_{2}}+\frac{S_{\Psi}^{2}}{2 I_{1}}\right)+\frac{G M m_{s}}{R_{o}}+\left(\frac{P_{\Phi}}{C_{\Theta}}-P_{\Psi} \tan \Theta\right)^{2} \\
& \left(\left(\frac{C_{\Psi}^{2}}{2 I_{1}}+\frac{S_{\Psi}^{2}}{2 I_{2}}\right)+P_{\Theta} S_{\Psi} C_{\Psi}\left(\frac{1}{I_{1}}-\frac{1}{I_{2}}\right)\right)+\frac{P_{\Theta} P_{\Psi} \tan \Psi}{3 I_{1} I_{2}}  \tag{7}\\
& \quad+\varepsilon_{1}\left\{\frac{1}{2} I_{1} C_{\Theta}^{2} C_{\Psi}^{2}-\frac{1}{2} I_{2} C_{\Theta}^{2} S_{\Psi}^{2}+\frac{1}{2} I_{3} S_{\Theta}^{2}\right\}+\varepsilon_{2} \\
& \left\{\frac{1}{2}\left(P_{\Phi}-P_{\mu_{0}}\right)^{2}+2 P_{\Phi} P_{\Psi} S_{\Theta}-4 P_{\Phi} P_{\Psi} S_{\Theta} S_{\Psi}^{2}-P_{\Phi}^{2} S_{\Psi}^{2}+P_{\Theta} P_{\Psi} S_{\Theta}^{2} C_{\Psi}\right\}
\end{align*}
$$

where $H_{n}$ is the new Hamiltonian in the principal coordinate, $I_{1}$ ,$I_{2}$ and $I_{3}$ are the principal inertia moments corresponding to the total GS equivalent to the eigenvalues of $\tilde{I}$, and $\varepsilon_{1}=\omega_{0}^{2}$ and
$\varepsilon_{2}=1 / m_{s} R_{O}^{2}$ are the perturbation parameters. The parameters $\Phi, \Theta, \Psi$ are the new set of Euler angles in the direction of the principal axes of $\tilde{I}$, coinciding with the $\varphi, \theta, \psi$ angles after transformation to the new principal coordinates. The perturbation terms appear as the coefficients of $\varepsilon_{1}$ and $\varepsilon_{2}$ are related to the orbital motion of the GS. This means that the translational motion of the spacecraft is affected on the dynamical system in the form of a perturbation. Although the Hamiltonian is written as a simple perturbation equation, the system is too complex for nonlinear studies. Therefore, the order of the dynamic system needs to be reduced. Since the open-loop system has the two constant of motion as the energy and total angular momentum, the Hamiltonian can be reduced using the extended Deprit transformation.

## Reduction of Hamiltonian

According to the Hamiltonian of the system based on equation (7), the coordinate $\Phi$ does not directly appear in the Hamiltonian equation, and it is a cyclic coordinate. Furthermore, the related generalized momentum $P_{\Phi}$ is a constant of motion. Therefore, the order of the system can be reduced using the canonical transformation. For this purpose, the new Serret-Andoyer variables are defined for the RotoTranslatory motion on the basis of the spherical triangle shown in figure 2. Using the modified Deprit transformation, the variables of the system consisting of $\Phi, \Theta, \Psi, \mu_{o}, R_{o}$ are transformed to new parameters as $\mathrm{h}, \mathrm{g}, \mathrm{l}, \lambda_{o}, r_{o}$ by the following relations.

$$
\begin{equation*}
P_{\Phi}=P_{h}, \quad P_{\Theta}=P_{g} S_{j} S_{\Psi-1}, \quad P_{\Psi}=P_{l}, \quad P_{\mu_{o}}=P_{\lambda_{o}}, \quad P_{r_{o}}=P_{R_{o}} \tag{8}
\end{equation*}
$$

