Research Journal of Mathematical and Statistical Sciences ISSN 2320-6047
Vol. **9(2),** 6-11, May **(2021)** *Res. J. Mathematical and Statistical Sci.*

Asymptotic variance and MSE of prametric functions in sampling from one – and two-truncation parameter families

Bhatt Milind B.* and Patel Shantilal R.
Department of Statistics, Sardar Patel University, Vallabhvidyanagar-388120, Anand, Gujarat, India
bhattmilind_b@yahoo.com

Available online at: www.iscamaths.com, www.isca.in, www.isca.me
Received 10th December 2020, revised 14th March 2021, accepted 19th April 2021

Abstract

Asymptotic distributions related to one- and two-truncation parameter family of distributions are presented in this article. Due to non-availability of variance and MSE for unbiased and biased estimator we use asymptotic theory. Using the asymptotic distributions performance of different estimators are compared and MSE stabilization transformation obtained. Strength and tress problem, $P[Y > X]$ also studied and performance of the UMVU estimator relative to MLE measured in terms of LRE and ARE and illustrated by example.

Keywords: Left-truncation, Right-truncation and two-truncation, asymptotic distributions, asymptotic variance, asymptotic mean square error, asymptotic relative efficiency, asymptotic limiting risk efficiency.

Introduction

One sample case has been considered by Tate[1] truncation, right-truncation (one-truncation) parameter family of distributions whereas Bar-Lev and Boukai[2] parameter family of distributions respectively and obtained the UMVU estimators by using probability integral transformation and solving integral equation (Widder[3] review of other methods Guenther[4]) but challenges of obtaining exact variance was not address. Krishnamoorthy, Rohatgi, and Blass[5] and Patel and Bhatt[6] obtained the UMVU estimator based on type-II censored sample from single one distribution and single two-truncation distribution respectively. Ferentinos[7] studied doubly type-II sample from truncation distribution for the UMVU estimator but attempt of obtaining exact variance was not made and Rohatgi[8] reported that exact variance of the UMVU estimator obtained by Tate[1] and Bar-Lev and Boukai[2] computed directly but expression is complicated. However computation of exact variance of the UMVU estimator of the specific function such as reliability, density, hazard rate are easy to compute for single one-truncation parameter family of distributions and single two-truncation parameter family of distributions. Bhatt and Pate[9,10] derived asymptotic variance of the UMVU estimator, MSE of MLE and LRE of the UMVU estimator relative to MLE for single onetwo-truncation parameter family of distributions.

Two sample problem (uncensored sample, type-II censored sample) studied by Rohatgi[11] and Selvavel[12,13] in sampling from two one-truncation parameter families of distributions whereas Patel and Bhatt[14] in sampling from one- and two- truncation parameter families of distributions and obtained explicit expression for the UMVU estimator of any U-estimable parametric function but exact variance of the UMVU estimators are not available. Patel and Bhatt[14] defined independent two-truncated family of distributions and independent one-truncated family of distributions with probability density function (pdf) as

$$f_1(x;\theta_1,\theta_2) = q_1(\theta_1,\theta_2)h_1(x);\ a<\theta_1<x<\theta_2<b, \qquad (1)$$

$$f_2(y;\theta_3) = q_2(\theta_3)h_2(y);\ a<y<\theta_3<b, \qquad (2)$$

$$f_3(z;\theta_4) = q_3(\theta_4)h_2(z);\ a<\theta_4<x<b, \qquad (3)$$

where $-\infty \le a < b \le \infty$ are known constants, $q_i\ (i=1,2,3)$ are everywhere differentiable functions, $h_i\ (i=1,2,3)$ are absolutely continuous functions for two sample problem and obtained the UMVU estimator of any U-estimable parametric function $g(\theta_1,\theta_2,\theta_3)$. Extensive study for unbiased estimator can found from Patel and Bhatt[15], Patel and Bhatt[14] and Selvavel[13] in which the UMVU estimators for doubly type-II censored, type-II censored and uncensored sample is obtained.

If $g(\theta_1,\theta_2,\theta_3) = \phi_1(\theta_1,\theta_2)\phi_2(\theta_3)$ then inference of g in particular the UMVU estimator of g is product of independent UMVU estimators of $\phi_1(\theta_1,\theta_2)$ and $\phi_2(\theta_3)$ derived by Tate[1] and Bar-Lev and Boukai[2]. If $g(\theta_1,\theta_2,\theta_3)$ are not above form, that is $g = EXP[-\theta_1,\theta_2\theta_3]$ or $g = P[Y>X]$ then explicit expression obtained by Patel and Bhatt[16] is applicable for estimation purpose only. One would like to estimate ratio of reliabilities; $EXP[-\theta_3(\theta_2-\theta_1)]$ when hypothesis regarding equality of the reliabilities at a given point t for two two-parameter negative exponential distributions with common scale parameter θ_3^{-1} and location parameters θ_1 and θ_2 rejected[16]. We can use knowledge of estimator with greater confidence if we have knowledge of exact or at the most asymptotic variance for unbiased estimator and MSE in the case of biased estimator and more preferably we can relatively compare performance of the UMVU estimator and MLE (traditional competitors).

In all above cases challenges of obtaining explicit expression of exact variance or MSE were not properly or optimally addressed and result of which we use estimate of variance of estimator because expression of exact variance is either complicated or intractable as reported by Krishnamoorthy and Rohatgi[8]. The application of asymptotic result derived in section 2 are therefore of interest which provide a novel method to solve above challenges.

Problem of obtaining the UMVU estimator of $P[Y > X]$ extensively studied by Beg, M. A.[17-20] when the independent random variables X and Y follows one-truncation parameter families. The problem obtaining unbiased of estimator of $P[Y > X]$ for X and Y follows Pareto distributions, one of special case two independent left truncated family of distributions studied by Beg and Singh[21] whereas for X and Y follows uniform distributions, one of special case two independent right truncated family of distributions studied by Dixit and Phal[22]. Many researchers investigated previous work and other aspect of this problem[23-25]. Asymptotic test for the parameter of two one truncation parameter families extensively studied by Bhatt, Pate, and Pratik Nandy[26].

