PT Symmetric Octic Potentials and Symmetry breaking

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Abstract

The study of PT-symmetric complex potentials has recently acquired very much attention in the field of quantum mechanics. In this article, we have solved the family of PT-symmetric octic potential to gain further insight in the eigen spectra of the complex quantum system. The potential parameters and the energy eigenvalue of the PT- symmetric octic potential have been obtained. The Hamiltonian retains the PT-symmetry but the wave function does not follow the PT- symmetry. This leads to breakdown of PT-symmetry.

Keywords: PT -symmetry, complex quantum system, space reflection, hermiticity, time reflection, symmetry breaking.

Introduction

The properties of parity operator P and time reflection operator T such that P is linear and has effect $P \to -P$ and $x \to -x$; T is anti-linear and has effect $P \to -P$, $x \to x$ and $i \to -i$.T reverses the sign of i because, like P, it conserves the elementary commutation relation of the quantum mechanics: $[x,p] = i$[1]. The PT -symmetry allows for the possibility of non-Hermitian and complex Hamiltonians but still leads to real eigenvalue of complex quantum mechanical problem. The condition of Hermiticity is an alternative of PT- symmetry; Thus, infinitely new complex Hamiltonians could be constituted which were ignored earlier as they were non-hermitian. The condition of PT -symmetry provides the prospect to study new quantum theories. For certain class of PT -symmetric Hamiltonians, the spectrum remains entirely real, discrete and bounded. The concept of PT-symmetry had its relations with some earlier independent works as well[2-7].

Solution for complex octic potential

Considering the general representation of octic potential

$$V(x) = \sum_{i=1}^{8} c_i x^i \tag{1}$$

The one dimensional Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \right]\psi(x) = E\psi(x)$$

But in units of $\hbar = m = 1$, the equation becomes

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + V(x) \right]\psi(x) = E\psi(x) \tag{2}$$

The conditions for the PT -symmetric potential are
$c_1, c_3, c_5, c_7 \in iR$
$c_2, c_4, c_6, c_8 \in R$

The wave function is expressed in the form of product of exponential and polynomial as:

$$\psi(x) = f(x)\exp\left(-\sum_{k=1}^{5} d_k x^k \right) \tag{3}$$

Where $f(x)$ is some polynomial function of x.
We focus on the following two choices of $f(x)$:
    a) $f(x) = 1$
    b) $f(x) = x + a_o$
But can be generalized for higher degree as well.

For f(x) = 1

$$\psi(x) = \exp(-d_1 x - d_2 x^2 - d_3 x^3 - d_4 x^4 - d_5 x^5) \tag{4a}$$
$$\psi'(x) = (-d_1 - 2d_2 x - 3d_3 x^2 - 4d_4 x^3 - 5d_5 x^4)\psi(x) \tag{4b}$$
$$\psi''(x) = [(-d_1 - 2d_2 x - 3d_3 x^2 - 4d_4 x^3 - 5d_5 x^4)^2 + (-2d_2 - 6d_3 x - 12d_4 x^2 - 20d_5 x^3)]\psi(x) \tag{4c}$$

Substituting and equating each power of x, we get

$$c_8 = \frac{25}{2}d_5^2$$
$$c_7 = 20d_4 d_5$$
$$c_6 = 8d_4^2 + 15d_3 d_5$$
$$c_5 = 10d_2 d_5 + 15d_3 d_4$$
$$c_4 = \frac{9}{2}d_3^2 + 5d_1 d_5 + 8d_2 d_4$$
$$c_3 = 4d_1 d_4 + 6d_2 d_3 - 10d_5$$
$$c_2 = 2d_2^2 + \frac{3}{2}d_1 d_3 - 6d_4$$
$$c_1 = d_1 d_2 - 3d_3$$

We fix the value of constant i.e. $c_8 = 1/2$. It gives $d_5 = \pm 1/5$. We chose the positive sign in order to normalize the wave function which is as;

$$\psi(x) = \exp\left(-d_1 x - d_2 x^2 - d_3 x^3 - d_4 x^4 - \frac{1}{5}x^5 \right) \tag{5}$$

The related energy level is determined by equation (6)

$$E = -\left(\frac{d_1^2}{2} - d_2\right) \qquad (6)$$

Since $c_1, c_3, c_5, c_7 \in iR$, this requires that d_1, d_3, d_5 must be imaginary. Even if b_1 is imaginary, the energy eigenvalue is real as well as the related wave function is PT –symmetric [8].

