Research Journal of Recent Sciences ISSN 2277-2502
Vol. **2(10),** 33-40, October **(2013)** *Res.J.Recent Sci.*

Hybrid Heuristic Computational approach to the Bratu Problem

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Available online at: www.isca.in, www.isca.me
Received 26th April 2013, revised 9th May 2013, accepted 3rd July 2013

Abstract

In this study a stochastic method based on the heuristic computation is applied for solving the Bratu boundary value problem and an initial value problem of the Bratu-type. A mathematical model consisting of unknown adaptable parameters has been developed using the linear combinations of log sigmoid basis functions. The Genetic algorithm (GA), Pattern Search (PS), Interior Point algorithm (IPA), Active Set algorithm (ASA), and three hybrid schemes combining GA with PS, IPA, and ASA have been employed for learning of the unknown adaptable parameters. To demonstrate the efficacy of the presented method, comparisons of the results are made with the some standard analytical methods as well as the exact solutions. The results from the proposed method are found to be satisfactory and comparable to the standard analytical methods.

Keywords: Bratu problem, Boundary value problem, Initial value problem, Genetic algorithm (GA), Pattern search (PS), Active set algorithm (ASA)

Introduction

Boundary value problems (BVPs) arise in numerous useful applications including fluid mechanics, thermodynamics, controls, beam deflection, optimization theory, and various other disciplines of engineering and science[1,2].

Motivated by the potential applications of BVPs a significantly great attention has been devoted toward the study of these nonlinear problems. Since many nonlinear problems either do not have an exact solution or obtaining the same is difficult analytically, therefore these problems are tackled using various approximate analytical and numerical techniques. Many methods including shooting method, finite difference method, adomian decomposition method (ADM), decomposition method, polynomial spline method, variational iteration method (VIM), and homotopy perturbation method (HPM) have been employed to deal with such nonlinear BVPs[1,2].

The classical Bratu problem in one-dimensional planner coordinates is represented by the boundary value problem of the form given by the following equation[3-9]

$$u''(x) + \lambda e^{u(x)} = 0, \quad 0 < x < 1 \qquad (1)$$
$$u(0) = 0, \text{ and } u(1) = 0$$

The classical Bratu problem (1) arises in diverse applications of engineering and science including the model of fuel ignition, chemical reaction theory, radiative heat transfer, Chandrasekhar model of the expansion of the universe, and nanotechnology[3-10].

Many authors have utilized a wealth of approximate techniques for the numerical solution of the standard Bratu and Bratu-type problems[3-10]. Deeba and Khuri[3] applied decomposition method (DM) to the Bratu boundary value problem. Khuri[4] used laplace transform decomposition (LTDM) method for solving Bratu's problem. Wazwaz[5] applied adomian decomposition method (ADM) to the Bratu-type boundary value and initial value problems. Recently Abukhaled[6] et al. implemented three different spline-based approaches such as cubic B-spline collocation, adaptive collocation, and optimal collocation for the solution of Bratu and Bratu-type problems. Vahidi, and Hasanzade[7] in their recent study applied restarted adomian decomposition method (RADM) to the initial value problem of the Bratu-type. Rashidinia and Jalilian[8] used a non-polynomial spline method for the numerical solution of the Bratu boundary value problem.

The standard approximate analytical and numerical techniques that have been applied to the Bratu and Bratu-type problems are all deterministic in nature. Further these deterministic standard methods give the solution on the predefined discrete points besides having some other limitations like the calculation of adomian polynomials involved in some standard techniques and complexity due to the increase in sampling points[11,12]. Recently stochastic methods such as evolutionary computing and neural network based techniques have received tremendous attention from the researchers and these methods have been exploited as powerful alternative tools for solving nonlinear problems in engineering and science. Although a rich variety of nonlinear problems in ordinary differential equations (ODEs) have been solved using evolutionary computing methods only a few are given here for the reference. Ibraheem and Khalaf[13] used neural network (NN) based method for solving boundary value problems. Khan[12,14] et al. used the Genetic algorithm (GA) and the Particle Swarm Optimization (PSO) based artificial neural network (ANN) model for the solution of the initial value problems in ODEs and the nonlinear ODE's. PSO based hybrid neural network method has also been used in[15] for the solution of the nonlinear differential equation. Malik[16] et al. applied the hybridization approach of GA and IPA to the numerical solution of force-free and forced Duffing-van der pol oscillator. The stochastic methods have been shown to provide the solution of the nonlinear problems in ODEs with greater accuracy besides offering some advantages over the standard numerical techniques[12, 15-16].