MSE and variance of MLE and the UMVU have not been obtained due to intractability of exact distributions to be needed. In this paper we deriving necessary asymptotic distributions at reasonable order in section 2 and one sample case [one-truncation (left-truncation / right-truncation) and two-truncation parameter family] is discuss as corollaries 2.1 and 2.2. In terms of ARE and LRE, performance for traditional competitor such as MLE and the UMVU estimator studied in section 3. Section 4 deals with problem of getting MSE stabilization transformation and section 5 devote to asymptotic MSE and variance of the MUVU estimator (traditional competitors) of reliability functional P[Y>X] to compared performance in terms of LRE and ARE and illustrated by example.

Asymptotic distributions [Main Results]

Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ and $Y_{1:n} < Y_{2:n} < \ldots < Y_{n:n}$ and $Z_{1:n} < Z_{2:n} < \ldots < Z_{n:n}$ be corresponding sets of order statistics of random samples $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ and $Z_1, Z_2, \ldots, Z_n$ from f_i; $i = 1, 2, 3$ then smallest order statistics $X_{1:n}$ and largest order statistics $X_{n:n}$ is complete sufficient statistics of θ_1 and θ_2 and largest order statistics $Y_{n:n}$ is complete sufficient statistics of θ_3 and smallest order statistics $Z_{1:n}$ is complete sufficient statistics of θ_4 respectively. Bhatt and Patel[9,10] have shown that at the order $o(n^{-1})$ random variables

$$U_1 = nf_1(\theta_1; \theta_1, \theta_2)(X_{1:n} - \theta_1) \to E[0,1]; \tag{4}$$
$$U_2 = nf_1(\theta_2; \theta_1, \theta_2)(\theta_2 - X_{n:n}) \to E[0,1] \tag{5}$$

$$U_3 = nf_2(\theta_3; \theta_3)(\theta_3 - Y_{n:n}) \to E[0,1] \tag{6}$$
$$U_4 = nf_3(\theta_4; \theta_4)(Z_{1:n} - \theta_4) \to E[0,1]; \tag{7}$$

and they are independent. If $T \to E[\mu, \sigma]$ denotes that T follows exponential distribution then it's pdf is

$$E[\mu, \sigma] = \sigma^{-1} EXP[\sigma^{-1}(t - \mu)];\ \mu < t < \infty,\ \mu > 0,\ \sigma > 0. \tag{8}$$

Also we note that

$$f_1(\theta_1; \theta_1, \theta_2) = q_1(\theta_1, \theta_2) h_1(\theta_1) = q_1^{-1}(\theta_1, \theta_2) \frac{d}{d\theta_1} q_1(\theta_1, \theta_2), \tag{9}$$
$$f_1(\theta_2; \theta_1, \theta_2) = q_1(\theta_1, \theta_2) h_1(\theta_2) = -\, q_1^{-1}(\theta_1, \theta_2) \frac{d}{d\theta_2} q_1(\theta_1, \theta_2), \tag{10}$$
$$f_2(\theta_3; \theta_3) = q_2(\theta_3) h_2(\theta_3) = -\, q_2^{-1}(\theta_3) \frac{d}{d\theta_3} q_2(\theta_3). \tag{11}$$
$$f_3(\theta_4; \theta_4) = q_3(\theta_4) h_3(\theta_4) = q_3^{-1}(\theta_4) \frac{d}{d\theta_4} q_3(\theta_4). \tag{12}$$

Theorem 2.1: Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ and $Y_{1:n} < Y_{2:n} < \ldots < Y_{n:n}$ be corresponding set of order statistic of random sample $X_1, X_2, \ldots, X_n$ and of random sample $Y_1, Y_2, \ldots, Y_n$ from f_1 and f_2 respectively. In the above set up if $g(\theta_1, \theta_2, \theta_3)$ be a real valued differentiable parametric function with non-zero first order partial derivative with respect to $\theta_1, \theta_2, \theta_3$ then asymptotic distribution of random variable

$$U = nf_1(\theta_1; \theta_1, \theta_2) f_1(\theta_2; \theta_1, \theta_2) f_2(\theta_3; \theta_3)[g(X_{1:n}, X_{n:n}, Y_{n:n}) - g(\theta_1, \theta_2, \theta_3)] \to P[0, \lambda_1, \lambda_2, \lambda_3] \tag{13}$$

where

$$\lambda_1 = \lambda_1(g) = f_1(\theta_2; \theta_1, \theta_2)\, f_2(\theta_3; \theta_3) \frac{\partial}{\partial\theta_1} g(\theta_1, \theta_2, \theta_3), \tag{14}$$
$$\lambda_2 = \lambda_2(g) = -f_1(\theta_1; \theta_1, \theta_2)\, f_2(\theta_3; \theta_3) \frac{\partial}{\partial\theta_2} g(\theta_1, \theta_2, \theta_3), \tag{15}$$
$$\lambda_3 = \lambda_3(g) = -f_1(\theta_2; \theta_1, \theta_2) f_1(\theta_2; \theta_1, \theta_2)\, \frac{\partial}{\partial\theta_3} g(\theta_1, \theta_2, \theta_3) \tag{16}$$

and $P[\xi, \eta\lambda_1, \eta\lambda_2, \eta\lambda_3]$ denoted by General Gamma Distribution[23] with pdf as

$$f_U(u; \xi, \eta\lambda_1, \eta\lambda_2, \eta\lambda_3) = \left(\frac{\lambda_1}{(\eta\lambda_1 - \eta\lambda_2)(\eta\lambda_1 - \eta\lambda_3)}\right) e^{-\frac{u-\xi}{\eta\lambda_1}} + \left(\frac{\lambda_2}{(\eta\lambda_2 - \eta\lambda_1)(\eta\lambda_2 - \eta\lambda_3)}\right) e^{-\frac{u-\xi}{\eta\lambda_2}} + \left(\frac{\lambda_3}{(\eta\lambda_3 - \eta\lambda_1)(\eta\lambda_3 - \eta\lambda_2)}\right) e^{-\frac{u-\xi}{\eta\lambda_3}};\quad u > \xi > 0,\ \eta > 0,\ \lambda_1 \neq \lambda_2 \tag{17}$$