In addition, the following relations are used in this transformation based on the spherical triangle $Q_{1} Q_{2} Q_{3}{ }^{8,15}$.
$\cos (\Theta)=\cos (I) \cos (j)-\sin (I) \sin (j) \cos (g)$
$\cos (\Psi-l)=\cos (\Phi-h) \cos (g)+\sin (\Phi-h) \sin (g) \cos (I)$
$\cos (\Phi-h)=\cos (\Psi-l) \cos (g)+\sin (\Psi-l) \sin (g) \cos (J)$
, where
$\cos (I)=\frac{P_{h}}{P_{g}}, \quad \cos (j)=\frac{P_{l}}{P_{g}}$
Substituting equations (8-12) into equation (7), the reduced Hamiltonian is obtained here in its simplest form as

$$
\begin{aligned}
& H_{n}^{\prime}=\frac{P_{1}^{2}}{2 I_{3}}+\frac{P_{r_{o}}^{2}}{2 m_{s}}+\frac{G M m_{s}}{r_{o}}+\frac{1}{2}\left(P_{g}^{2}-P_{l}^{2}\right)\left(\frac{S_{l}^{2}}{I_{1}}+\frac{C_{l}^{2}}{I_{2}}\right)+ \\
& \varepsilon_{1}\left\{\frac{1}{2} I_{3}-\frac{1}{2} I_{3} C_{I}^{2} C_{J}^{2}-\frac{1}{2} I_{3} S_{I}^{2} S_{J}^{2} C_{g}^{2}+\frac{1}{2}\left(I_{1}-I_{2}\right)\left(S_{I}^{2} S_{J} C_{g}+C_{I} S_{J} C_{l}\right)\right. \\
& \left.\quad-\frac{1}{2} I_{1} S_{J}^{2}+\frac{1}{2} I_{2} C_{I}^{2} S_{g}+\frac{1}{4}\left(I_{3}-I_{1}\right) S_{2 I} S_{2 J}\right\}+ \\
& \varepsilon_{2}\left\{\frac{1}{2} P_{\lambda_{o}}^{2}+P_{\lambda_{0}}\left(S_{J}^{2} S_{I}^{2}+\frac{1}{2} S_{2 I} C_{2 J}\right)+P_{l}\left(S_{I}^{2} S_{J}+S_{I} S_{g} S_{l}\right)+S_{g} C_{l}+S_{I}^{2}-S_{I} S_{j}\right\}
\end{aligned}
$$

Consequently, according to the reduced Hamiltonian, the equations of motion of the Kelvin-type GS under the gravity gradient perturbation torques are derived as
$\dot{q}=\frac{\partial H_{n}^{\prime}}{\partial P_{q}}, \quad \dot{P}_{q}=-\frac{\partial H_{n}^{\prime}}{\partial q}+\vec{N}_{g g}$
, where $\vec{N}_{g g}$ is the gravity gradient torques after the transformation by the Serret-Andoyer variables.
$\vec{N}_{g g}=\frac{3 \mu}{2 R_{o}^{3}}\left\{\left(I_{3}-I_{2}\right) S_{2 \varphi} C_{\theta}^{2} \hat{i}+\left(I_{3}-I_{1}\right) S_{2 \theta} C_{\varphi} \hat{j}+\left(I_{1}-I_{2}\right) S_{2 \theta} C_{\theta} \hat{k}\right\}$

In accordance with equations (14) in the unperturbed system, the parameters $h, \lambda_{o}$ are constant parameters and $P_{h}, P_{\lambda_{o}}$ are the constant of motion. As a result, the system with order 5 is reduced to the system with order 3 in the unperturbed system using the new Deprit transformation.

Since the orbital motion affects on the system through perturbation terms, and because the parameter $h$ is constant, the main core of the dynamical model is formed by the $l-P_{l}$ and $g-P_{g}$ dynamics as
$\dot{g}=\frac{\partial H_{n}^{\prime}}{\partial P_{g}}=P_{g}\left(\frac{S_{l}^{2}}{I_{1}}+\frac{C_{l}^{2}}{I_{2}}\right)$
$\dot{P}_{g}=-\frac{\partial H_{n}^{\prime}}{\partial g}=0+\varepsilon_{1}\left\{-I_{3} S_{I}^{2} S_{J}^{2} S_{g} C_{g}+\frac{1}{2}\left(I_{1}-I_{2}\right) S_{I}^{2} S_{J} S_{g}-\frac{1}{2} I_{2} C_{I}^{2} C_{g}\right\}$
$+\varepsilon_{2}\left\{-P_{l}\left(S_{l} C_{g} S_{l}\right)-C_{g} C_{l}\right\}+\varepsilon_{g}\left(N_{g g, g}\right)$

