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A synthetic control chart for generalized exponential distribution

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Abstract

This article proposes the synthetic control chart for the generalized exponential distribution. The generalized exponential distribution has two parameters. A process generates an out-of-control signal when there is a shift in any of the parameter of the generalized exponential distribution. To measure the performance of the proposed synthetic control chart, the popular measures such as average run length, standard deviation of run length, median run length and inter-quartile range are used. The changes in parameters affect the average run length, standard deviation of run length, median run length and inter-quartile range. The performance of the proposed synthetic generalized exponential chart is compared with the chart for monitoring parameters of the generalized exponential distribution. The proposed chart is more efficient than the existing chart in term of the average run length, standard deviation of run length, median run length and inter-quartile range.

Keywords: Control chart, synthetic, parameter, standard deviation of run length and average run length.

Introduction

Time between-events (TBE) control charts were introduced as alternatives for monitoring high quality processes. Instead of monitoring the number or proportion of events occurring in a certain sampling interval, the TBE control charts monitor the time between successive occurrences of events. The TBE control chart was proposed by Calvin[1] and further it is studied by Goh[2] as cumulative count of conforming (CCC) chart. The conforming run length control chart was proposed by Bourke[3]. The observed TBE follows an exponential distribution and the suggested chart is the t-chart or exponential chart or T chart.

This article presents the development of the upper-sided synthetic control chart for monitoring parameters of the generalized exponential (GE) distribution to increase the sensitivity of the control chart based on the GE distribution.
In the literature, Gupta and Kundu[4] proposed the two-parameter GE distribution. Gupta and Kundu[5] developed the GE distribution which is a best alternative to the gamma distribution as well as Weibull distribution. Gupta and Kundu[6] studied the different methods of estimation of the GE distribution. Gupta and Kundu[7] studied the statistical inference of generalized exponential distribution.

In the literature, Chiang et al.[8] suggested the bootstrap control charts for the GE distribution. Khilare and Shirke[9] developed the control charts for monitoring parameters of the GE distribution.

The rest of the paper is organized as: Section 2 gives GE chart. In Section 3 CRL chart is explained. Section 4 describes the synthetic control chart based on the two parameter GE distribution. Section 5 gives performance study of the synthetic GE chart. Conclusions are given in Section 6.

The Generalized Exponential chart

Suppose that the random variable X has the GE distribution. The distribution function of X is as follows:

$$F(x;\alpha,\lambda)=\left(1-e^{-\lambda x}\right)^{\alpha};\ \alpha,\lambda,x>0$$

The density function of X is

$$f(x;\alpha,\lambda)=\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}e^{-\lambda x};\alpha,\lambda,x>0$$

Here, α is the shape parameter and λ is the scale parameter.

The upper control limit (a) of a chart to detect upward shift in the process parameter is given by: $\Pr(X\geq a)=\alpha_1$, $a=\frac{1}{\lambda_0}\log\left(1-\left(1-\alpha_1\right)^{\frac{1}{\alpha_0}}\right)$.

Where α_1 tolerable false alarm rate, α_0 and λ_0 are the in-control shape and scale parameters respectively.

The average run length (ARL) of the GE chart is as follows:

$$ARL(\delta)=\frac{1}{\Pr(X\geq a)}, \tag{1}$$

where δ is the shifts in the shape parameter or a scale parameter. The control chart for monitoring parameters of the GE distribution is the GE chart.

In Section 3, the conforming run length control chart is explained.

Conforming Run Length Chart

The conforming run length (CRL) control chart is an attributes control chart to detect shifts in the fraction nonconforming (p) was first introduced by Bourke[3]. Suppose Y is the number of conforming units observed between two consecutive nonconforming units including ending nonconforming unit and which is the CRL. The random variable $Y \to G(p)$, then the distribution function of Y is as follows

$$F(y) = 1 - (1 - p)^y$$

If $CRL \le L$, the process signals an out-of-control status in upward direction, where L is the lower control limit of the CRL chart. Therefore, to detect an increase in the fraction nonconforming (p), only L of the CRL control chart is enough and it is given by

$$L = \frac{\ln(1-\alpha_Y)}{\ln(1-p_0)},$$

where, α_Y and p_0 are respectively false alarm rate and in-control fraction nonconforming. L must be rounded to a nearest integer.

