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***Short Communication***

# Discovery of New Classes of AG-groupoids

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## Abstract

*We discover eight new subclasses of AG-groupoids namely; anti-commutative AG-groupoid, transitively commutative AG-groupoid, self-dual AG-groupoid, unipotent AG-groupoid, left alternative AG-groupoid, right alternative AG-groupoid, alternative AG-groupoid and flexible AG-groupoid. We prove their existence by providing examples to these classes. We also prove some basic results of these classes and present a table of their enumeration up to order 6.*

**Keywords:** AG-groupoid, LA-group, AG-group, types of AG-groupoid, enumeration.

## Introduction

AG-groupoids have been enumerated[1] up to order 6. Using GAP[2] the Cayley tables have been obtained in the above-mentioned enumeration. We investigate eight new interesting subclasses of AG-groupoids. These classes are anti-commutative AG-groupoid, transitively commutative AG-groupoid, self-dual AG-groupoid, unipotent AG-groupoid, left alternative AG-groupoid, right alternative AG-groupoid, alternative AG-groupoid and flexible AG-groupoid. We prove here the existence of the above classes by providing their cayley tables. We investigate some relations between them and to some other known classes of AG-groupoid.

Table 1 provides counting of the newly found classes of AG-groupoids for the non-associative AG-groupoids. Section 2 is about anti-commutative AG-groupoids and transitively commutative AG-groupoids. Here we prove that every anti-commutative AG-groupoid and every cancellative AG$^{**}$-groupoid $S$ is transitively commutative. Also the equivalence of AG-band and locally associativity is proved for anti-commutative AG-groupoid. In section 3 we introduce the concept of alternativity and flexibility from loop theory into AG-groupoids. Here we prove two basic facts that every AG-3-band is flexible and that in a right alternative AG-groupoid, square of every element commute with every element. Section 4 is about the existence of self-dual AG-groupoid and unipotent AG-groupoids, where we prove that a self-dual AG-groupoid with left identity becomes commutative monoid and also that in a left alternative self-dual AG-groupoid, square of every element commutes with every element.

**Preliminaries:** A groupoid is called AG-groupoid if it satisfies the left invertive law[3]: $(ab)c = (cb)a$. An AG$^{**}$-groupoid is an AG-groupoid satisfying the identity $a(bc) = b(ac)$. An AG-groupoid with left identity is called AG-monoid. Every AG-monoid is AG$^{**}$- groupoid. An AG-groupoid $S$ always satisfies the medial law[4;Lemma 1.1 (i)]: $(ab)(cd) = (ac)(bd)$, while an AG-monoid satisfies paramedial law[4;Lemma1.1 (ii)]: $(ab)(cd) = (db)(ca)$. An AG-groupoids $S$ with left identity e is an AG$^{**}$-groupoid. An AG-groupoid $S$ which satisfy $a(bc) = b(ac)$, for all $a,b,c \in S$, is called AG$^{*}$-groupoid. An AG-groupoid is called Bol$^{*}$-groupoid if it satisfies the identity $(ab \cdot c)d = a \cdot (bc \cdot d)$. An element $a$ of an AG-groupoid $S$ is called idempotent if $a^2 = a$. An AG-groupoid $S$ is called idempotent or AG-2-band or simply AG-band[5] if its every element is idempotent. An AG-groupoid $S$ is called AG-3-band[6] if its every element satisfies $a(aa) = (aa)a = a$. Some applications of AG-groupoid in theory of flocks are given in[7] and some of its applications in geometry have been investigated by Shah[8]. AG-groupoid (also called LA-semigroup), is the generalization of commutative semigroups. For additional sources on AG-groupoids, we suggest arcticles[9,10], while for the semigroup concept we refer the reader to a book by Howie[11]. We present counting of the new subclasses of AG-groupoids in table 1. Note that only the number of non-associative AG-groupoids is shown.

## Anti-commutativity and Transitively Commutativity of AG-groupoids

Actually the notion of anti-commutativity and transitively commutativity had been defined for AG-bands[5], which is a very small class of AG-groupoids. We make these definitions global for the whole AG-groupoids and prove their existence in example 1 and example 2.

**Definition 1:** An AG-groupoid $S$ is called anti-commutative if for all $a,b \in S, ab = ba$, implies that $a = b$.

**Table-1**
**Classification and enumeration results for new subclasses of AG-groupoids of orders 3–6**

| Order | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| Total | 20 | 331 | 31913 | 40104513 |
| Anti-commutative AG-groupoids | 1 | 2 | 4 | 0 |
| Transitively commutative AG-groupoids | 3 | 61 | 2937 | 1239717 |
| Self-dual AG-groupoids | 0 | 8 | 133 | 4396 |
| Left alternative AG-groupoids | 0 | 5 | 171 | 12029 |
| Right alternative AG-groupoids | 2 | 33 | 997 | 139225 |
| Alternative AG-groupoids | 0 | 2 | 59 | 4447 |
| Flexible AG-groupoids | 1 | 19 | 447 | 32770 |
| Unipotent AG-groupoids | 5 | 74 | 3946 | 1739186 |

**Example-1:** An anti-commutative AG-groupoid of order 4.

| · | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 3 | 4 | 2 |
| 2 | 4 | 2 | 1 | 3 |
| 3 | 2 | 4 | 3 | 1 |
| 4 | 3 | 1 | 2 | 4 |

**Example 2:** A transitively commutative AG-groupoid (a non-anticommutative AG-groupoid) of order 4.

| · | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 |
| 4 | 2 | 2 | 2 | 1 |

We now give an interesting relation of $AG^{**}$-groupoid with transitively commutative AG-groupoid.