$$
\begin{align*}
& \dot{l}=\frac{\partial H_{n}^{\prime}}{\partial P_{l}}=P_{l}\left(\frac{1}{I_{3}}-\frac{S_{l}^{2}}{I_{1}}-\frac{C_{l}^{2}}{I_{2}}\right)+\varepsilon_{2}\left\{S_{I}^{2} S_{J}+S_{l} S_{g} S_{l}\right\}  \tag{18}\\
& \dot{P}_{l}=-\frac{\partial H_{n}^{\prime}}{\partial l}=-\left(P_{g}^{2}-P_{l}^{2}\right) S_{l} C_{l}\left(\frac{1}{I_{1}}-\frac{1}{I_{2}}\right)+ \\
& \varepsilon_{1}\left\{\frac{1}{2}\left(I_{1}-I_{2}\right) C_{I} S_{J} S_{l}\right\}+\varepsilon_{2}\left\{S_{g} S_{l}-P_{l} S_{I} S_{g} C_{l}\right\}+\varepsilon_{g}\left(N_{g g, l}\right)
\end{align*}
$$

, where $\varepsilon_{g}=3 \mu / 2 r_{o}^{3}$ is the perturbation parameter relative to the gravity gradient torques and $N_{g g, g}, N_{g g, l}$ are the components of the gravity gradient perturbations in the $g$ and $l$ directions based on equation (15) after completing the transformation by the Serret-Andoyer variables.

## Bifurcation and Chaos studies

In order to study the GS system, the nonlinear attitude equations of motion for the GS on the basis of equations $(18,19)$ under the influences of the translational motion and gravity gradient perturbation are rewritten as follows. These equations consist of two parts as perturbed and unperturbed. Thus

$$
\begin{equation*}
\dot{l}=P_{l}\left(\frac{1}{I_{3}}-\frac{S_{l}^{2}}{I_{1}}-\frac{C_{1}^{2}}{I_{2}}\right)+\varepsilon\left\{\frac{1}{m_{s}}\left(S_{I}^{2} S_{J}+S_{I} S_{g} S_{l}\right)\right\} \tag{20}
\end{equation*}
$$



Figure-2
Definition of Serret-Andoyer variables in Deprit Transformation ${ }^{8}$
$\dot{P}_{l}=-\left(P_{g}^{2}-P_{l}^{2}\right) S_{l} C_{l}\left(\frac{1}{I_{1}}-\frac{1}{I_{2}}\right)+\varepsilon\left\{\begin{array}{l}v_{0}^{2}\left(\frac{1}{2}\left(I_{1}-I_{2}\right) C_{l} S_{J} S_{l}\right)+\frac{1}{m_{s}}\left(S_{g} S_{l}-P_{l} S_{l} S_{g} C_{l}\right) \\ +\frac{2 \mu}{3 r_{o}}\left(S_{h} S_{g} C_{l}^{2}-2 S_{J} C_{l} C_{h}^{2}\right)\end{array}\right\}$
, where, $\varepsilon=1 / r_{o}^{2}$ is a new perturbation parameter involving the combination of the parameters $\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{g}$, and $v_{0}$ is the orbital velocity of the GS. Based on equations (20, 21), in the unperturbed system for $\varepsilon=0$, the system has five fixed points. The fixed point $(0,0)$ is a hyperbolic saddle point corresponding to the rotation about the intermediate axis $\zeta_{2}$, the two equilibrium points $( \pm \pi / 2,0)$ indicate the border of stability margin with respect to the spin around the minor $\zeta_{3}$ and major $\zeta_{1}$ axes, respectively, and the two fixed points $\left(\sin ^{-1} \sqrt{\left(I_{1} I_{2}-I_{1} I_{3}\right) /\left(I_{2}^{2}-I_{2} I_{1}\right)}, \pm P_{g}\right)$ are related to the upper and lower limits of the phase portrait which are beyond the region of interest.

