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An Inverse Optimization Model for Linear Fractional Programming

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Abstract

In this paper, we have proposed an inverse model for linear fractional programming (LFP) problem in which the parameters associated with the numerator of the objective function in the given LFP are adjusted as little as possible(under l_1 norm) so that the given feasible solution become optimal. We formulate this problem as a linear programming problem. A numerical example is given in the last to show, how this model can apply for production planning problem to bring down the level of unemployment.

Keywords: Inverse optimization, linear fractional programming, duality.

Introduction

In an optimization problem, we assume that all the parameters associated with the decision variables in the objective function or in constraint set are known and we need to find an optimal solution to it. However in practice, it is difficult to determine all model parameters with precision but we have some estimates of these parameters and also have certain optimal solutions from experience or experiments. An inverse optimization is to adjust the values of parameters as little as possible so that the known solution becomes an optimal solution of that problem.

Burton and Toint[1] were the first who investigate the inverse optimization for shortest path problem under l_2 norm, since then a lot of work has been done on inverse optimization but most of the work is based on combinatorial optimization problems. Zhang and Liu have first been calculated some inverse linear programming problem[2] and further investigated inverse linear programming problems[3]. Ahuja and Orlin[4] provide various references in the area of inverse optimization and compile several applications in network flow problems with unit weight and develop combinatorial proofs of correctness. Huang and Liu[5] and Amin and Emrouznejad[6], have considered applications of inverse problem. Yibing, Tiesong and Zhongping worked on inverse optimal value problem[7], Zhang and Zhang worked on inverse quadratic programming problems[8-10], and Wang has given the cutting plane algorithm for inverse integer programming problem[11]. Milan Hladik have first been considered inverse problem for generalized linear fractional programming[12] and Jaing, Xiao, Zhang and Zhang worked on inverse linear programming[13].

The linear fractional programming problem seeks to optimize the objective function of non-negative variables of quotient form with linear functions in numerator and denominator subject to a set of linear and homogeneous constraints. Bajanilov[14] compiled the literature of Linear Fractional Programming: Theory, Methods, Applications and Software in the form of book. Charnes-Cooper[15], Kantiswarup[16], Chadha[17, 18], Jain and Mangal[19, 20], Jain, Mangal and Parihar[21], Borza, Rambely, and Saraj[22] and many researchers gave different methods for solving linear fractional programming problem.

In the following section, we fix a feasible solution x^0 and objective value z^0 for the given LFP problem and obtained the inverse problem of it to make the solution x^0 an optimal with the objective value z^0. In the proposed method, first the dual of given LFP is obtained and parameters associated with the numerator in the objective function are adjusted. Inverse problem is obtained by applying optimality conditions to the adjusted problem in such a way that the total adjustment in the parameters should be minimized. Here we are considering l_1 measure for the adjustment of parameters. Then we formulated the above obtained inverse problem as a linear programming problem with a large number of variables, which is solvable by any existing method or by optimization tools like LINGO, EXCEL SOLVER etc. We also applied this model to a practical problem.

Inverse LFP Problem

Let *lfp(A,b,p,d)* denote the linear fractional programming problem

$$\text{Max}(z) = (px + p_0)/(dx + d_0)$$
$$\text{s.t.} \quad Ax \leq b$$
$$\text{and} \quad x \geq 0 \qquad (1)$$

Where $A \in R^{m\times n}, b \in R^m, p \in R^n, d \in R^n$, $S = \{x: Ax \leq b; x \geq 0\}$ is the set of feasible solutions and p_0, d_0 are scalars.

If it is assumed that $dx + d_0 > 0$ in the feasible region, objective function is continuously differentiable and the feasible set S is regular, then the dual of above *lfp* given by Chadha[18] is the following linear programming problem:

Min $g(y, z) = z$

s.t. $A^t y + d^t z \geq p^t$

$-b^t y + d_0 z = p_0$

$y \geq 0$ (2)

Feasible solution of dual is given by

$T = \{y, z : A^t y + d^t z \geq p^t, -b^t y + d_0 z = p_0 ;\ y \geq 0\}$

Let $x^0 \in S$ is the known feasible solution and z^0 is the desired objective value, then the inverse problem is to find $(p_0', p') \in R^{n+1}$ in such a way that $\|(p_0', p') - (p_0, p)\|$ is minimum and x^0 is optimal to *lfp(A,b,p',d)* with the objective value z^0. Where $\|.\|$ is the l_1 norm defined as $\|p' - p\|_p = \sum |p_j' - p_j|$.

Define

$$F(x^0, z^0) = [(p_0', p') \in R^{n+1} : \max\left\{\frac{p'x + p_0'}{dx + d_0} : Ax \leq b; x \geq 0\right\} = \frac{p'x^0 + p_0}{dx^0 + d_0} = z^0]$$ (3)

Then the inverse *lfp(A,b,p',d)* can be express as:

$$\text{Min}\{\|(p_0', p') - (p_0, p)\| : (p_0', p') \in F(x^0, z^0)\}$$ (4)

According to the optimality condition of linear fractional programming, x solve *lfp(A,b,p,d)* if and only if it satisfies the following conditions: i. $Ax \leq b$, $x \geq 0$ (primal feasibility), ii. $A^t y + d^t z \geq p^t$, $-b^t y + d_0 z = p_0$; $y \geq 0$ (dual feasibility), iii. $(px + p_0)/(dx + d_0) = g(y, z)$ (strong duality).