Motivated by the continuing research of the Bratu-type problems and the strengths of the evolutionary computing techniques we consider the Bratu boundary value problem (1) and an initial value problem of the Bratu-type model given by the following equation[5-7]

$$u''(x) - 2e^{u(x)} = 0, \ 0 < x < 1 \qquad (2)$$
$$u(0) = 0, and\ u'(0) = 0$$

The main goal of this study is to apply the heuristic computing approach for the solution to the Bratu boundary value problem (1) and an initial value problem of the Bratu-type (2). Four different heuristic techniques such as Genetic Algorithm (GA), Pattern Search (PS), Interior Point Algorithm (IPA), and Active Set Algorithm (ASA) have been employed in this study. Moreover three hybrid schemes that combine the GA with, PS, IPA, and ASA called as GA-PS, GA-IPA, and GA-ASA have been applied to obtain the approximate numerical solution of the Bratu boundary value problem (1) and an initial value problem of the Bratu-type (2). To prove the efficacy and the accuracy of the our method, comparisons of the results are made with the exact solutions and some standard approximate analytical and numerical techniques such as decomposition method (DM)[3], laplace transform decomposition method (LTDM)[4], B-spline method[8], non-polynomial cubic spline method[8], perturbation iteration algorithm (PIA)[6], adomian decomposition method (ADM)[7], and restarted adomian decomposition method (RADM)[7].

Material and Methods

An overview of search optimization algorithms such as genetic algorithm (GA), interior point algorithm (IPA), active set algorithm (ASA), and pattern search algorithm (PS) used in this work is presented. The procedural steps of hybrid schemes combining GA with IPA, ASA, and PS are also given. The description of the proposed methodology is also presented.

Genetic Algorithm (GA): Genetic Algorithm (GA) is one of the widely used global search technique which is based on the evolutionary principles of Darwin's theory. The GA operates on a population of individual solutions. The algorithm evolves population toward an optimal solution over the successive generations by using selection, crossover, and mutation operations[17].

In recent years the hybridization of GA with PS, IPA, and other local search methods has been widely used to solve many problems in engineering and science. It has been demonstrated that the heuristic hybrid algorithms provide improved performance. In this work GA has been hybridized with PS, IPA, and ASA. The GA has been used as global optimizer while PS, IPA, and ASA have been utilized for local search refinement. The procedural steps of the hybrid schemes are given as follows while the parameter settings of these algorithms used in this work are given in table- 1 and table- 2.

Pseudo code: GA hybridized with PS, IPA, and ASA

Step 1: (Population Initialization)
A population of N chromosomes or individuals is generated using random number generator. Each population consists of M number of genes. The number of genes is equal to the number of unknown adaptable parameters.

Step 2: (Fitness Evaluation)
Compute the fitness of each chromosome in the current population using the fitness function (FF). Rank the individuals according their fitness values.

Step 3: (Termination Criteria)
The algorithm terminates if the maximum number of cycles has exceeded or a predefined fitness value is achieved. If the termination criterion is satisfied then go to step 6 for local search refinement, else continue and repeat steps 2 to 5.

Step 4: (Reproduction)
A new generation is populated using the crossover operation. Parents are selected on the basis of their fitness which produces offspring (children) to act as parents for the next generation.

Step 5: Mutation
Mutation operation is optional and it is carried if there is no improvement in the fitness in a generation.

Step 6: (Local Search Fine Tuning)
The best chromosome achieved by the GA is fed to the PS, IPA, and ASA as a starting point for fine tuning and improvement.

