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The Solution of Nonlinear Klein - Gordon Equation using Reduced Differential Transform Method

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Abstract

In this paper, a seemingly new mathematical technique, the reduced differential transform method (RDTM) has been applied to solve the nonlinear Klein - Gordon equation of the form: $u_{tt} - u_{xx} = -u^2 + f(t)$ *subject to the general initial conditions. The results indicate this method (RDTM) to be very effective and simple. It is also consistent with the homotopy analysis method (HAM).*

Keywords: Klein-Gordon equation, reduced differential transform method, homotopy analysis method.

Introduction

Nonlinear phenomena that are encountered in many areas of science such as fluid dynamics, chemical kinetics, crack propagation and quantum mechanics can be modeled by partial differential equations. The Klein – Gordon (KG) equation is known to model many problems in classical and quantum mechanics, particularly in the area of relativity, solitons and condensed matter physics[1]. The reduced differential transform method has been used by several authors to solve linear and nonlinear partial differential equations[2–14].

In this paper, we have successfully solved the nonlinear KG equation of the form:

$$u_{tt} - u_{xx} = -u^2 + f(t) \quad (1)$$

Subject to the general initial conditions, using the reduced differential transform method (RDTM) and obtained results comparable to that from HAM method[1]

The reduced differential transform method (RDTM)

The basic definitions of the RDTM are well elaborated to include[7 – 10]

Definition 1: If the function $f(x, t)$ is analytic and continuously differentiable with respect to both space and time, then we write that:

$$F_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} f(x,t)\right]_{t=0} \quad (2)$$

In equation (3), the t-dimensional spectrum function $F_k(x)$ is the transformed function whereas $f(x, t)$ is the original function.

Definition 2: [7 – 10]: The differential inverse of $F_k(x)$ is defined as:

$$f(x,t) = \sum_{k=0}^{\infty} F_k(x) t^k \quad (3)$$

A combination of equations (3) and (4) then gives:

$$f(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} f(x,t)\right]_{t=0} t^k \quad (4)$$

As stated by Taha[14], the concept of the reduced differential transform method is derived from the power series expansion. Moreover, the fundamental theorems of reduced differential transform method (RDTM) stated without proofs are given below[14]

Theorem 1: If $m(x,t) = f(x,t) \pm g(x,t)$ is the original function, then the transformed function will be given as $M_k(x) = F_k(x) \pm G_k(x)$.

Theorem 2: If the original function is given as $m(x,t) = \beta f(x,t)$ then the transformed function will be gives as $M_k(x) = \beta F_k(x)$; β is a constant.

Theorem 3: If the original function is given as $m(x,t) = x^a t^b$ then the transformed function will be given as $M_k(x) = x^a \delta(k-b), \delta(k) = \begin{cases} 1, k = 0 \\ 0, k \neq 0 \end{cases}$.

Theorem 4: If the original function is given as $m(x,t) = x^a t^b f(x,t)$ then the transformed function will be given as $M_k(x) = x^a F_{k-b}(x)$.

Theorem 5: If the original function is given as $m(x,t) = f(x,t)\, g(x,t)$ then the transformed function will be given as

$$M_k(x) = \sum_{r=0}^{k} G_r F_{k-r}(x) = \sum_{r=0}^{k} F_r G_{k-r}(x).$$

Theorem 6: If the original function is given as $m(x,t) = \frac{\partial^r}{\partial t^r} f(x,t)$ then the transformed function will be given as $M_k(x) = (k+1)\dots(k+r)F_{k+r}(x)$.

Theorem 7: If the original function is given as $m(x,t) = \frac{\partial}{\partial x} f(x,t)$ then the transformed function will be given as $M_k(x) = \frac{\partial}{\partial x} F_k(x)$.

Theorem 8: If the original function is given as $m(x,t) = \frac{\partial^2}{\partial x^2} f(x,t)$ then the transformed function will be given as $M_k(x) = \frac{\partial^2}{\partial x^2} F_k(x)$.

Solution of the KG equation using RDTM

We solve first the KG equation of the form[1]

$$u_{tt} - u_{xx} = -u^2 \quad (5)$$

Subject to the general initial conditions given as:

$$(a) u(x,0) = 1 + \sin(x), u_t(x,0) = 0 \quad (6)$$

The recursion formula for the reduced differential transform of equation (6) is:

$$(k+1)(k+2)U_{k+2}(x) = \frac{\partial^2}{\partial x^2} U_k(x) - U_k^2(x) \quad (7)$$

Where $U_k(x)$ is the transformed function. The initial condition (7) reveals that

$$U_0 = 1 + \sin(x) \quad (8a)$$

$$U_1(x) = 0 \quad (8b)$$

$$U_2(x) = \frac{1}{2}[-3\sin(x) - 2 + \cos(x)^2] \quad (8c)$$

The results of equations (8a – 8c) reveal that

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k \quad (9)$$

yields:

$$u(x,t) = 1 + \sin(x)t + \frac{t^2}{2}[-3\sin(x) - 2 + \cos(x)^2] + \cdots \quad (10)$$