The average run length of the CRL chart (ARL_{CRL}) is given by

$$ARL_{CRL} = \frac{1}{p * \left(1 - (1-p)^L\right)}.$$

The operations of the synthetic GE chart are outlined below

The Synthetic Generalized Exponential Chart

In the literature to detect small to moderate shifts, Wu and Spedding[10] suggested the synthetic control chart which is the combination of the Shewhart's type $\overline{X}$ chart and CRL chart to detect shifts in the process location. In literature, to monitor high quality processes using synthetic control chart, Yen et al.[11] developed synthetic type control charts for high quality process. Khilare and Shirke[12] proposed the synthetic control chart based on the Weibull distribution. The synthetic chart GE chart is proposed by integrating the operations of the GE chart and CRL chart. The operations of the synthetic GE chart are outlined – i. Determine upper control limit (b) of the GE chart and L. ii. Obtain time between events (X) by taking a sample of n units. iii. If $X \le b$, a sample is a non-defective and move to second step. iv. If $X > b$, a sample is a defective and move to next step. v. Y is the number of samples between the present defective sample and preceding defective sample including present defective sample. vi. If $Y > L$, then the process is said to be under control and move back to second step. vii. If $Y \le L$, the process is an out-of-control state and continue to next step. viii. If out-of-control signal is investigated and no assignable causes founded then the signal is considered as false alarm and then move back to step (2). Otherwise assignable causes must be eliminated.

The ARL of the synthetic GE chart is denoted by $ARL_1(\delta)$ and its formula is as follows

$$ARL_1(\delta) = \frac{1}{P * [1 - (1-P)^L]}, \tag{2}$$

Where δ is a shift in the shape or the scale parameters of the GE distribution and P is a probability of the nonconforming sample. The in-control ARL of the synthetic GE chart is denoted by $ARL_1(\delta_0)$ and its formula is as follows:

$$ARL_1(\delta_0) = \frac{1}{P(\delta_0)\left(1 - (1 - P(\delta_0))^L\right)}, \tag{3}$$

Where,

$$P(\delta_0) = P(X \ge b),$$
$$P(\delta_0) = 1 - P(X < b).$$

The synthetic GE chart is properly designed by solving an optimization problem.
The objective function is to be minimize

$$ARL_1(\delta^*) = \frac{1}{P(\delta^*)\left(1 - (1 - P(\delta^*))^L\right)},$$

subject to the equality constraint

$$ARL_1(\delta_0) = \frac{1}{P(\delta_0)\left(1 - (1 - P(\delta_0))^L\right)},$$

The two optimal parameters (L, b) of the synthetic GE chart to be chosen in such a way that $ARL_1(\delta^*)$ should be minimum with an optimal shift size subject to a specified $ARL_1(\delta_0)$.

The optimal design procedure of the synthetic GE chart is given below: i. Set α_0, λ_0 and $ARL_1(\delta_0)$. ii. Take beginning value of L as 1. iii. Compute 'b' using equation (2). iv. Using equation (2) obtain ARL of the synthetic GE chart ($ARL_1(\delta^*)$) from the current pair of (L, b). v. If ARL of the synthetic GE chart decreased then increase L by one and then go back to step (3). Otherwise, take the current L and 'b' as the final values in the synthetic GE chart.

Table-1: Different sets of values of *L* and *b* for the synthetic GE chart. ($\alpha_0 = 0.5,\ \lambda_0 = 1.5,\ \alpha^* = 6,\ \lambda^* = 0.5$ and $ARL_1(\delta_0) = 350$)

L	UCLs	$ARL_1(\delta^*)$
1	2.40566	1.283
2	2.63292	1.210
3	2.76516	1.221
4	2.85858	1.242
5	2.93077	1.261

From the Table-1 it is observed that the $ARL_1(\delta^*)$ first declines then increases. It reaches its minimum value 1.210 when L= 2 and UCLs = 2.63292. Therefore, the design parameters of the synthetic GE chart are *L* = 2 and *b* = 2.651756.

The following section gives performance study of the proposed synthetic GE chart.

Performance Study of the Synthetic GE Chart

The ARL, standard deviation of run length (SDRL), median run length (MRL) and inter-quartile range (IQR) are used to evaluate the performance of the proposed synthetic GE chart. The ARL is the average number of the TBE sample points required to signal an out-of-control status. However, the run length random variable takes only positive integer values and the shape of this probability distribution is significantly right skewed. If run length distribution is right skewed, ARL alone cannot provide complete information about a chart performance. In this case, the percentiles of the run length distribution provide important information about a control chart performance. Therefore, we compute first quartile (Q_1), second quartile (Q_2= MRL) (MRL is more accurate indicator of a typical control chart performance) and third quartile (Q_3) to assess the performance of the proposed control chart. The changes in parameters affect the ARL, SDRL, MRL and IQR of the control charts.