Proof: Consider Young's form of Taylor series expansion of $g(X_{1:n}, X_{n:n}, Y_{n:n})$ around θ_1, θ_2 and θ_3 such as

$$g(X_{1:n}, X_{n:n}, Y_{n:n}) = g(\theta_1, \theta_2, \theta_3) + [X_{1:n} - \theta_1] \frac{\partial}{\partial\theta_1} g(\theta_1, \theta_2, \theta_3) + [X_{n:n} - \theta_2] \frac{\partial}{\partial\theta_2} g(\theta_1, \theta_2, \theta_3) + [Y_{n:n} - \theta_3] \frac{\partial}{\partial\theta_3} g(\theta_1, \theta_2, \theta_3) + \sum_{k=2}^{\infty} \left(\frac{r^k}{k!}\right) \left[\alpha_1 \left(\frac{\partial}{\partial\theta_2}\right) + \alpha_2 \left(\frac{\partial}{\partial\theta_4}\right) + \alpha_3 \left(\frac{\partial}{\partial\theta_3}\right)\right]^k g(\theta_1, \theta_2, \theta_3)$$

Where $\alpha_1 r = (X_{1:n} - \theta_1)$, $\alpha_2 r = (\theta_2 - X_{n:n})$, $\alpha_3 r = (\theta_3 - Y_{n:n})$ and simplifying we get

$$U = nf_1(\theta_1; \theta_1, \theta_2) f_1(\theta_2; \theta_1, \theta_2) f_2(\theta_3; \theta_3)[g(X_{1:n}, X_{n:n}, Y_{n:n}) - g(\theta_1, \theta_2, \theta_3)] \to P[0, \lambda_1, \lambda_2, \lambda_3] =$$

$\left[f_1(\theta_2;\theta_1,\theta_2)f_2(\theta_3;\theta_3)\frac{\partial}{\partial\theta_1}g(\theta_1,\theta_2,\theta_3)\right]nf_1(\theta_1;\theta_1,\theta_2)(X_{1:n}-\theta_1)$
$+\left[-f_1(\theta_1;\theta_1,\theta_2)f_2(\theta_3;\theta_3)\frac{\partial}{\partial\theta_2}g(\theta_1,\theta_2,\theta_3)\right]f_1(\theta_2;\theta_1,\theta_2)(\theta_2-X_{n:n})$
$+\left[-f_1(\theta_1;\theta_1,\theta_2)f_1(\theta_2;\theta_1,\theta_2)\frac{\partial}{\partial\theta_3}g(\theta_1,\theta_2,\theta_3)\right]nf_2(\theta_3;\theta_3)(\theta_3-Y_{n:n})$
$+\sum_{k=2}^{\infty}\left(\frac{r^k}{k!}\right)\left[\alpha_1\left(\frac{\partial}{\partial\theta_1}\right)+\alpha_2\left(\frac{\partial}{\partial\theta_2}\right)+\alpha_3\left(\frac{\partial}{\partial\theta_3}\right)\right]^k g(\theta_1,\theta_2,\theta_3).$

Using $U_j;\, j=1,2,3$ defined in (4), (5) and (6) and substituting λ_1, λ_2 and λ_3 given in (14), (15) and (16), U will be

$U=nf_1(\theta_1;\theta_1,\theta_2)f_1(\theta_2;\theta_1,\theta_2)f_2(\theta_3;\theta_3)[g(X_{1:n},X_{n:n},Y_{n:n})-g(\theta_1,\theta_2,\theta_3)]$
$=\lambda_1U_1+\lambda_1U_2+\lambda_3U_3\to P[0,\lambda_1,\lambda_2,\lambda_3].$

and using (17), theorem 2.1 follows.A Similar argument leads to the following theorem 2.2.

Theorem 2.2: Let $X_{1:n}<X_{2:n}<\ldots<X_{n:n}$ and $Z_{1:n}<Z_{2:n}<\ldots<Z_{n:n}$ be corresponding set of order statistic of random sample $X_1,X_2,\ldots,X_n$ and of random sample $Z_1,Z_2,\ldots,Z_n$ from f_1 and f_3 respectively. In the above set up if $g(\theta_1,\theta_2,\theta_4)$ be a real valued differentiable parametric function with non-zero first order partial derivative with respect to $\theta_1,\theta_2,\theta_4$ then asymptotic distribution of random variable

$U=nf_1(\theta_1;\theta_1,\theta_2)f_1(\theta_2;\theta_1,\theta_2)f_3(\theta_4;\theta_4)[g(X_{1:n},X_{n:n},Z_{1:n})-g(\theta_1,\theta_2,\theta_4)]\to P[0,\lambda_1,\lambda_2,\lambda_4]$ (18)

$=\lambda_1U_2+\lambda_1U_2+\lambda_4U_4\to P[0,\lambda_1,\lambda_2,\lambda_4]$ where
$\lambda_4=\lambda_4(g)=f_1(\theta_2;\theta_1,\theta_2)f_1(\theta_2;\theta_1,\theta_2)\ \frac{\partial}{\partial\theta_4}g(\theta_1,\theta_2,\theta_4)$ (19)

and $P[0,\lambda_1,\lambda_2,\lambda_4]$ is general gamma distribution defined in (17).