Therefore the feasible solution x^0 will solve the inverse *lfp(A,b,p',d)* with the objective value z^0 if and only if the following conditions are satisfied: i. $A^t y + d'^t z^0 \geq p'^t$, $-b^t y + d_0 z^0 = p_0'$; $y \geq 0$, ii. $(p'x^0 + p_0)/(d'x^0 + d_0) = g(y, z^0) = z^0$.

Using these conditions the inverse problem can be formulated as:

Min $\|(p_0', p') - (p_0, p)\|$

s. t. $A^t y + d'^t z^0 \geq p'^t$

$-b^t y + d_0 z^0 = p_0'$

$(p'x^0 + p_0)/(d'x^0 + d_0) = z^0$ $y \geq 0$ (5)

If we define $e, f \in R^n_+$ such that $p' - p = e - f$; $ef = 0$ and $p_0' - p_0 = e_0 - f_0$; $e_0 f_0 = 0$ then the inverse problem under l_1 norm can be expressed as:

Min $\|(e, f, e_0, f_0)\|_1$

s. t. $A^t y + dz^0 \geq (p + e - f)^t$

$-b^t y + d_0 z^0 - e_0 + f_0 = p_0$

$[(p + e - f)x^0 + (p_0 + e_0 - f_0)]/[dx^0 + d_0] = z^0$

$y, e, f, e_0, f_0 \geq 0$ (6)

Numerical Example

Consider the following problems

$P(x) = 2.5x_1 + x_2 + 0.5x_3 + 4 \longrightarrow \max$ (a)

$x \in S$

$D(x) = x_1 + 1.5x_2 + 2x_3 + 6 \longrightarrow \max$ (b)

$x \in S$

Where S is the following feasible set

$x_1 + 2x_2 + 4x_3 \leq 24$

$4x_1 + 2x_2 + x_3 \leq 12$

$x_1, x_2, x_3 \geq 0$

Let P(x) denotes the profit and D(x) denote the manpower requirement of the company. If x' solve the problem (a) and x^0 solve the problem (b) then the production plan x' is best from the company's point of view, as it maximize company's profit P(x) and the production plan x^0 is best from the society's point of view, as it maximize manpower requirement D(x).

Solving the problems (a) and (b), we have x' = (3, 0, 0) and x^0 = (0, 4, 4). Clearly both the problems have different optimal solutions. If the unemployment present in the society and our aim is to bring down the level of unemployment then the company may choose the production plan which maximizes manpower requirement D(x) as well as the profit P(x) and/or profit per manpower requirement P(x)/D(x).

We now consider the LFP problem

$$\text{Max }(z) = \frac{P(x)}{D(x)} = \frac{2.5x_1 + x_2 + 0.5x_3 + 4}{x_1 + 1.5x_2 + 2x_3 + 6}$$

s.t. $x_1 + 2x_2 + 4x_3 \leq 24$

$4x_1 + 2x_2 + x_3 \leq 12$

$x_1, x_2, x_3 \geq 0$

If we consider x^0 = (0, 4, 4) as a feasible solution to the LFP and $z^0 = P(x^0)/D(x^0) = 0.5$ is the desired objective value then by using inverse optimization we can adjust the parameters associated with $P(x)$ so that x^0 become an optimal solution to the LFP problem.

Substituting $A = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 2 & 1 \end{pmatrix}$, $b = (24 \quad 12)^t$, $p = (2.5 \quad 1 \quad 0.5)$, $d = (1 \quad 1.5 \quad 2)$, $e = (e_1 \quad e_2 \quad e_3)$ and $f = (f_1 \quad f_2 \quad f_3)$ in (6), the inverse problem is as follows:

Min $(e_0 + f_0 + e_1 + f_1 + e_2 + f_2 + e_3 + f_3)$

s.t. $y_1 + 4y_2 - e_1 + f_1 \geq 2$

$2y_1 + 2y_2 - e_2 + f_2 \geq 0.25$

$4y_1 + y_2 - e_3 + f_3 \geq -0.5$

$-24y_1 - 12y_2 - e_0 + f_0 = 1$

$4e_2 - 4f_2 + 4e_3 - 4f_3 + e_0 - f_0 = 0$

$y_1, y_2 \geq 0$ and $e_i, f_i \geq 0$ for i = 0,1,2,3

An optimal solution using TORA is $f_1 = 2, f_2 = 0.25, e_3 = 0.5, f_0 = 1$.

Using these values the modified objective function is

$$\max\ (z) = \frac{0.5x_1 + 0.75x_2 + x_3 + 3}{x_1 + 1.5x_2 + 2x_3 + 6}$$

The solution x^0 = (0, 6.4, 1.2, 0) is an optimal solution of the LFP problem, it is also observed that x^0 also maximize the modified profit function.

Conclusion

A class of inverse linear fractional programming has been studied here. The new approach gives the best compromise solution for company which is best for society's point of view. An illustration observation used to demonstrate the advantage of the new approach.

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