In order to numerically analyze the dynamical system, the equations of motion in (14) are simulated for the case of $I_{1}>I_{2}>I_{3}$. For this purpose, the Poincare map is first used for the nonlinear studies based on equations $(20,21)$. The Poincare map of the system is generated by intersecting the trajectories of the phase portrait $l-g-P_{l}$ with plane $g=\pi$. In the numerical solutions, the constant values are assumed $I_{1}=1.2, I_{2}=0.8$, $I_{3}=0.6, m_{s}=4, P_{h}=0.7$ and the initial conditions are $l_{0}=0$,
$g_{0}=0.01, h_{0}=0.1, r_{O_{0}}=6675, \lambda_{o_{0}}=1, P_{l_{0}}=0, P_{g_{0}}=1, P_{h_{0}}=1$, $P_{r_{00}}=2, \quad P_{x_{0}}=0.3$. In the unperturbed case for $\varepsilon=0$, the Poincare map of the system is demonstrated in figure 3, indicating the regularity in the unperturbed system clearly. The Poincare section of the perturbed system under the translational effects and gravity gradient torques for $\varepsilon=2 \times 10^{-8}$ is shown in figure 4. The intersection of stable and unstable manifolds in the heteroclinic orbits is revealed in the Poincare plane of figure 4. Therefore, the occurrence of chaos in the perturbed system is verified.

Heteroclinic intersection in the Poincare section can be described by a horseshoe map based on the Smale-Brikhoff theorem. The trajectory on the unstable manifold joins another trajectory on the stable manifold due to the heteroclinic intersection. This would means that the stable manifold makes the trajectory approach to the saddle point and the unstable manifold get the trajectory away from the saddle point. Hence, a heteroclinic tangling exists around the saddle point and the trajectories can seem to wander randomly near to it, introducing a basin of attraction in some dissipative systems. Therefore, the heteroclinic intersection leads to the stretching, compression, and folding in the trajectories of the system around the saddle points. Based on the heteroclinic tangling and horseshoe map behaviour, chaos occurs near the saddle point. According to figure 4(b), the chaotic band with chaos windows is clearly obvious ${ }^{9,10}$.


Figure-3
Poincare section of the unperturbed system for $\varepsilon=0$

(a)

(b)

Figure-4
Poincare section of the perturbed system for $\varepsilon=2 \times 10^{-8}$, (a) chaotic behaviour (b) chaotic band near the saddle point

In the Hamiltonian GS dynamic, chaos basically appears in the system due to the effects of the perturbations such as translational motion and gravity gradient torques. The effects of perturbation on the chaotic system can also be analyzed using the global heteroclinic bifurcation. Therefore, the behaviour of the $l-j$ phase trajectory is examined with respect to the changes of the perturbation parameter as shown in figure 5. In the unperturbed system for $\varepsilon=0$, the $l-j$ phase portrait is demonstrated in Fig 5a, indicating a regular limit cycle with periodic behaviour. Increasing the value of the perturbation parameter to $\varepsilon=0.8 \times 10^{-8}$ makes the limit cycle to be irregular, leading to a quasi-periodic behaviour, as illustrated in figure 5b. According to the actual value of the perturbation parameter as $\varepsilon=2 \times 10^{-8}$, the irregularity in the responses is due to the occurrence of chaos in the perturbed system based on figure 5 c . Therefore, figure 5 illustrates the heteroclinic bifurcation route to chaos, converting from the periodic to quasi-periodic, finally leading to a chaotic behaviour with respect to growing perturbation parameter value.

## Lyapunov Exponent

The Lyapunov exponent (LE) criterion is a strong tool for the analysis of chaos numerically. The LE measures exponentially
the attraction or divergence of two close trajectories in time in the phase portrait with close initial conditions. Furthermore, the LE displays the extreme effects of the initial conditions into sensitiveness of the system. The LE ( $\lambda$ ) for each state variable $x_{i}(t)$ is mathematically defined as
$\lambda=\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} E_{i}(x(\tau)) d \tau=\lim _{t \rightarrow \infty} \frac{1}{} \ln \left|\frac{\delta x_{i}(t)}{\delta x_{i}(0)}\right|$
, where, $x_{i}(0)$ is the initial condition, and $E_{i}(x(t))$ is the real eigenvalue of the Jacobian matrix related to the divergence rate of the system. The negative value of the LE indicates the orbits converge together and, the positive value shows the growing of the distance between the adjacent orbits exponentially. Due to positive value of the LE, the system exhibits a sensitive dependence on the initial condition. As a result, if the system has at least one positive LE, it usually indicates chaos ${ }^{3,9}$. The value of the LE for each state is calculated in the perturbed Roto-Translatory GS under the gravity gradient perturbation with the effects of the constant rotation of the wheels for $\varepsilon=2 \times 10^{-8}$ and depicted in table 1 .