Corollary 2.1: In the set up of theorem 2.1 suppose $g(\theta_1,\theta_2,\theta_3)=k_1(\theta_3)$ function of θ_3 only and in the set up of theorem 2.2 $g(\theta_1,\theta_2,\theta_4)=k_2(\theta_4)$ function of θ_4 only then $\tilde{k}_1(\theta_3)=k_1(Y_{n:n})$ and $\tilde{k}_2(\theta_4)=k_2(Z_{1:n})$ are MLE of $k_1(\theta_3)$ and $k_2(\theta_4)$ whereas the UMVU estimator $k_1(\theta_3)$ and $k_2(\theta_4)$ obtained by Tate[1] are

$$\hat{k}_1(\theta_3)=k_1(Y_{n:n})+\frac{\frac{\partial}{\partial Y_{n:n}}k_1(Y_{n:n})}{f_2(Y_{n:n};Y_{n:n})}$$
$$\hat{k}_2(\theta_4)=k_1(Z_{1:n})-\frac{\frac{\partial}{\partial Z_{1:n}}k_2(Z_{1:n})}{f_3(Z_{1:n};Z_{1:n})}.$$

Tacking $g(\theta_1,\theta_2,\theta_3)=k_1(\theta_3)$ and $g(\theta_1,\theta_2,\theta_4)=k_2(\theta_4)$ in theorem 2.1 and theorem 2.2 following asymptotic distributions for one sample [left/right] truncation parameter family of distributions reduced from general gamma distribution defined in (17) as special cases

$\left[\tilde{k}_2(\theta_4)-k_2(\theta_4)\right]\to E[0,n^{-1}\Delta_1]$; if $k_2'(\theta_4)>0$,
$\left[\hat{k}_2(\theta_4)-k_2(\theta_4)\right]\to E[-n^{-1}\Delta_1,n^{-1}\Delta_1]$; if $k_2'(\theta_4)>0$,
$\left[\tilde{k}_2(\theta_4)-k_2(\theta_4)\right]\to E[0,-n^{-1}\Delta_1]$; if $k_2'(\theta_4)<0$,
$\left[\hat{k}_2(\theta_4)-k_2(\theta_4)\right]\to E[-n^{-1}\Delta_1,n^{-1}\Delta_1]$; if $k_2'(\theta_4)<0$,
$\left[\tilde{k}_1(\theta_3)-k_1(\theta_3)\right]\to E[0,n^{-1}\Delta_2]$; if $k_1'(\theta_3)>0$,
$\left[\hat{k}_1(\theta_3)-k_1(\theta_3)\right]\to E[-n^{-1}\Delta_2,n^{-1}\Delta_2]$; if $k_1'(\theta_3)>0$,
$\left[\tilde{k}_1(\theta_3)-k_1(\theta_3)\right]\to E[0,n^{-1}\Delta_2]$; if $k_1'(\theta_3)<0$,
$\left[\hat{k}_1(\theta_3)-k_1(\theta_3)\right]\to E[n^{-1}\Delta_2,-n^{-1}\Delta_2]$; if $k_1'(\theta_3)<0$,

$\Delta_1=[f_3(\theta_4;\theta_4)]^{-1}\frac{\partial}{\partial\theta_4}k_2(\theta_4)$ and

$\Delta_2=[f_2(\theta_3;\theta_3)]^{-1}\frac{\partial}{\partial\theta_3}k_1(\theta_3)$ and agreed to asymptotic distributions derived in Bhatt and Pate[9].

Corollary 2.2: In the set up of theorem 2.1 if $g(\theta_1,\theta_2,\theta_3)=k(\theta_1,\theta_2)$ or in the set up of theorem 2.2 if $g(\theta_1,\theta_2,\theta_4)=k(\theta_1,\theta_2)$ function of θ_1 and $\theta_2$only then $\tilde{k}(\theta_1,\theta_2)=k(X_{1:n},X_{n:n})$ is MLE of $k(\theta_1,\theta_2)$ whereas the UMVU estimator of $k(\theta_1,\theta_2)$ obtained by Bar-Lev and Boukai[2] is

$$\hat{k}(\theta_1,\theta_2)=k(X_{1:n},X_{n:n})+\frac{\frac{\partial}{\partial X_{n:n}}k(X_{1:n},X_{n:n})}{(n-1)f(X_{n:n};X_{1:n},X_{n:n})}-\frac{\frac{\partial}{\partial X_{1:n}}k(X_{1:n},X_{n:n})}{(n-1)f(X_{1:n};X_{1:n},X_{n:n})}-\frac{\frac{\partial^2}{\partial X_{1:n}\partial X_{n:n}}k(X_{1:n},X_{n:n})}{n(n-1)f(X_{1:n};X_{1:n},X_{n:n})f(X_{n:n};X_{1:n},X_{n:n})}.$$

For two-truncation distribution, the asymptotic distribution of random variable

$V=nf_1(\theta_1;\theta_1,\theta_2)f_1(\theta_2;\theta_1,\theta_2)[k(X_{1:n},X_{n:n})-k(\theta_1,\theta_2)]\to P[0,\lambda_1,\lambda_2]$ as special cases and agreed to that of asymptotic distributions derived by Bhatt and Pate[10].