Table-1
Parameter Settings of GA and IPA

GA			IPA		
Parameters	**Settings**		**Parameters**	**Settings**	
	Example 1	**Example 2**		**Example 1**	**Example 2**
Population size	240	240	Start point	best chromosome from GA	best chromosome from GA
Chromosome size	30	30	Maximum iterations	1000	1000
Selection function	Stochastic uniform	Stochastic uniform	Maximum function evaluations	48000	200000
Mutation function	Adaptive feasible	Adaptive feasible	Function tolerance	1e-22	1e-18
Crossover function	Heuristic	Heuristic	Nonlinear constraint tolerance	1e-10	1e-18
Hybridization	IPA/PS/ASA	IPA/ASA/PS	Derivative type	Forward differences	Central differences
No. of generations	1500	2000	Hessian	BFGS	BFGS
Function tolerance	1e-22	1e-18	Subproblem algorithm	ldl fctorization	ldl factorization
Bounds	-20, +20	-20, +20	Bounds	-20, +20	-20, +20

Table-2
Parameter Settings of PS and ASA

PS			ASA		
Parameters	**Settings**		**Parameters**	**Settings**	
	Example 1	**Example 2**		**Example 1**	**Example 2**
Start point	Optimal chromosome from GA	Optimal chromosome from GA	Start point	best chromosome from GA	best chromosome from GA
Poll method	GPS positive basis 2N	MADS positive basis 2N	Maximum iterations	400	400
Polling order	Random	consecutive	Maximum function evaluations	48000	200000
Maximum iterations	3000	4000	Function tolerance	1e-22	1e-18
Maximum function evaluation	200000	230000	Nonlinear constraint tolerance	1e-10	1e-18
Function tolerance	1e-22	1e-22	SQP constraint tolerance	1e-6	1e-6

Pattern Search (PS): The Pattern Search (PS) method belongs to the direct search optimization techniques that explore a series of points that may reach to the best point. The algorithm initiates the search by creating a mesh from a set of points, around the current point. The algorithm looks for a point in the mesh that gives improvement in the objective function value. If the PS algorithm discovers such a point in the mesh that point turns into the current point in the next step. This process continues until the optimal value of the objective function is attained by the algorithm. The PS algorithm also possesses enhanced adaptability for local search refinement[18,19].

Interior Point Algorithm (IPA): Interior Point Algorithm (IPA) computes iterates that lie in the feasible interior region. The algorithm reaches the optimal solution by computing and following a continuous central path of the feasible region while reducing the barrier parameter[20].

Active Set Algorithm: Active Set Algorithm (ASA) is an iterative method that solves constrained optimization problems by searching solutions in the feasible sets. The main objective of the algorithm is to estimate the active set at the solution of the problem. The standard AS method works in two phases called as the feasibility phase and the optimality phase. In the feasibility phase the algorithm finds a feasible point for the constraints while neglecting the objective. In the optimality phase the algorithm performs the minimization of the objective while feasibility is retained[21].

Proposed Methodology: We may assume that the approximate numerical solution $u(x)$ and its first and second derivatives, $u'(x)$, and $u''(x)$ can be represented by a linear combination of some basis functions as follows.

$$u(x) = \sum_{i=1}^{m} a_i\, \varphi(b_i x + c_i) \qquad (3)$$

$$u'(x) = \sum_{i=1}^{m} a_i b_i \varphi'(b_i x + c_i) \quad (4)$$

$$u''(x) = \sum_{i=1}^{m} a_i b_i^2 \varphi''(b_i x + c_i) \quad (5)$$

where $\varphi(x)$ is taken as the log sigmoid function which is given by

$$\varphi(x) = \frac{1}{1+e^{-x}} \quad (6)$$

$a_i, b_i, \text{ and } c_i$, are real valued unknown adaptable parameters, and m is the number of basis functions.