For f(x) = x + a_0

The wave function and their derivatives are expressed as:

$$\psi(x) = (x + a_0) \exp(-d_1x - d_2x^2 - d_3x^3 - d_4x^4 - d_5x^5) \qquad (7a)$$

$$\psi`(x) = \exp(-d_1x - d_2x^2 - d_3x^3 - d_4x^4 - d_5x^5) + (-d_1 - 2d_2x - 3d_3x^2 - 4d_4x^3 - 5d_5x^4)\,\psi(x) \qquad (7b)$$

$$\psi`(x) = [(-d_1 - 2d_2x - 3d_3x^2 - 4d_4x^3 - 5d_5x^4)^2 + (-2d_2 - 6d_3x - 12d_4x^2 - 20d_5x^3)]\,\psi(x) + 2(-d1 - 2d_2x - 3d_3x^2 - 4d_4x^3 - 5d_5x^4)\exp(-d_1x - d_2x^2 - d_3x^3 - d_4x^4 - d_5x^5) \qquad (7c)$$

Substituting and equating each power of x, we get

$$c_8 = \frac{1}{2}$$
$$c_7 = 4d_4$$
$$c_6 = 8d_4^2 + 5d_3$$
$$c_5 = 2d_2 + 12d_3d_4$$
$$c_4 = \frac{9}{2}d_3^2 + 8d_2d_4 + d_1$$
$$c_3 = 4d_1d_4 + 6d_2d_3 - 3$$
$$c_2 = 2d_2^2 + 3d_1d_3 - 10d_4 + a_0$$
$$c_1 = 2d_1d_2 - 6d_3 + 4d_4a_0 - a_0^2$$

The condition on a_0 as:
$$a_0^4 - 4d_4a_0^3 + 3d_3a_0^2 - 2d_2a_0 + d_1 = 0 \qquad (8)$$

Associated energy eigenvalue is given by
$$E = -\frac{d_1^2}{2} + 3d_2 - 3d_3a_0 + 4d_4a_0^2 - a_0^3 \qquad (9)$$

Case I : $d_1 = d_3 = d_4 = 0$
The condition on a_0 in equation (8)
$$a_0(a_0^3 - 2d_2) = 0$$

And energy eigenvalue of the system is
$$E = 3d_2 - a_0^3$$
i) $a_0 = 0$

The energy becomes
$$E = 3d_2$$
$$c_5 \neq 0, \qquad c_3 \neq 0, \qquad c_2 \neq 0, \qquad c_6 \neq 0,$$
Potential turns out to be complex and energy eigenvalues are real and single valued.

ii) $(a_0^3 - 2d_2) = 0$
The energy eigenvalue is $E = d_2$

The linear terms present in the potential are calculated as:

$$c_8 = \frac{1}{2}$$
$$c_7 = 0$$
$$c_6 = 0$$
$$c_5 = 2d_2$$
$$c_4 = 0$$
$$c_3 = -3$$
$$c_2 = 2d_2^2 + a_0$$
$$c_1 = -a_0^2$$

Thus a_0 will have one real and two imaginary values. Thus we have one real and two imaginary potentials (complex conjugate) for the same energy eigenvalues. In the energy formulae as the sign of the energy components appears explicitly, the occurrence of complex conjugate energy pair is necessary[9].

Case II: $d_2 = d_4 = 0$
The condition on a_0 in equation (8) becomes
$$a_0^4 + 3d_3a_0^2 + d_1 = 0 \qquad (10)$$

And energy eigenvalue in equation (9) becomes
$$E = -\frac{d_1^2}{2} - 3d_3a_0 - a_0^3 \qquad (11)$$
Solving for a_0^2 $\quad a_0^2 = \frac{1}{2}\left(-3d_3 \pm \sqrt{9d_3^2 - 4d_1}\right)$
And

$$a_0 = \pm\left[\frac{1}{2}\left(-3d_3 \pm \sqrt{9d_3^2 - 4d_1}\right)\right]^{\frac{1}{2}}$$
$$E = -\frac{d_1^2}{2} \mp \left[\frac{1}{2}\left(-3d_3 \pm \sqrt{9d_3^2 - 4d_1}\right)\right]^{\frac{1}{2}}\left[\frac{1}{2}\left(3d_3 \pm \sqrt{9d_3^2 - 4d_1}\right)\right] \qquad (12)$$

This gives
$$V(x) = \frac{1}{2}x^8 + 5d_3x^6 + \left(\frac{9}{2}d_3^2 + d_1\right)x^4 - 3x^3 + (3d_1d_3 + a_0)x^2 - (6d_3 + a_0^2)x \qquad (13)$$

The PT-symmetric holds good in case of real values of d's for the wave function as well as for the potential. On the other hand if d's is imaginary, it is seen as symmetry breaking. We also find here that PT invariant potentials are not always leads to complete real spectrum but complex energy solutions are also obtained. The Hamiltonian retains the PT -symmetry but the wave function does not follow the PT –symmetry[10]. This leads to break down of PT - symmetry.

Conclusion

We have solved PT -symmetric octic potential with assumptions and obtained the wave function and corresponding energy levels which turn out to be real. In section 2.1, single levels energy is obtained and PT -symmetry is conserved. In section 2.2 (case I), real and complex conjugate energy pair exists. In case II, more than one energy level is obtained and PT -symmetry breaks down when the values of d's are imaginary. We have solved for the linear terms in the Polynomial function as a factor of the exponential representing wave function but the quadratic and higher order wave function can also be obtained in a similar way.

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