ARE and LRE of MLE and the UMVU Estimator

One sample In the set up of theorem 2.1 the UMVU estimator of $g=g(\theta_1,\theta_2,\theta_3)$ obtained by is given by

$$\hat{g}(\theta_1,\theta_2,\theta_3)=g(X_{1:n},X_{n:n},Y_{n:n})-\frac{\frac{\partial}{\partial X_{1:n}}g(X_{1:n},X_{n:n},Y_{n:n})}{(n-1)f_1(X_{1:n};X_{1:n},X_{n:n})}+\frac{\frac{\partial}{\partial X_{n:n}}g(X_{1:n},X_{n:n},Y_{n:n})}{(n-1)f_1(X_{n:n};X_{1:n},X_{n:n})}+\frac{\frac{\partial}{\partial Y_{n:n}}g(X_{1:n},X_{n:n},Y_{n:n})}{nf_2(Y_{n:n};Y_{n:n})}-\frac{\frac{\partial^2}{\partial X_{1:n}\partial X_{n:n}}g(X_{1:n},X_{n:n},Y_{n:n})}{n(n-1)f_1(X_{1:n};X_{1:n},X_{n:n})f_1(X_{n:n};X_{1:n},X_{n:n})}-\frac{\frac{\partial^2}{\partial Y_{n:n}\partial X_{1:n}}g(X_{1:n},X_{n:n},Y_{n:n})}{n(n-1)f_1(X_{1:n};X_{1:n},X_{n:n})f_2(Y_{n:n};Y_{n:n})}+\frac{\frac{\partial^2}{\partial Y_{n:n}\partial X_{n:n}}g(X_{1:n},X_{n:n},Y_{n:n})}{n(n-1)f_1(X_{n:n};X_{1:n},X_{n:n})f_2(Y_{n:n};Y_{n:n})}-\frac{\frac{\partial^3}{\partial X_{1:n}\partial X_{n:n}\partial Y_{n:n}}g(X_{1:n},X_{n:n},Y_{n:n})}{n^2(n-1)f_1(X_{1:n};X_{1:n},X_{n:n})f_1(X_{n:n};X_{1:n},X_{n:n})f_2(Y_{n:n};Y_{n:n})}. \quad (20)$$

In order to compare the performance of MLE and the UMVU estimator, the following theorems have been established.

Theorem 3.1: In the set up of theorem 2.1, if $\tilde{g} = \tilde{g}(\theta_1, \theta_2, \theta_3) = g(X_{1:n}, X_{n:n}, Y_{n:n})$ is MLE estimator and $\hat{g} = \hat{g}(\theta_1, \theta_2, \theta_3)$ is the UMVU estimator of U-estimable parametric function $g = g(\theta_1, \theta_2, \theta_3)$ then at the order $o(n^{-3})$ the LRE of $\hat{g}$ relative to $\tilde{g}$ is given by

$$LRE[\hat{g}, \tilde{g}] = 2\left[1 + \frac{\beta\gamma - \alpha\beta - \alpha\gamma}{\alpha^2 + \beta^2 + \gamma^2}\right] \quad (21)$$

where

$$\alpha = \frac{\frac{\partial}{\partial\theta_1} g(\theta_1, \theta_2, \theta_3)}{n f_1(\theta_1; \theta_1, \theta_2)}, \quad (22)$$

$$\beta = \frac{\frac{\partial}{\partial\theta_2} g(\theta_1, \theta_2, \theta_3)}{n f_1(\theta_2; \theta_1, \theta_2)}, \quad (23)$$

$$\gamma = \frac{\frac{\partial}{\partial\theta_3} g(\theta_1, \theta_2, \theta_3)}{n f_2(\theta_3; \theta_3)} \quad (24)$$

Proof: Using theorem 2.1 at the order $o(n^{-1})$ the asymptotic distribution of $[\tilde{g} - g]$ is

$$[\tilde{g} - g] \to P[0, \alpha, -\beta, -\gamma], \quad (25)$$

with bias $B[.]$ as

$$B[\tilde{g}] = \beta + \gamma - \alpha, \quad (26)$$

at the order $o(n^{-2})$ and MSE as

$$MSE[\tilde{g}] = 2[\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma - \alpha\beta - \alpha\gamma] \quad (27)$$

at the order $o(n^{-3})$.

Taking $G(X_{1:n}, X_{n:n}, Y_{n:n}) = \hat{g}(\theta_1, \theta_2, \theta_3)$ given in (20) then U will be

$$U = [G(X_{1:n}, X_{n:n}, Y_{n:n}) - G(\theta_1, \theta_2, \theta_3)] = \lambda_1(G)V_2 + \lambda_2(G)V_4 + \lambda_4(G)V_4$$

which reduces to

$$Z = [\hat{g}(\theta_1, \theta_2, \theta_3) - g(\theta_1, \theta_2, \theta_3)] = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_4 V_4 \quad (28)$$

at the order $o(n^{-2})$ and at the order $o(n^{-1})$ and the asymptotic distribution of $[\hat{g} - g]$ is

$$[\hat{g} - g] \to P[\beta + \gamma - \alpha,\ \alpha,\ -\beta,\ -\gamma] \quad (29)$$

with variance of $\hat{g}$ as

$$V[\hat{g}] = \alpha^2 + \beta^2 + \gamma^2 \quad (30)$$

at the order $o(n^{-3})$. Straight forward computation leads to the theorem 3.1.

Theorem 3.2: In the set up of theorem 2.2, if $\tilde{g} = \tilde{g}(\theta_1, \theta_2, \theta_4) = g(X_{1:n}, X_{n:n}, Z_{1:n})$ is MLE and $\hat{g} = \hat{g}(\theta_1, \theta_2, \theta_4)$ is the UMVU estimator of U-estimable parametric function $g = g(\theta_1, \theta_2, \theta_4)$ then at the order $o(n^{-3})$ the LRE of $\hat{g}$ relative to $\tilde{g}$ is given by

$$LRE[\hat{g}, \tilde{g}] = 2\left[1 + \frac{\alpha\zeta - \alpha\beta - \beta\zeta}{\alpha^2 + \beta^2 + \zeta^2}\right], \quad (31)$$

where

$$\zeta = \frac{\frac{\partial}{\partial\theta_4} g(\theta_1, \theta_2, \theta_4)}{f_3(\theta_4; \theta_4)} \quad (32)$$

ARE of the UMVU estimator with respect MLE does not exist.

