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On a Generalization of Landsberg Space

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Abstract

In this present work, a special tensor $V_{ijk} = \lambda L_{ijk} + a_i h_{jk} + a_j h_{ik} + a_k h_{ij}$ is introduced and some properties of Finsler space admitting the so-called tensor is studied.

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Introduction

Let $F^n (n \geq 3)$ be an n-dimensional Finsler space with metric function $L(x, y)$. There are five kinds of torsion tensors in the theory of Finsler space based on Cartan's connection, out of which

$$P_{ijk} = y^h P_{hijk} \quad \text{and} \quad C_{ijk} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}$$

As (v)hv-torsion tensor and (h)hv-torsion tensor are of great importance tensors for the present study, where P_{hijk} is as hv-curvature tensor.

Various interesting forms of these tensors have been studied by many geometers[1-4]. One of them is a C-reducible Finsler space in which the torsion tensor C_{ijk} is of the form[5]

$$C_{ijk} = \frac{1}{n+1}(C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) \tag{1}$$

where h_{ij} is the angular metric tensor and $C_i = C_{ijk} g^{jk}$, where g^{jk} is reciprocal of the metric tensor g_{jk}.

Izumi[2,3] introduced P^*-Finsler space in which P_{ijk} is of the form

$$P_{ijk} = \lambda C_{ijk} \tag{2}$$

where λ is a scalar homogeneous function of degree zero in y^i. In a P-reducible Finsler space the tensor P_{ijk} is of the form[6]

$$P_{ijk} = \frac{1}{n+1}(G_i h_{jk} + G_j h_{ik} + G_k h_{ij}) \tag{3}$$

where $G_i = C_{i|0} = C_{i|j}$. A Finsler space in which $P_{ijk} = 0$ is called a Landsberg space [7]. If $C_{ijk|h} = 0$, then F^n is called a Bewald's affinely connected space[8, 9].

Prasad[10] introduced a special form of torsion tensor P_{ijk} as follows

$$P_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ik} + a_k h_{ij} \tag{4}$$

Where $\lambda = \lambda(x, y)$ is a scalar homogenous function of degree 1 and $a_i = a_i(x)$ is a homogenous function of degree 0 with respect to y^i. He then studied some properties of F^n satisfying (4). Peyghan et. al.[11] studied F^n satisfying (4) as generalized P-reducible Finsler space.

Preliminaries

Let M be an n-dimensional C^∞ manifold, By $T_x M$ we mean the tangent space at $x \in M$ and by $TM \setminus 0$ the slit tangent bundle of M.

A Finsler metric on M is a function $L: TM \to [0, \infty)$ which has the following properties: i. L is C^∞ on $TM \setminus 0$. ii. L is positively homogenous function of degree 1 on TM. iii. For each $y \in T_x M$, the metric tensor g_{ij}, the angular metric tensor h_{ij} are respectively given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j}$$

The angular metric tensor h_{ij} can also be written in terms of the normalized element of support[12] $l_i = \frac{1}{L} g_{ij} y^i y^j$, as $h_{ij} = g_{ij} - l_i l_j$.
For $y \in T_x M \setminus 0$ define Cartan torsion tensor vector as:
$C_i := g^{jk} C_{ijk}$.

According to Deicke's theorem, $C_i = 0$ is the necessary and sufficient condition for F^n to be Riemannian.

Let $F^n = (M^n, L)$ be a Finsler space . For $y \in T_x M \setminus 0$, we define

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}(C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) \tag{5}$$

A Finsler space F^n is said to be C-reducible if $M_{ijk} = 0$.
Next, we define the tensor

$$L_{ijk} := C_{ijk|l} y^l \tag{6}$$

where the '|' means h-covarient differentiation with respect to Cartan connection.

A Finsler space F^n is called a Landsberge space if $L_{ijk} = 0$.
Define

$$L_i := g^{jk} L_{ijk} \tag{7}$$

A Finsler space F^n is said to be weakly Landsberg space if $L_i = 0$ [13].
Moreover,we define

$$\overline{M}_{ijk} := L_{ijk} - \frac{1}{n+1}(L_i h_{jk} + L_j h_{ik} + L_k h_{ij}) \tag{8}$$

A Finsler space F^n is said to be P-reducible if $\overline{M}_{ijk} = 0$.
It is obvious that every C-reducible Finsler space is P-reducible, but the converse is not true. Peyghan et. al. proved the following theorem[11].

Lemma 1: Let (M,L) be a generalized P-reducible Finsler manifold. Then
$\overline{M} = \lambda M$
We define

$$\overline{\overline{M}}_{ijk} := V_{ijk} - \frac{1}{n+1}(V_i h_{jk} + V_j h_{ik} + V_k h_{ij}) \tag{9}$$

where

$$V_{ijk} := \lambda L_{ijk} + a_i h_{jk} + a_j h_{ik} + a_k h_{ij} \tag{10}$$

and

$$V_i := g^{jk} V_{ijk} \tag{11}$$

The purpose of the present paper is to study F^n satisfying (10).