The values of unknown adaptable parameters are acquired by formulating a problem exclusive fitness function which consists of the sum of two parts. The first part represents the mean square error (ε_1) of the given ODE without boundary conditions, and the second part represents the mean square error (ε_2) linked with the boundary and/or initial conditions, given by (7) and (8) respectively.

$$\varepsilon_1 = \frac{1}{m+1} \sum_{i=0}^{m} \left[\ddot{u}(x_i) + \lambda\left(e^{u(x_i)}\right)\right]^2 \quad (7)$$

$$\varepsilon_2 = \frac{1}{2}\{(u(0))^2 + (u(1))^2\} \quad (8)$$

The fitness function is accordingly given as follows

$$\varepsilon_j = \varepsilon_1 + \varepsilon_2 \quad (9)$$

where j is the cycle index.

The minimization of the fitness function given by equation (9) is executed using the heuristic search methods by suitable learning of the equations given by equation (3) to equation (5). Consequently the optimal values of the adaptable parameters are acquired. The optimal values of the adaptable parameters are used in equation (3) which provide the approximate numerical solution $u(x)$ of the given problem conveniently.

Results and Discussion

The proposed methodology is applied to the Bratu boundary value problem and an initial value problem of the Bratu-type. For the effectiveness and the reliability of the proposed method, comparisons of the results are made with the exact solutions and some standard numerical and analytical methods.

Example 1: We consider the Bratu problem (1) with two special cases: $\lambda = 1 \text{ and } \lambda = 2$. The exact solution of equation (1) for $\lambda > 0$ is given by following equation[5-10].

$$u(x) = -2\ln\left[\frac{\cosh\left(\frac{\theta}{2}\left(x-\frac{1}{2}\right)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right] \quad (10)$$

where θ satisfies

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right) \quad (11)$$

The literature[3-10] has reported that there exist two solutions of the Bratu problem when $\lambda = \lambda_c$, one solution when $\lambda < \lambda_c$ and no solution when $\lambda > \lambda_c$, where λ_c = 3.513830719, is known as critical value.

The approximate numerical solution of equation (1) using the proposed method is obtained by developing its fitness function for each case $\lambda = 1 \text{ and } \lambda = 2$. The number of basis functions is taken equal to 10. Therefore the mean square errors ε_1 and ε_2 associated with (1) for $\lambda = 1$ are given by

$$\varepsilon_1 = \frac{1}{11}\sum_{i}^{11}\left[\ddot{u}(x_i) + 1\left(e^{u(x_i)}\right)\right]^2 \quad (12)$$

$$\varepsilon_2 = \frac{1}{2}\{(u(0))^2 + (u(1))^2\} \quad (13)$$

Therefore the fitness function ε_j is given as follows

$$\varepsilon_j = \frac{1}{11}\sum_{i=1}^{11}\left(\ddot{u}(x_i) + 1\left(e^{u(x_i)}\right)\right)^2 + \frac{1}{2}\{(u(0))^2 + (u(1))^2\}\Big|_j \quad (14)$$

Similarly the fitness function for $\lambda = 2$ is formulated which is give by

$$\varepsilon_j = \frac{1}{11}\sum_{i=1}^{11}\left(\ddot{u}(x_i) + 2\left(e^{u(x_i)}\right)\right)^2 + \frac{1}{2}\{(u(0))^2 + (u(1))^2\}\Big|_j \quad (15)$$

where $u(x)$ and $\ddot{u}(x)$ are given by (3) and (5) respectively.

The fitness functions given by (14) and (15) are minimized by applying GA, PS, IPA, ASA, and three hybrid schemes GA-PS, GA-IPA, and GA-ASA for the learning of the unknown adaptable parameters. For the implementation Matlab 7.6 has been utilized.

The parameter settings used for the implementation of the algorithms are given in table-1 and table-2. The length of chromosome i.e. the number of unknown adaptable parameters (a_i, b_i, c_i) are chosen equal to 30. The values of these unknown adaptable parameters are restricted between -20 and + 20. This was observed by several simulations that by bounding these unknown adaptable parameters to the specified interval we get good results. The optimal values of the unknown adaptable parameters achieved after the execution of the heuristic schemes according to the prescribed parameter settings are provided in table-3. The approximate solution $u(x)$ of the Bratu problem equation (1) is obtained by using the values of unknown adaptable parameters from table-3. The results obtained using the proposed schemes are given in table- 4.