Example 3.1: Consider uniform distribution on interval (θ_1, θ_2) and on interval $(0, \theta_3)$ with pdf as

$$f_1(x; \theta_2) = \frac{1}{\theta_2 - \theta_1}; \theta_1 < x < \theta_2 \quad (33)$$

$$f_2(x; \theta_3) = \frac{1}{\theta_3}; 0 < y < \theta_3 \quad (34)$$

for X's and Y's respectively. Let $g = g(\theta_1, \theta_2, \theta_3) = e^{-\theta_1\theta_2\theta_3}$ then using section 3

$$[\tilde{g} - g] \to P\left[0, -\frac{\theta_2\theta_3(\theta_2 - \theta_1)g}{n}, \frac{\theta_1\theta_3(\theta_2 - \theta_1)g}{n}, \frac{g\theta_1\theta_2\theta_3}{n}\right],$$

$$[\hat{g} - g] \to P\left[\frac{g\theta_1\theta_2\theta_3[\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2 + 1]}{n}, -\frac{\theta_2\theta_3(\theta_2 - \theta_1)g}{n}, \frac{\theta_1\theta_3(\theta_2 - \theta_1)g}{n}, \frac{g\theta_1\theta_2\theta_3}{n}\right].$$

.

MSE stabilized transformation

The asymptotic inference regarding the parameters $(\theta_1, \theta_2, \theta_3)$ and $(\theta_1, \theta_2, \theta_4)$ will require the asymptotic distributions of $(X_{1:n} - \theta_1, \theta_2 - X_{n:n}, \theta_3 - Y_{n:n})$ and $(X_{1:n} - \theta_1, \theta_2 - X_{n:n}, Y_{1:n} - \theta_4)$ respectively. In this section attempt have been made to resolve dependency of MSE on parameters by using suitable MSE stabilization transformation.

Theorem 4.1: In the set up of theorem 2.1 if $g(\theta_1, \theta_2, \theta_3)$ is MSE stabilized transformation then

$$g(\theta_1, \theta_2, \theta_3) = \pm(\log[q_1(\theta_1, \theta_2)] \pm \log[q_2(\theta_3)])$$

Proof: After obtaining solution of

$$\frac{\partial}{\partial\theta_1} g(\theta_1, \theta_2, \theta_3) = \pm f_1(\theta_1; \theta_1, \theta_2),$$

$$\frac{\partial}{\partial\theta_2} g(\theta_1, \theta_2, \theta_3) = \pm f_1(\theta_2; \theta_1, \theta_2), \text{ and}$$

$\frac{\partial}{\partial\theta_3} g(\theta_1, \theta_2, \theta_3) = \pm f_2(\theta_3; \theta_3)$ and using theorems 2.1 theorems 4.1 follows. Analogous arguments lead to following theorem 4.2.

Theorem 4.2: In the set up of theorem 2.2 if $g(\theta_1, \theta_2, \theta_4)$ is MSE stabilized transformation then

$$g(\theta_1, \theta_2, \theta_4) = \pm(\log[q_1(\theta_1, \theta_2)] + \log[q_3(\theta_4)]).$$

Asymptotic MSE and variance of $P[Y > X]$

Statistical inference concerning the reliability functional $P[Y > X]$ has been of great interest to research in reliability theory and survival analysis. Asymptotic MSE and variance of biased estimator, MLE and the UMVU estimator of reliability functional has been obtained and compared their performance in terms of LRE. In the set up of theorem 2.1, let

$$g = g(\theta_1, \theta_2, \theta_3) = P_{(\theta_1,\theta_2,\theta_3)}[Y > X]$$
$$= \int_{\theta_1}^{min(\theta_2,\theta_3)} P_{\theta_3}(Y > x)\, f_1(X; \theta_1, \theta_2) dx$$
$$= \begin{cases} q_1(\theta_1,\theta_2) q_2(\theta_3) \int_{\theta_1}^{\theta_3} h_1(x) \left[\int_x^{\theta_3} h_2(u)\right] du; \theta_1 < \theta_3 < \theta_2 \\ q_1(\theta_1,\theta_2) q_2(\theta_3) \int_{\theta_1}^{\theta_2} h_1(x) \left[\int_x^{\theta_3} h_2(u)\right] du; \theta_1 < \theta_2 < \theta_3 \end{cases}$$

and let $\tilde{g}$ and $\hat{g}$ be the MLE and the UMVU estimator of g. Using Theorem 3.1 the asymptotic distribution of $[\tilde{g} - g]$ is

$$[\tilde{g} - g] \to \begin{cases} P\left[0, \frac{g + F_{21} - 1}{n}, \frac{g}{n}, -\frac{F_{13} - g}{n}\right]; \theta_1 < \theta_3 < \theta_2 \\ -\frac{g + F_{22} - 1}{n} \\ P\left[0, \frac{g + F_{21} - 1}{n}, -\frac{1 - g - F_{22}}{n}, -\frac{1-g}{n}\right]; \theta_1 < \theta_2 < \theta_3, \end{cases}$$

at the order $o(n^{-1})$ with bias, $B[.]$ as

$$B[\tilde{g}] = \begin{cases} \frac{1 + F_{13} - F_{21} - 3g}{n}; \theta_1 < \theta_3 < \theta_2 \\ \frac{3(1-g) - F_{21} - F_{22}}{n}; \theta_1 < \theta_2 < \theta_3, \end{cases}$$

at the order $o(n^{-2})$ and MSE as

$$MSE[\tilde{g}] = \begin{cases} 2\left[\frac{1+6g^2+F_{13}^2+F_{21}(4g+F_{21}-2)-F_{13}(4g+F_{21}-1)-4g}{n^2}\right]; \theta_1 < \theta_3 < \theta_2 \\ 2\left[\frac{6(g-1)^2+F_{21}^2+(F_{21}+F_{22})(4g+F_{22}-4)}{n^2}\right]; \theta_1 < \theta_2 < \theta_3 \end{cases}$$

at the order $o(n^{-3})$ and at the order $o(n^{-1})$ the asymptotic distribution of $[\hat{g} - g]$ is