Properties of F^n admitting the tensor V_{ijk}

Let F^n be a Finsler space satisfying (10), when $\lambda = 1$, and a_i vanishes then F^n is a Landsberg space. Again if $V_{ijk} = 0$, then from (10) it follows that
$L_{ijk} = -\frac{1}{\lambda}(a_i h_{jk} + a_j h_{ik} + a_k h_{ij})$, and $a_i = \frac{\lambda}{n+1} C_i$.

Hence we have the following propositions:
Proposition 1: A Finsler space satisfying (10) is a C-reducible Finsler space iff $V_{ijk} = 0$.

Proposition 2: Let a Finsler space F^n satisfying (10) be a weakly Landsberg space, then, $a_i = \frac{1}{n+1} V_i$.
The proof immediately follows from contraction of (10) with g^{jk}. We prove the following theorem.

Theorem 1. Let (M,L) be a Finsler manifold satisfying (10), then

$$\overline{\overline{M}} = \lambda \overline{M} \tag{12}$$

Proof: Suppose (10) is satisfied then contracting it by g^{ij} and using $g^{ij} h_{ij} = n-1$ and $g^{ij}(a_i h_{jk}) = g^{ij}(a_j h_{ik}) = a_k$, we get: $V_k = \lambda L_k + (n+1) a_k$,

from which

$$a_k = \frac{1}{n+1} V_k - \frac{\lambda}{n+1} L_k \tag{13}$$

Substituting (13) in (10) we obtain

$$V_{ijk} = \lambda L_{ijk} + \frac{1}{n+1}(V_i h_{jk} + V_j h_{ik} + V_k h_{ij}) - \frac{\lambda}{n+1}(L_i h_{jk} + L_j h_{ik} + L_k h_{ij}) \tag{14}$$

The relation (14) , may be put as

$$V_{ijk} - \frac{1}{n+1}(V_i h_{jk} + V_j h_{ik} + V_k h_{ij}) = \lambda[L_{ijk} - \frac{1}{n+1}(L_i h_{jk} + L_j h_{ik} + L_k h_{ij})] \tag{15}$$

The relation (15) is equivalent to (12) by which the proof of the theorem is completed.

Corollary 1: If (M,L) a Finsler manifold satisfying (10) is a generalized P-reducible Finsler manifold as well, then
$\overline{\overline{M}} = M$.

Proof: The proof immediately follows from the theorem (3.1) and lemma (1).

Theorem 2: Let (M,L) be a Finsler manifold satisfying (10) and $V_{ijk|l} = V_{ijl|k}$,

Then F^n is a P-reducible Finsler space .

Proof: Given that

$$V_{ijk|l} - V_{ijl|k} = 0 . \quad (16)$$

Contracting (16) with y^l yield

$$V_{ijk|l}\, y^l = 0 \quad (17)$$

Taking h-covarient derivative of (10) and then contracting by y^l ,we get

$$V_{ijk|l}\, y^l = \bar{\lambda} L_{ijk} + \lambda V_{ijk} + \bar{a}_i h_{jk} + \bar{a}_j h_{ik} + \bar{a}_k h_{ij} , \quad (18)$$

where $\bar{\lambda} = \lambda_{|l}\, y^l , \bar{a}_i = a_{i|l}\, y^l$. By replacing (17) into (10), we obtain

$$V_{ijk|l}\, y^l = (\bar{\lambda} + \lambda) L_{ijk} + (\lambda a_i + \bar{a}_i) h_{jk} + (\lambda a_j + \bar{a}_j) h_{ik} + (\lambda a_k + \bar{a}_k) h_{ij} \quad (19)$$

From (17) and (19), we get

$$L_{ijk} = \frac{-1}{\bar{\lambda} + \lambda^2} [(\lambda a_i + \bar{a}_i) h_{jk} + (\lambda a_j + \bar{a}_j) h_{ik} + (\lambda a_k + \bar{a}_k) h_{ij}] \quad (20)$$

Contracting (20) with g^{ij} and using the relation

$$g^{ij} h_{ij} = n - 1 \quad \text{and} \quad g^{ij}(a_i h_{jk}) = a_k = g^{ij}(a_j h_{ik}) ,$$

implies that $L_k = \dfrac{-(n+1)}{\bar{\lambda} + \lambda^2} (\lambda a_k + \bar{a}_k)$,

or equivalently $\lambda a_k + \bar{a}_k = -\dfrac{\bar{\lambda} + \lambda^2}{n+1} L_k$ (21)

Putting the relation (21) into (20) and simplifying we have

$$L_{ijk} = \frac{1}{n+1} (L_i h_{jk} + L_j h_{ik} + L_k h_{ij}) .$$

Conclusion

Finsler space is a natural extension of the Riemannian space in which all geometric objects depends not only in positional coordinates as in Riemannian geometry but also in directional arguments. There are Finsler spaces with special structures called as special Finsler spaces such as Berwald's space, Landsberg space and so on. In this paper Finsler space with a special tensor as a generalized Landsberg space is introduced and obtained some results.

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