The absolute errors by the proposed heuristic schemes have been computed relative to the exact solution and these have been presented in table-5. For the accuracy and the efficacy of the proposed method comparisons are carried out with the exact solutions and some standard analytical methods such as decomposition method (DM)[3], Laplace transform decomposition method (LTDM)[4], optimal spline[6], non-polynomial cubic spline[8], and B-spline[8]. Comparison of the absolute errors in table-5 reveals that the proposed method based on the heuristic computational approach yields the results of the Bratu problem equation (1) for two special cases $\lambda = 1$ and $\lambda = 2$ with the significantly greater accuracy. Further it is established from the comparison of table-5 that the absolute errors relative to the exact solutions obtained from the proposed schemes are significantly smaller as compared to the standard methods including decomposition method (DM)[3], laplace transform decomposition method (LTDM)[4], non-polynomial cubic spline method[8] and B-spline method[8]. Moreover the better performance of hybrid schemes is quite evident from table- 5.

Table-3
Optimal values of unknown adaptable parameters acquired by hybrid schemes

		GA-PS			GA-ASA			GA-IPA		
	i	a_i	b_i	c_i	a_i	b_i	c_i	a_i	b_i	c_i
λ=1	1	-1.2026	-0.9780	4.7183	-1.1873	-0.9927	4.3567	-0.9870	-0.8866	4.0878
	2	0.0636	3.1262	1.7586	0.0532	2.8125	1.3375	0.1371	2.5885	1.3474
	3	3.3970	0.0286	-2.8143	3.3774	0.0092	-2.8024	3.1203	-0.1468	-2.6795
	4	-2.4075	-1.4589	-0.3726	-2.3967	-1.4546	-0.3605	-2.2438	-1.5079	-0.2057
	5	-1.9894	-0.2529	1.6472	-1.9747	-0.2414	1.5959	-1.7019	-0.2075	1.4432
	6	0.4889	-0.8683	3.0274	0.4933	-0.7706	2.8568	0.4568	-0.7396	2.6768
	7	-2.7638	1.2354	-1.6975	-2.7251	1.2393	-1.7020	-2.5884	1.1878	-1.5518
	8	-2.0535	1.6769	3.3943	-2.0717	1.5492	3.2516	-1.7665	1.4070	3.1085
	9	2.6244	-1.4582	4.0706	2.6098	-1.4604	3.9947	2.3478	-1.4449	3.6874
	10	3.0827	2.0025	3.2143	3.0735	2.0118	3.1644	2.6066	2.0180	2.9024
λ=2	1	2.0497	2.0166	0.2739	1.9681	1.9209	0.2519	1.9109	1.9301	0.2668
	2	-0.2482	-0.8913	0.1337	-0.3238	-0.1363	0.0881	-0.2924	-0.1368	0.0864
	3	1.0784	3.5803	2.3166	1.1145	3.4939	2.3152	1.0969	3.4901	2.3596
	4	-0.5860	-0.0930	1.0323	-0.5827	0.0042	0.8449	-0.5221	0.0024	0.8409
	5	-2.2732	0.7748	-0.1741	-2.3987	0.7396	-0.4401	-2.3208	0.6664	-0.4480
	6	0.4298	-2.3960	5.2107	0.4101	-1.8643	4.5666	0.3630	-1.9026	4.5434
	7	-0.5210	2.5587	-2.2779	-0.8380	2.3214	-2.1652	-0.8158	2.3580	-2.1672
	8	-4.0139	1.6085	-2.7957	-4.0283	1.5534	-3.0031	-4.0585	1.5711	-2.9750
	9	-1.7985	-2.0488	-0.0763	-1.7691	-2.0924	-0.0573	-1.7780	-2.0643	-0.0848
	10	0.4287	-1.5663	-0.3880	0.2635	-0.6650	-0.1063	0.2335	-0.6549	-0.1033