$$[\hat{g} - g] \to \begin{cases} P\left[\frac{1 + F_{13} - F_{21} - 3g}{n}, \frac{g + F_{21} - 1}{n}, \frac{g}{n}, -\frac{F_{13} - g}{n}\right]; \theta_1 < \theta_3 < \theta_2 \\ P\left[\frac{3 - 3g - F_{21} - F_{22}}{n}, \frac{g + F_{21} - 1}{n}, -\frac{1-g-F_{22}}{n}, -\frac{1-g}{n}\right]; \theta_1 < \theta_2 < \theta_3, \end{cases}$$

with asymptotic variance as

$$V[\hat{g}] = \begin{cases} \frac{g^2+(g-F_{13})^2+(g+F_{21}-1)^2}{n^2}; \theta_1 < \theta_3 < \theta_2 \\ \frac{3(g-1)^2+2(g-1)F_{21}+F_{21}^2+F_{22}(2g+F_{22}-2)}{n^2}; \theta_1 < \theta_2 < \theta_3 \end{cases}$$

at the order $o(n^{-3})$, where $F_{21} = F_2(\theta_1; \theta_3)$, $F_{22} = F_2(\theta_2; \theta_3)$ and $F_{13} = F_1(\theta_3; \theta_1, \theta_2)$ are distribution function of Y and X at θ_1, θ_2 and θ_3 respectively. Hence at the order $o(n^{-3})$ asymptotic LRE of $\hat{g}$ relative to $\tilde{g}$ is

$$LRE[\hat{g}] = \begin{cases} 2\left[1 + \frac{g(3g+2F_{21}-2)-F_{13}(2g+F_{21}-1)}{g^2+(g-F_{13})^2+(g+F_{21}-1)^2}\right]; \theta_1 < \theta_3 < \theta_2 \\ 2\left[1 + \frac{F_{21}(2g+F_{22}-2)+(g-1)(3g+2F_{22}-3)}{3(g-1)^2+2(g-1)F_{21}+F_{21}^2+F_{22}(2g+F_{22}-2)}\right]; \theta_1 < \theta_2 < \theta_3 \end{cases}$$

Example 5.1: Consider uniform distribution on interval (θ_1, θ_2) and uniform distribution on interval $(0, \theta_3)$ with probability density function (pdf) given in (33) and (34) and suppose

$$g = g(\theta_2, \theta_4) = P_{\theta_2,\theta_4}[Y > X]$$
$$= \begin{cases} \frac{(\theta_3-\theta_1)^2}{2(\theta_2-\theta_1)\theta_3}; \theta_1 < \theta_3 < \theta_2 \\ 1 - \frac{\theta_1 + \theta_2}{2\theta_3}; \theta_1 < \theta_2 < \theta_3. \end{cases}$$

Then using the results of section 6 following results are obtainable.

$$[\tilde{g} - g] \to \begin{cases} P\left[0, \frac{(\theta_1-\theta_3)(\theta_1-2\theta_2+\theta_3)}{2n(\theta_1-\theta_2)\theta_3}, -\frac{(\theta_1-\theta_3)^2}{2n(\theta_1-\theta_2)\theta_3}, -\frac{\theta_1^2-\theta_3^2}{2n\theta_3(\theta_1-\theta_2)}\right]; \theta_1 < \theta_3 < \theta_2 \\ P\left[0, \frac{\theta_1-\theta_2}{2n\theta_3}, -\frac{\theta_1-\theta_2}{2n\theta_3}, -\frac{\theta_1+\theta_2}{2n\theta_3}\right]; \theta_1 < \theta_2 < \theta_3, \end{cases}$$

$$B[\tilde{g}] = B[\tilde{g}] = \begin{cases} \frac{(\theta_1-\theta_3)(\theta_1+2\theta_2-\theta_3)}{2n(\theta_1-\theta_2)\theta_3}; \theta_1 < \theta_3 < \theta_2 \\ \frac{\theta_1+\theta_2}{2n\theta_3}; \theta_1 < \theta_2 < \theta_3 \end{cases}$$

$$MSE[\tilde{g}] = \begin{cases} \left[\frac{(\theta_1-\theta_3)^2(\theta_1^2+2\theta_2^2+\theta_3^2-2\theta_2\theta_3)}{n^2\theta_3^2(\theta_1-\theta_2)^2}\right]; \theta_1 < \theta_3 < \theta_2 \\ \frac{\theta_1^2+\theta_2^2}{n^2\theta_3^2}; \theta_1 < \theta_2 < \theta_3 \end{cases}$$

$$[\hat{g} - g] \to \begin{cases} P\left[\frac{(\theta_1-\theta_3)(\theta_1+2\theta_2-\theta_3)}{2n(\theta_1-\theta_2)\theta_3}, \frac{(\theta_1-\theta_3)(\theta_1-2\theta_2+\theta_3)}{2n(\theta_1-\theta_2)\theta_3}, -\frac{(\theta_1-\theta_3)^2}{2n(\theta_1-\theta_2)\theta_3}, -\frac{\theta_1^2-\theta_3^2}{2n\theta_3(\theta_1-\theta_2)}\right]; \theta_1 < \theta_3 < \theta_2 \\ P\left[\frac{\theta_1+\theta_2}{2n\theta_3}, \frac{\theta_1-\theta_2}{2n\theta_3}, -\frac{\theta_1-\theta_2}{2n\theta_3}, -\frac{\theta_1+\theta_2}{2n\theta_3}\right]; \theta_1 < \theta_2 < \theta_3, \end{cases},$$

$$V[\hat{g}] = \begin{cases} \frac{(\theta_1-\theta_3)^2[3\theta_1^2+4\theta_2^2+3\theta_3^2-4\theta_2\theta_3+2\theta_1(\theta_3-2\theta_2)]}{4n^2\theta_3^2(\theta_1-\theta_2)^2}; \theta_1 < \theta_3 < \theta_2 \\ \frac{3\theta_1^2-2\theta_1\theta_2+3\theta_2^2}{4n^2\theta_3^2}; \theta_1 < \theta_2 < \theta_3 \end{cases}$$

$$LRE[\hat{g}, \tilde{g}] = \begin{cases} 2\left[1 - \frac{\theta_1^2+\theta_3^2+2\theta_1(\theta_3-2\theta_2)}{3\theta_1^2+4\theta_2^2+3\theta_3^2-4\theta_2\theta_3+2\theta_1(\theta_3-2\theta_2)}\right]; \theta_1 < \theta_3 < \theta_2 \\ 2\left[1 - \frac{(\theta_1-\theta_2)^2}{3\theta_1^2-2\theta_1\theta_2+3\theta_2^2}\right]; \theta_1 < \theta_2 < \theta_3 \end{cases}$$