**Theorem 1.** Every cancellative $AG^{**}$-groupoid S is transitively commutative.

**Proof.** Let $a,b,c\in S$ such that $ab=ba, bc=cb$, Then consider $b(ac)=a(bc)=a(cb)=c(ab)=c(ba)=b(ca)$, which by left cancellativity implies that $ac=ca$.

**Corollary 1:** Every AG-group is transitively commutative. The following theorem shows that the class of transitively commutative AG-groupoids always contain the class of anti-commutative AG-groupoids.

**Theorem 2.** Every anti-commutative AG-groupoid S is transitively commutative.

**Proof.** Let S be an anti-commutative AG-groupoid and let $a,b,c\in S$, such that $ab=ba, bc=cb.$ Then by definition of anti-commutativity, this implies that $a=b, b=c.$, But this implies that $a=c$ and which further implies that $ac=ca.$ Hence S is transitively commutative.

**Conjecture 1:** Every anti-commutative AG-groupoid S is cancellative but the converse is not true.

We now give the following result to show that each AG-band is locally associative.

**Theorem 3:** Let S be an anti-commutative AG-groupoid. Then the following are equivalent. i. S is AG-band; ii. S is locally associative.

**Proof:** (i) ⇒ (ii) is always true, ii. ⇒ (i). By definition of locally associativity and anti-commutativity, for every $a\in S$, we have $aa^2=a^2a\Rightarrow a^2=a.$

## Alternative and Flexible AG-groupoids

In an attempt to bring AG-groupoids a bit closer to quasigroups and loops, the concept of nucleus of AG-groupoids was introduced by Shah[8] and by doing so, six new classes of AG-groupoids have been defined. Here we introduce the concept of flexibility and alternativity from loops. This will give us four more classes of AG-groupoids.

**Definition 3:** An AG-groupoid is called flexible if it satisfies the identity, $xy\cdot x=x\cdot yx.$

**Definition 4:** An AG-groupoid is called left alternative if it satisfies the identity, $xx\cdot y=x\cdot xy.$

**Definition 5:** An AG-groupoid is called right alternative if it satisfies the identity, $xy.y=x.yy.$

**Definition 6:** An AG-groupoid is called alternative if it is both left alternative and right alternative.

**Example 3:** i. A left alternative AG-groupoid of order 4 and ii. A right alternative AG-groupoid of order 3, iii. An alternative AG-groupoid of order 4.

| · | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 3 | 1 |

(i)

| · | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 |
| 3 | 1 | 2 | 1 |

(ii)

| · | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 2 |
| 4 | 3 | 1 | 1 | 1 |

(iii)

The following results illustrate some basic properties of these new classes.

**Proposition 1:** Every AG-3-band is flexible.

**Proof:** Let $S$ be an AG-3-band and $x, y \in S$. Then $x \cdot yx = xx^2 \cdot yx = xy \cdot x^2 x = xy \cdot x$, Hence $S$ is flexible.

**Proposition 2:** In a right alternative AG-groupoid, square of every element commute with every element.

**Proof:** Let S be an AG-groupoid and $x, y \in S$. Then $x^2 y = yx \cdot x = yx^2$.

The following now easily follows.

**Corollary 2:** (i) Every right alternative AG-groupoid is locally associative. (ii) A right alternative AG-monoid is commutative monoid.

Though a non-associative left alternative can be AG-monoid (see the following Example) but then it cannot contain inverses because a left alternative AG-group is abelian group[8].

**Example 4.** A left alternative AG-monoid of order 4.

| · | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 3 |
| 3 | 1 | 1 | 1 | 2 |
| 4 | 1 | 2 | 3 | 4 |

**Theorem 4.** *A right alternative AG-groupoid having left identity is commutative semigroup.*

**Proof:** Let $S$ be a right alternative AG-groupoid, and let $a, b \in S$ then by definition of right alternative AG-groupoid, we have, $ab \cdot b = ab^2 \Rightarrow b^2 a = ab^2$, by left invertive law. Now let $b = e$, we have $e^2 a = ae^2 \Rightarrow a = ae$. Thus $S$ has right identity and hence is commutative semigroup.

## Self-dual AG-groupoids and Unipotent AG-groupoids

Here we introduce the notion of Self-duality from the theory of semi-group into AG-groupoids. (Left) AG-groupoid and right AG-groupoid can easily be seen dual to each other. Thus the transpose of the multiplication table of an AG-groupoid becomes right AG-groupoid. There are AG-groupoids whose transpose is also an AG-groupoid. In this section we discuss such AG-groupoids.

**Definition 7.** An AG-groupoid is called self-dual if it is also right AG-groupoid.

**Note:** Though not studied as a class but the name "almost semigroup" has been used for what we call self-dual AG-groupoid.
It is easy to prove that:

**Proposition 3.** A self-dual AG-groupoid with left identity becomes commutative monoid.

**Definition 8.** An AG-groupoid $S$ is called unipotent if for every $a, b \in S$, we have $a^2 = b^2$.

**Example 5:** A self-dual AG-groupoid that is also unipotent.

| · | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 2 |
| 4 | 1 | 3 | 1 | 1 |

**Theorem 5:** *In a left alternative self-dual AG-groupoid, square of every element commutes with every element.*

**Proof:** Let $S$ be an AG-groupoid and $x, y \in S$. Then $x^2 y = x \cdot xy = yx^2$.

## Conclusion

New eight classes of AG-groupoids have been discovered and investigated. Enumeration of each class up to order 6 has also been provided in a table. Some of the interesting relations of these new discovered classes with each other and with other previously known classes have been investigated. The researchers of the field are motivated to investigate these new classes more in detail.

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