Figure-5
Heteroclinic bifurcation as growth of the irregularity with increase in $\varepsilon$ (a) periodic behaviour for $\varepsilon=0$, (b) quasi-periodic behaviour for $\varepsilon=0.8 \times 10^{-8}$, (c) chaotic behaviour for $\varepsilon=2 \times 10^{-8}$

The sign of LE based on the values in table 1 is an indication of the strange attractor and chaos in the perturbed system. Also, the diagram for the largest LE relative to the perturbation parameter $\varepsilon$ is demonstrated in figure 6 under the initial conditions and constant values as used in the nonlinear studies. According to figure 6 , increasing the perturbation parameter results in the sign of the LE change to positive and increases the probability of chaos occurrence. The results obtained from figure 6 completely coincide with the heteroclinic bifurcation results as shown in figures 3-6. This good agreement is because the positive value for the Largest LE indicates the chaotic attraction, and the growing irregularity in the responses of the system with
increasing $\varepsilon$ demonstrates the heteroclinic bifurcation route to chaos.

## Simulation results

In order to study the influences of the orbital motion on the rotational dynamics, the Roto-Translatory motion of the GS system is analyzed. Therefore, the coupled equations of motion based on equations (14) are simulated numerically with $\varepsilon=2 \times 10^{-8}$ for initial values and constant parameters as those given in the previous section under the gravity gradient perturbation ${ }^{16-18}$.

Table-1
The values of Lyapunov Exponents for $\boldsymbol{\varepsilon}=2 \times 10^{-8}$


Figure-6
Largest Lyapunov exponent of the system relative to increasing the $\varepsilon$

(a)

(c)

(b)

(e)
(d)

Figure-7
Trajectories of the system phase portrait (a) Trajectory $\omega_{x}-\omega_{y}$, (b) Trajectory $l-i-j$, (c) Trajectory $i-j$, (d) Trajectory $l-i$, (e) Trajectory l-J

The quasi-periodic and chaotic responses consisting of the phase portrait trajectories, Poincare' section, and time-history responses of the system are demonstrated in figure 7-9. According to figure 7, the stretching, compressing, and folding in the phase space trajectories indicate chaos in the perturbed system. Based on the Fradkov definition of chaos, the chaotic attraction can be illustrated using the trajectory behaviour as globally bounded and locally unstable in accordance with the Lyapunov stability method. The Poincare section of the chaotic GS system is demonstrated in figure 8. The Poincare section due to the intersection of the $l-I-j$ trajectory with the plane $l=3.1$ in figure 8 a , with the plane $j=2.7$ as in figure 8 b , and with the plane $I=0.8$ as in Fig. 8c verifies the chaotic behaviour in the system. Based on the Devaney definition of chaos, distinct set of points in the Poincare' section of the system are indications of the existence of chaos in the GS model. Moreover, the time series responses of the system are depicted in figure 9, a clear verification of chaos in the system.
studied in this work. After modelling the system based on the Hamiltonian approach under the gravity gradient perturbations, the Hamiltonian is reduced using the modification of the Deprit canonical transformation due to the complexity in model. The reduction method is applied based on the definition of SerretAndoyer variables in the Roto-Translatory dynamics. Without taking into consideration the perturbation effects involving the orbital motion and gravity gradient, the simulation results demonstrate the regular periodic responses. The effects of the perturbation on the dynamical system lead to an increase of the irregularity and route to chaos via heteroclinic bifurcation. Bifurcation and chaos occur in the GS due to the intersection of the stable and unstable manifolds in the heteroclinic orbits around the saddle point. Chaos is also verified in the perturbed system using the Poincare section, phase portrait trajectories, and the time series responses along with the Lyapunov exponent criterion. The above Chaos analysis can offer an appropriate model for the attitude control of chaos in the GS system.

## Conclusion

The heteroclinic bifurcation and chaotic dynamics of the Rototranslatory motion for a triaxial asymmetric Kelvin-type GS is


Figure-8
Poincare section of the system (a) in $i-j$ plane, (b) in $l-i$ plane, (c) in l-j plane


Figure-9
Time series response of the system (a) coordinate $\boldsymbol{l}$, (b) generalized momentum $P_{\text {, }}$

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