Conclusion

Asymptotic theory develop for one and two-truncation parameter family of distributions virtue of which asymptotic variance of the estimator of U-estimable parametric function or MSE in the case of biased estimator derived (generally not available) and performance compared in terms of LRE.

Abbreviations: MSE; Mean Square Error, UMVU; Uniformly Minimum Variance Unbiased, MLE; Maximum Likelihood Estimator, LRE; Limiting Risk Efficiency, ARE; Asymptotic Relative Efficiency.

References

1. Tate, R. F. (1959). Unbiased estimation; function of location and scale parameters. *Ann. Math. Statist*., 30, 341-366.

2. Bar-Lev and Boukai (1985). Minimum variance unbiased estimation for families of distributions involving two-truncation parameters. *J. Statistical Planning and Inference*, 12, 379-384.

3. Widder. D. (1961). Advanced Calculus. Prentice-Hall, Englewood cliffs, N.J.

4. Guenther, W. C. (1978). Some easily found minimum variance unbiased estimators. *American Statistician* (Amer. Statist.), 32, 29-34.

5. Krishnamoorthy, K., Rohatgi, V. K. and Blass, J. (1989). Unbiased estimation in Type- II censored samples from one-truncation density. *Communication in Statistics Theory and methods*, 18(3), 1023-1030.

6. Patel, S. R. and Bhatt, M. B., (1990). Unbiased estimation in Type II censored samples from two-truncation parameter families. *Communication in Statistics Theory and methods*, 19(7), 3809-3832.

7. Ferentinos, K. K. (1990). Minimum variance unbiased estimation in doubly type - II censored samples from families of distributions involving two-truncation parameters. *Communication in Statistics Theory and methods*, 19, 847-856.

8. Krishnamoorthy, K. and Rohatgi, Vijay. K. (1988). Minimum variance unbiased estimator in some nonregular families. *Communication in Statistics Theory and methods*, 17(11), 3757-3765.

9. Bhatt, M. B. and Pate, S.R. (1999). Asymptotic distributions for order statistics of truncated one parameter family of distributions. *Communication in Statistics Theory and methods*, 8, 327-345.

10. Bhatt, M. B., and Pate, S.R. (2001) A asymptotic distributions for order statistics of truncated one parameter family of distributions. *Journal of planning and inferecr*, 92(8), 39-46.

11. Rohatgi, V. K. (1989). Unbiased estimation of parametric function in sampling from two one-truncation parameter families. *Austral. J. Statist*., 31(2), 327-332.

12. Selvavel, K., (1989). Unbiased estimation in sampling from two one-truncation parameter families when both samples are type-II censored. *Communication in Statistics Theory and methods*, 18, 3519-3531.

13. Selvavel, K., (1990). Statistical inference for truncated parameter families. Ph. D. Thesis Bowling Green Stat University, U. S. A.

14. Patel, S. R. and Bhatt, M. B., (1991a). Unbiased estimation in doubly Type II censored samples from two two-truncation parameter families. *Communication in Statistics Theory and methods*, 20(7), 2119-2148.

15. Patel, S. R. and Bhatt, M. B. (1991a). Unbiased estimation in Type II censored samples from two- truncation parameter families. *Communication in Statistics Theory and methods*, 19(10), 3808-3832.

16. Patel, S. R. and Bhatt, M. B., (1992a). Unbiased estimation of parametric function in sampling from one- and two-truncation parameter families. *Austral. J. Sttist*, 34(1), 47-54.

17. Beg, M.A. (1980a). Estimation of P(Y<X) for truncation parameters distributions. *Communication in Statistics Theory and methods*, 9, 327-345.

18. Beg, M.A. (1980b). On the Estimation of P(Y<X) for two parameter exponential distribution. *Metrika.*, 27, 29-34.

19. Beg, M.A. (1980c). Estimation of P(Y<X) for exponential family. *Trans. Reliab*., 29, 158-159.

20. Beg, M.A. (1983). Unhiased estimators for truncation and scale parameters. *Amer. J. Math. Mgmt. Sci*., 3, 251-274.

21. Beg, M.A. and Singh, N. (1979). Estimation of P(Y<X) for Pareto distribution. *Trans. Reliab*., 28, 411-414.

22. Dixit, U. J and Phal, K. D. (2009). Estimation of P[Y < X] for the uniform distribution in the Presence of Outliers. *Journal of Probability and Statistical Science*, 7(1), 11-18.

23. Johnson, N.L. (1970). Continuous univariate distributions. No. 04; QA273. 6, J64. John Wiley & Sons Inc.

24. Simonoff, J. S., Hochberg, Y. and Reiser, B. (1986). Alternative estimation producers for P(Y<X) in categorized data. *Biometrics*, 42, 895–907.

25. Samuel, Kotz. S., Yan Lumelskii, Y. and Marianna, Pensky (2002). The Stress-Strength Model and its Generalizations Theory and Applications. World Scientific, London.

26. Bhatt, M. B., Pate, S.R. and Pratik, B. Nandy (2018). Asymptotic test for the parameter of two one truncation parameter families of distributions. *Communication in Statistics*–Simulation and Computation, 47(10).