Equation (11) is the exact solution to the problem and is the same as the one obtained by HAM method[1]

Example 2: Here we take as initial condition

$$(b) u(x,0) = 1 + \cos x; \frac{du(x,0)}{dt} = 1$$

The following coefficients have been obtained:

$$u_2 = \frac{-1}{2}[3\cos x + \cos^2 x + 1]\, u_3 = 0; u_4 = \frac{1}{24}\cos x - \frac{1}{12}\sin^2 x + \frac{1}{12}(1+\cos x)\cos x + \frac{1}{12}(1 + 3\cos x + \cos^2 x)(1+\cos x) - \frac{1}{12}$$

Example 3: Here we take as initial condition

c) $u_0 = 1 - x$; $u_1=0$

$$u_2 = -\frac{1}{2}[1-x]^2; u_3 = 0; u_4 = \frac{-1}{12} + \frac{1}{12}(1-x)^2$$

Example 4: Here we take as initial condition

U(x, 0)=xexp(-x); $u_t(x,0)=0$

The following coefficients have been evaluated using MATLAB software:

U_2= -2 exp(-x)+xexp(-x)

U_3=0

U_4=-1/3 exp (-x) +1 / 12 xexp (-x)-1/12 (-4exp(-x) + 2xexp (-x)) x exp (-x)

We can then evaluate the solution by the inverse formula (4)

The truncated solutions of order 4 for examples 1 and 2 are plotted below together with the exact solution.

The simulations have been done and the graph for exact and RDTM solutions are given below.

We observe for the two examples (examples 1 and 2) that for the plots in a range centered at 0 the convergence is very accurate and the range grows with the order of truncation. This observation proves that the method works very well in approximating these equations with arbitrary order of accuracy**.**

Conclusion

In this paper, we have successfully applied a seemingly novel technique, the reduced differential transform method to solve the initial value nonlinear KG equation with examples. The result obtained is exactly the same as that obtained by Alomari[1] using HAM method. We also state therefore that the reduced differential transform method is effective, reliable and easy to use.

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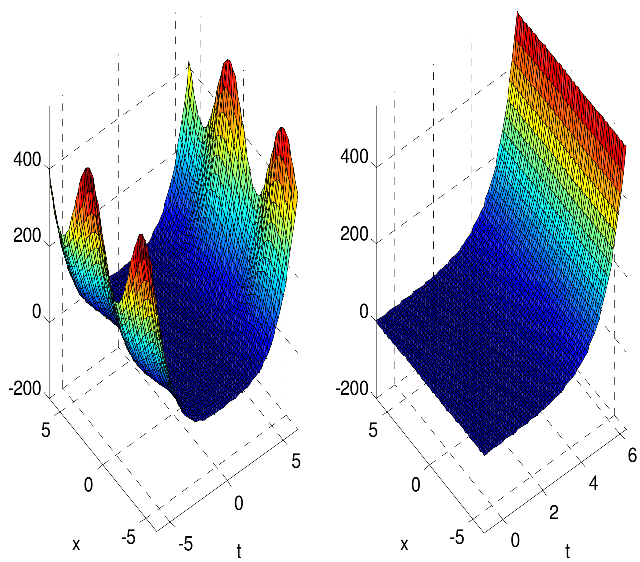

Figure-1
(a) exact solution example 1
(b) Solution by RDTM example 1 order 4 in t,

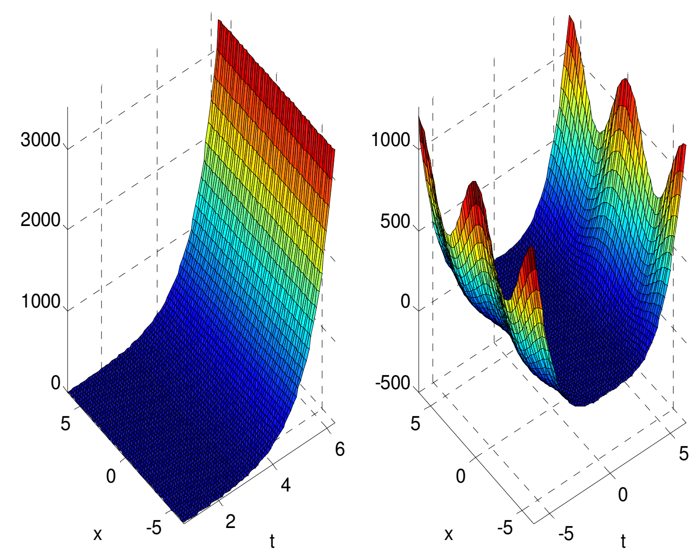

Figure-2
(a) exact solution example 2
(b) RDTM solution example 2