Table- 4
Approximate numerical results of Bratu problem (1) obtained using proposed method

	x	Exact[4]	Proposed Method			
			GA	GA-IPA	GA-ASA	GA-PS
λ=1	0.1	0.0498467900	0.0498451673	0.0498466792	0.0498471487	0.0498468057
	0.2	0.0891899350	0.0891873130	0.0891899115	0.0891905624	0.0891898725
	0.3	0.1176090956	0.1176055844	0.1176092506	0.1176099669	0.1176090691
	0.4	0.1347902526	0.1347853879	0.1347904232	0.1347907729	0.1347903249
	0.5	0.1405392142	0.1405326815	0.1405391460	0.1405389459	0.1405392814
	0.6	0.1347902526	0.1347823411	0.1347899189	0.1347893717	0.1347902151
	0.7	0.1176090956	0.1176005172	0.1176087951	0.1176082114	0.1176090023
	0.8	0.0891899350	0.0891812134	0.0891899996	0.0891894806	0.0891899220
	0.9	0.0498467900	0.0498377453	0.0498470933	0.0498465067	0.0498468702
λ=2	0.1	0.1144107440	0.1144422632	0.1144110417	0.1144112628	0.1144295992
	0.2	0.2064191156	0.2064542183	0.2064188630	0.2064187710	0.2064350302
	0.3	0.2738793116	0.2739240847	0.2738795293	0.2738795487	0.2738928867
	0.4	0.3150893646	0.3151461668	0.3150897674	0.3150900761	0.3151016345
	0.5	0.3289524214	0.3290141917	0.3289522874	0.3289525021	0.3289633331
	0.6	0.3150893646	0.3151488825	0.3150890217	0.3150888736	0.3150984878
	0.7	0.2738793116	0.2739372331	0.2738793427	0.2738790792	0.2738860159
	0.8	0.2064191156	0.2064804683	0.2064192665	0.2064192346	0.2064227941
	0.9	0.1144107440	0.1144741929	0.1144105967	0.1144106945	0.1144114262

Example 2: Consider the Bratu-type problem equation (2).

The exact solution of (2) is given by the following equation[5,6].

$$u(x) = -2\ln(\cos(x)) \qquad (16)$$

To apply the proposed scheme to the initial value problem of Bratu-type equation (2) fitness function is developed as follows.

$$\varepsilon_j = \frac{1}{11}\sum_{i=1}^{11}\left(\ddot{u}(x_i) - 2\left(e^{u(x_i)}\right)\right)^2 + \frac{1}{2}\left\{\left(u(0)\right)^2 + \left(u'(0)\right)^2\right\}\Big|_j \qquad (17)$$

The parameter settings for the implementation of the heuristic schemes used for this problem are given in table-1 and table-2. The optimal values of unknown adaptive parameters acquired are given in table-6. The approximate numerical results obtained using GA and hybrid schemes for different values of x are given in table-7. The absolute errors have been computed relative to the exact solution and are provided in table-8. Comparisons of the absolute errors are made with adomian decomposition method (ADM)[7], Restarted adomian decomposition method (RADM)[7], optimal spline method[6], and perturbation iteration algorithm (PIA (1, 3))[6]. It is observed from the comparisons that the proposed method provides satisfactory results of the Bratu-type problem equation (2). The comparison further shows that the absolute errors obtained from the proposed method based on the hybrid approaches are comparable with those ADM, RADM, and Optimal Spline methods, whereas they are relatively smaller than PIA(1, 3). The effectiveness of the hybrid schemes is also evident in this problem.