The change in the shape parameter

To assess the performance of the proposed synthetic GE chart when there is change in the shape parameter, the ARL and SDRL are computed and they are compared with the ARL and SDRL of the GE chart. The ARL and SDRL values of the synthetic GE chart and GE chart are presented in Table-2. SDRL values of both control charts are presented in parenthesis in Table-2.

Table-2: ARL and SDRL values of the GE and the synthetic GE charts for shifts in the shape parameter (α) with the scale parameter fixed at $\lambda_0 = 1.5$

α	GE chart ARL (UCL= 4.36691)	Synthetic GE chart ARL (L=2, UCLs=2.632920)
0.1	6990.50 (6989.00)	132491.97 (132491.47)
0.4	1748.00 (1747.50)	8353.46 (8352.96)
0.7	999.07 (998.57)	2751.57 (2751.07)
1	699.50 (699.00)	1360.07 (1359.57)
1.3	538.19 (537.69)	811.81(811.31)
1.6	437.37 (436.87)	540.60 (540.10)
2	350.00 (349.50)	350.00 (349.50)
4	175.25 (174.75)	92.67 (92.17)
6	117.00 (116.50)	43.59 (43.09)
8	87.88 (87.38)	25.94 (25.44)
10	70.40 (69.90)	17.55 (17.04)
12	58.75 (58.25)	12.87 (12.36)
14	50.43 (49.93)	9.99 (9.48)
16	44.19 (43.69)	8.07 (7.55)
18	39.34 (38.84)	6.72 (6.20)
20	35.45 (34.95)	5.74 (5.22)

From Table-2, it is seen that the ARL and SDRL values of the synthetic GE chart are considerably smaller than the ARL and SDRL values of the GE chart for all increasing shifts in the shape parameter (α). For decreasing shifts in the shape parameter from its in-control value $\alpha_0 = 2$, the performance of both control charts is poor. In this case the synthetic GE chart performs better than the GE chart.

From Table-3 it is observed that the Q_1, Q_2, Q_3, IQR values of the synthetic GE chart are significantly less than the GE chart for all shifts in the shape parameter (α) and which indicates superiority of the synthetic GE chart over the GE chart.

The change in the scale parameter

From Table-4 and Table-5 it is concluded that when there is a decreasing shift in the scale parameter from in-control value $\lambda_0 = 1.5$, ARL, SDRL, MRL and IQR values of the GE chart and the synthetic GE chart declines. For decreasing shifts in the scale parameter the synthetic GE chart performs better than the GE chart. When the scale parameter increases from $\lambda_0 = 1.5$, ARL, SDRL, MRL and IQR values of the GE chart and the synthetic GE chart goes on increasing.

Table-3: Quartiles and inter-quartile range (IQR) of the GE and the synthetic GE charts for shifts in the shape parameter (α) with scale parameter fixed at $\lambda_0 = 1.5$

Shift in α	GE Chart				Synthetic GE Chart			
	Q1	Q2	Q3	IQR	Q1	Q2	Q3	IQR
2	100.54	242.25	484.51	383.96	100.54	242.25	484.51	383.96
4	50.27	121.13	242.25	191.98	26.52	63.89	127.77	101.26
6	33.51	80.75	161.50	127.99	12.40	29.87	59.73	47.34
8	25.14	60.57	121.13	96.00	7.32	17.63	35.26	27.95
10	20.11	48.45	96.90	76.79	4.90	11.81	23.63	18.73
12	16.76	40.37	80.75	63.99	3.56	8.57	17.14	13.58
14	14.36	34.61	69.22	54.85	2.73	6.57	13.14	10.42
16	12.57	30.28	60.56	48.00	2.17	5.24	10.48	8.30
18	11.17	26.92	53.84	42.67	1.79	4.30	8.60	6.82
20	10.05	24.22	48.45	38.39	1.50	3.62	7.24	5.74

Table-4: ARL profile of GE and synthetic GE charts for shifts in scale parameter (λ) with shape parameter fixed at $\alpha_0 = 2$

λ	GE chart UCL= 4.36691	Synthetic GE chart L=2, UCLs=2.632920
0.1	1.14 (0.40)	1.06 (0.25)
0.5	4.70 (4.17)	3.02 (2.47)
0.7	10.89 (10.38)	6.88 (6.36)
0.9	25.71 (25.21)	17.27 (16.76)
1.1	61.23 (60.73)	45.80 (45.30)
1.3	146.28 (145.78)	125.42 (124.92)
1.5	350.00 (349.50)	350.00 (349.50)
1.6	541.52 (541.02)	587.32 (586.82)
1.7	837.90 (837.40)	987.60 (987.10)
1.8	1296.58 (1296.08)	1663.33 (1662.83)
1.9	2006.42 (2005.92)	2804.86 (2804.36)
2	3104.95 (3104.45)	4734.23 (4733.73)

The change in both the shape and scale parameters: To study the ARL and SDRL behavior of the proposed control chart when there is a simultaneous shift either increasing or decreasing in both parameters, the ARL and SDRL values are presented in Table-6. The performance of the proposed synthetic control is superior for upward shifts in the shape parameter and down ward shifts in the scale parameter. It is also remarked that if shifts are of equal magnitude in both the parameters scale and shape, both the control charts are insensitive to detect out-of-control signal.