Table- 5
Comparison of absolute errors between proposed method and standard numerical methods given in[3, 4, 8]

	Proposed Method					**Other Methods**			
	x	**GA**	**GA-IPA**	**GA-ASA**	**GA-PS**	**DM [3]**	**LTDM [4]**	**N. Cubic spline [8]**	**B-spline [8]**
	0.1	1.62E-06	1.11E-07	3.59E-07	1.57E-08	2.68E-03	6.25E-07	9.26E-08	2.98E-06
	0.2	2.62E-06	2.35E-08	6.27E-07	6.25E-08	2.02E-03	4.36E-07	1.75E-07	5.46E-06
	0.3	3.51E-06	1.55E-07	8.71E-07	2.65E-08	1.52E-04	2.26E-07	2.39E-07	7.33E-06
	0.4	4.86E-06	1.71E-07	5.20E-07	7.23E-08	2.20E-03	4.76E-07	2.81E-07	8.50E-06
λ=1	0.5	6.53E-06	6.82E-08	2.68E-07	6.72E-08	3.01E-03	8.06E-08	2.95E-07	8.89E-06
	0.6	7.91E-06	3.34E-07	8.81E-07	3.75E-08	2.20E-03	8.76E-07	2.81E-07	8.50E-06
	0.7	8.58E-06	3.01E-07	8.84E-07	9.33E-08	1.52E-04	1.01E-06	2.39E-07	7.33E-06
	0.8	8.72E-06	6.46E-08	4.54E-07	1.30E-08	2.02E-03	3.14E-07	1.75E-07	5.46E-06
	0.9	9.04E-06	3.03E-07	2.83E-07	8.02E-08	2.28E-03	2.18E-07	9.26E-08	2.98E-06
	0.1	3.15E-05	2.98E-07	5.19E-07	1.89E-05	1.52E-02	2.13E-03	1.03E-06	1.72E-05
	0.2	3.51E-05	2.53E-07	3.45E-07	1.59E-05	1.47E-02	4.21E-03	2.09E-06	3.26E-05
	0.3	4.48E-05	2.18E-07	2.37E-07	1.36E-05	5.89E-03	6.19E-03	3.02E-06	4.49E-05
	0.4	5.68E-05	4.03E-07	7.12E-07	1.23E-05	3.25E-03	8.00E-03	3.66E-06	5.28E-05
λ=2	0.5	6.18E-05	1.34E-07	8.07E-08	1.09E-05	6.98E-03	9.60E-03	3.89E-06	5.56E-05
	0.6	5.95E-05	3.43E-07	4.91E-07	9.12E-06	3.25E-03	1.09E-02	3.66E-06	5.28E-05
	0.7	5.79E-05	3.11E-08	2.32E-07	6.70E-06	5.89E-03	1.19E-02	3.01E-06	4.49E-05
	0.8	6.14E-05	1.51E-07	1.19E-07	3.68E-06	1.47E-02	1.24E-02	2.09E-06	3.26E-05
	0.9	6.34E-05	1.47E-07	4.95E-08	6.82E-07	1.52E-02	1.09E-02	1.03E-06	1.72E-05

Table-6
Optimal values of unknown adaptable parameters acquired by hybrid schemes

	GA-PS			**GA-ASA**			**GA-IPA**		
i	a_i	b_i	c_i	a_i	b_i	c_i	a_i	b_i	c_i
1	-1.574	-4.3007	-5.3328	-0.7058	-2.9094	-6.5500	-0.3453	-1.5097	-2.5190
2	5.0112	6.1085	-9.9512	5.6296	6.1657	-10.000	5.9163	5.1799	-8.0063
3	3.2318	-2.0084	-1.787	3.5729	-1.6589	-1.3569	1.5015	-2.7157	-2.4613
4	9.9683	1.5681	-3.2017	9.7022	0.9337	-2.0803	1.6462	0.4369	-0.6561
5	2.2362	-3.2781	-2.8949	2.9962	-4.6725	-5.6242	1.5190	-2.4307	-1.8001
6	5.0632	2.0861	-6.0192	5.0085	1.3898	-6.2992	2.1032	0.3709	-2.5095
7	-5.178	-2.058	-2.8193	-4.8552	-0.5026	-3.1264	-2.0347	-0.6000	-1.5870
8	-7.2745	0.6289	-3.9587	-7.3139	0.3284	-3.5974	-1.9510	-0.8493	-2.6926
9	-0.5584	-3.154	3.9479	-1.4498	-3.3044	4.2272	-0.6270	-4.6463	4.6490
10	0.1407	3.2648	-9.9211	0.1360	3.2655	-9.9216	0.6655	3.3812	-2.3110