Table-5: Quartiles and inter-quartile range (IQR) of GE and synthetic GE charts for shifts in the scale parameter (λ) with the shape parameter fixed at $\alpha_0 = 2$.

Shift in λ	GE Chart				Synthetic GE Chart			
	Q1	Q2	Q3	IQR	Q1	Q2	Q3	IQR
0.1	0.14	0.33	0.66	0.52	0.10	0.24	0.48	0.38
0.5	1.20	2.90	5.79	4.59	0.72	1.72	3.45	2.73
0.7	2.99	7.20	14.39	11.41	1.83	4.41	8.83	6.99
0.9	7.25	17.47	34.94	27.69	4.82	11.62	23.24	18.42
1.1	17.47	42.09	84.19	66.72	13.03	31.40	62.80	49.77
1.3	41.94	101.05	202.09	160.16	35.94	86.59	173.17	137.24
1.5	100.54	242.25	484.51	383.96	100.54	242.25	484.51	383.96
1.6	155.64	375.01	750.01	594.37	168.82	406.75	813.51	644.69
1.7	240.90	580.44	1160.88	919.98	283.97	684.21	1368.41	1084.44
1.8	372.86	898.37	1796.75	1423.89	478.37	1152.59	2305.17	1826.81
1.9	577.07	1390.40	2780.80	2203.73	806.76	1943.83	3887.67	3080.90
2	893.09	2151.84	4303.68	3410.59	1361.81	3281.17	6562.34	5200.53

Table-6: ARL and SDRL of the GE and the synthetic GE charts for simultaneous shifts in both the scale parameter (λ) and the shape parameter (α).

α	λ	GE chart ARL (a= 4.36691)	Synthetic GE chart ARL (L=2, b=2.632920)
2	1.5	350.00 (349.50)	350.00 (349.50)
2.5	1	31.82 (31.32)	18.89 (18.38)
3	0.8	11.31(10.80)	5.73 (5.21)
3.5	0.7	6.44 (5.92)	3.15 (2.60)
4	0.6	3.83(3.29)	1.97 (1.38)
4.5	0.5	2.40 (1.83)	1.41 (0.76)
5	0.4	1.62 (1.00)	1.15 (0.42)
5.5	0.3	1.22 (0.52)	1.04 (0.20)

Table-7: Quartiles and IQR of the GE and synthetic GE charts for simultaneous shifts in the scale parameter (λ) and the shape parameter (α).

Shift in α	Shift in λ	GE Chart				Synthetic GE Chart			
		Q1	Q2	Q3	IQR	Q1	Q2	Q3	IQR
2	1.5	100.54	242.25	484.51	383.96	100.54	242.25	484.51	383.96
2.5	1	9.01	21.71	43.42	34.41	5.29	12.74	25.49	20.20
3	0.8	3.11	7.49	14.98	11.87	1.50	3.61	7.23	5.73
3.5	0.7	1.70	4.11	8.22	6.51	0.75	1.81	3.63	2.88
4	0.6	0.95	2.29	4.58	3.63	0.41	0.98	1.96	1.55
4.5	0.5	0.53	1.29	2.57	2.04	0.23	0.56	1.12	0.89
5	0.4	0.30	0.72	1.44	1.14	0.14	0.34	0.68	0.54
5.5	0.3	0.17	0.40	0.81	0.64	0.09	0.21	0.43	0.34

Table-7 gives the quartiles and IQR of the proposed synthetic chart and GE chart and it shows that the synthetic control chart has higher power of detecting out-of-control signal in the process.

Conclusion

In this article the upper-sided synthetic control chart is proposed for monitoring parameters of the GE distribution. The performance of the proposed synthetic GE chart evaluated using ARL, SDRL, MRL and IQR values. These measures revealed that the proposed synthetic control chart for monitoring parameters of the GE distribution perform significantly better than the GE chart. The proposed synthetic GE chart detect early out-of-control signal a process. In general the proposed synthetic GE chart has a higher power of detecting an out-of-control signal. The proposed chart is simple and easy for practitioners.

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