Table-7
Approximate numerical results of the Bratu-type problem (2) obtained using proposed method

x	Exact	Proposed Method			
		GA	GA-IPA	GA-ASA	GA-PS
0	0	-0.000127233161791	0.000007240636863	0.000003621877791	-0.000073310095726
0.1	0.010016711246471	0.009902535662837	0.010021817918891	0.010011664524820	0.009946544501952
0.2	0.040269546104817	0.040198068621833	0.040274535411218	0.040255332920456	0.040231682985543
0.3	0.091383311852116	0.091354600664354	0.091382251837821	0.091362635059961	0.091377334308956
0.4	0.164458038150111	0.164447022159040	0.164453197408385	0.164428805539579	0.164459012798273
0.5	0.261168480887445	0.261159660847400	0.261164991524772	0.261124406291459	0.261161793365549
0.6	0.383930338838876	0.383935547889938	0.383922020710567	0.383870252913944	0.383929256733938
0.7	0.536171515135862	0.536212209029791	0.536154228580198	0.536102367276458	0.536199059283903
0.8	0.722781493622688	0.722853017760055	0.722764899238615	0.722703491886817	0.722834154891636
0.9	0.950884887171629	0.950961648843694	0.950859279059219	0.950781673289569	0.950937295838080
1.0	1.231252940772020	1.231366858940850	1.231233754079120	1.231132983521460	1.231336236641780

Table-8
Comparison of absolute errors between proposed method and standard numerical methods given in [6,7]

x	Proposed Method				Other Methods			
	GA	GA-IPA	GA-ASA	GA-PS	ADM[7]	RADM[7]	Optimal Spline[6]	PIA(1,3)[6]
0.1	1.14E-04	5.11E-06	5.05E-06	7.02E-05	4.39E-13	9.30E-14	7.25E-07	6.71E-06
0.2	7.15E-05	4.99E-06	1.42E-05	3.79E-05	4.54E-10	9.72E-11	1.61E-06	9.55E-06
0.3	2.87E-05	1.06E-06	2.07E-05	5.98E-06	2.66E-08	5.78E-09	2.38E-06	3.31E-06
0.4	1.10E-05	4.84E-06	2.92E-05	-9.75E-07	4.85E-07	1.07E-07	3.02E-06	8.04E-06
0.5	8.82E-06	3.49E-06	4.41E-05	6.69E-06	4.67E-06	1.06E-06	3.45E-06	8.48E-06
0.6	5.21E-06	8.32E-06	6.01E-05	1.08E-06	3.01E-05	7.07E-06	3.51E-06	2.03E-05
0.7	4.07E-05	1.73E-05	6.91E-05	-2.75E-05	1.48E-04	3.62E-05	2.78E-06	7.15E-05
0.8	7.15E-05	1.66E-05	7.80E-05	-5.27E-05	6.00E-04	1.54E-04	7.06E-07	2.91E-04
0.9	7.68E-05	2.56E-05	1.03E-04	-5.24E-05	2.11E-03	5.74E-04	6.86E-06	1.05E-03
1.0	1.14E-04	1.92E-05	1.20E-04	-8.33E-05	6.65E-03	1.95E-04	3.28E-06	3.53E-03

Conclusion

An alternate method based on hybrid heuristic computing has been applied for the solution to the Bratu boundary value problem and an initial value problem of the Bratu-type. On the basis of the simulation results and comparisons made with some standard analytical methods and exact solutions, it can be concluded that the presented heuristic computing method is effective and handy for solving the Bratu-type problems. The accuracy and the reliability of the proposed method are illustrated by solving two special cases of the Bratu boundary value problem and an initial value problem of the Bratu-type. It is observed that the proposed technique shows supremacy on some of the standard analytical techniques in comparison with the exact solutions for the Bratu problem. Moreover the proposed method can provide the approximate solution of the given problem conveniently and on the continuous grid of time once the learning of the unknown adaptable weights has been achieved.

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