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# A new generalized-G class of distributions and its applications with Dagum distribution

**Noman Rasheed**
Department of Statistics, Punjab College Hasilpur, Pakistan
nomanrasheed163@yahoo.com

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## Abstract

*The primary motive of this research is to familiarize new continuous distribution so- called Topp-Leone-Exponentiated generalized Dagum (TLEGDa) or Topp-Leone-Extended Dagum distribution; this distribution is based on two G classes namely Topp-Leone (TL-G) and Exponentiated generalized (EG-G) but this new class has not been introduced by any author till now. Therefore it is necessary to suggest new class Topp-Leone-Exponentiated generalized G class using the Exponentiated generalized G class as baseline distribution to the type-I Topp-Leone G class, after that Dagum distribution is used as example to new proposed class with applications to practical data sets. In this research we also obtain some basic properties of TLEGDa and compare its superiority to Exponentiated generalized Topp-Leone Dagum distribution (EGTLDa),its sub which are Dagum, Kumaraswamy Dagum, Extended Dagum, Exponentiated Kumaraswamy Dagum and other models in which Type II Topp Leone inverse Rayleigh, Zubair Weibull and Kumarswamy Weibull Poisson are include.*

**Keywords:** Topp-Leone G class, Exponentiated Generalized G class, Exponentiated generalized Topp-Leone, Method of Maximum likelihood, failure rate.

## Introduction

Researchers are very keen to find more flexible and attractive distributions having vast applications to the real life data sets, for this purpose they use G family as baseline distribution into other G class. In this context Handique et al.[1] developed Marshall-Olkin-Kumarswamy-G family of distributions, an extension in Marshall-Olkin family of distributions with Kumarswamy-G class having cdf and pdf

$$F_{MOKwG}(x)=\frac{1-[1-G(x)^{a}]^{b-1}}{1-\bar{\beta}[1-G(x)^{a}]^{b}}, \qquad \bar{\beta}=1-\alpha$$

$$f_{MOKwG}(x)=\frac{\beta abg(x)G(x)^{a-1}[1-G(x)^{a}]^{b-1}}{\left\{1-\bar{\beta}[1-G(x)^{a}]^{b}\right\}^{2}}, -\infty<x<\infty, a,b,\beta>0.$$

Recently, similarly Reyad at al.[2] introduced the transmuted Generalized Odd Generalized exponential –G Family of Distribution, in the same way Jamal et al[3]. developed Marshall-Olkin Odd Lindley-G family of distributions, another example is very import to mention here, same authors Reyad at al[4]. (2019) gave the idea about "The Exponentiated Generalized Topp Leone-G family of Distributions", extending the work on Topp-Leone G (TL-G) family of distributions provided by Al-Shomrani et al[5]. using Exponentiated generalized-G (EG-G) class of distributions suggested by Cordeiro et al[6]., cdf and pdf of TL-G is

$$F_{TL-G}(x,\xi)=[G(x,\xi)]^{\alpha}[2-G(x,\xi)]^{\alpha}=[1-\{1-G(x,\xi)\}^{2}]^{\alpha} \quad \alpha>0, x\in R \tag{1}$$

$$f_{TL-G}(x,\xi)=2\alpha g(x,\xi)\bar{G}(x,\xi)[G(x,\xi)]^{\alpha-1}[2-G(x,\xi)]^{\alpha-1} \tag{2}$$

While $\bar{G}(x,\xi)=1-G(x,\xi)$

Above equation can easily be written as

$$f_{TL-G}(x,\xi)=2\alpha g(x,\xi)[1-G(x,\xi)][1-\{1-G(x,\xi)\}^{2}]^{\alpha-1}$$

Furthermore, Cordeiro et al[6]. introduced EG-G class of distributions with cdf and pdf

$$F_{EG-G}(x,\xi)=\{1-[1-G(x,\xi)]^{a}\}^{b} \quad a,b>0, x\epsilon R \tag{3}$$

And

$$f_{EG-G}(x,\xi)=abg(x,\xi)[1-G(x,\xi)]^{a-1}\{1-[1-G(x,\xi)]^{a}\}^{b-1} \tag{4}$$

Using TL-G model as baseline model Reyad et al.[4] introduced new model as

$$F_{EGTL-G}(x,\xi)=\{1-[1-F_{TL}(x,\xi)]^{a}\}^{b}$$
$$=\{1-[1-[1-\{1-G(x,\xi)\}^{2}]^{\alpha}]^{a}\}^{b}$$

and

$$f_{EGTL-G}(x) = 2ab\alpha g(x,\xi)[1 - F_{TL}(x,\xi)]^{a-1}\{1 - [1 - F_{TL}(x,\xi)]^{a}\}^{b-1}$$

or

$$f_{EGTL-G}(x) = 2ab\alpha g(x,\xi)[1 - [1 - \{1 - G(x,\xi)\}^2]^{\alpha}]^{a-1}\{-[1 - [1 - \{1 - G(x,\xi)\}^2]^{\alpha}]^{a}\}^{b-1}$$

From above mentioned ideas we present a new G class of distributions so-called Topp-Leone-Exponentiated generalized-G or (TLEG-G) class of distributions using EG-G class as baseline distribution of TL-G class.

## Topp-Leone-Exponentiated generalized-G class of distributions

Inserting equation (3) in (1) we get the cumulative distribution function of TLEG-G as

$$F_{TLEG-G}(x,\xi) = [1 - \{1 - \{1 - [1 - G(x,\xi)]^{a}\}^{b}\}^2]^{\alpha} \quad a,\alpha,b > 0 \ , x \epsilon R \qquad (5)$$

Having pdf with additional shaper parameter $\alpha$

$$f_{TLEG-G}(x,\xi) = 2\alpha abg(x,\xi)[1 - G(x,\xi)]^{a-1}\{1 - [1 - G(x,\xi)]^{a}\}^{b-1} \times (1 - \{1 - [1 - G(x,\xi)]^{a}\}^{b})[1 - (1 - \{1 - [1 - G(x,\xi)]^{a}\}^{b})^2]^{\alpha-1} \qquad (6)$$

It's necessary to mention here that both TL-G and EG-G are same with little bit difference, in EG-G class if $a = 2$ then EG-G becomes TL-G class, furthermore setting $a = 2$ we obtain EGTL-G class provided by Reyal at al. through this research we can cover two more G classes namely TLTL-G and EGEG-G class.

**Sub Models:** The TLEG-G class has different sub models when the parametric values changes, here we discuss them i. If $a = 1$ then (5) called Topp-Leone generalized –G class of distributions with cdf $[1 - \{1 - G(x,\xi)^{b}\}^2]^{\alpha}$. ii. If $b = 1$ then (5) called Topp-Leone G class of distributions with additional shape parameter "$a$" cdf is $[1 - \{1 - G(x,\xi)\}^{2a}]^{\alpha}$. iii. If $a = b = 1$ then (5) reduces to Topp-Leone G class of distribution.

## Some useful Expansions

Probability density function of TLEG-G model expand by different binomial expansions as

$$[1 - (1 - \{1 - [1 - G(x,\xi)]^{a}\}^{b})^2]^{\alpha-1} = \sum_{j=0}^{\infty} (-1)^{f} \binom{\alpha-1}{f} (1 - \{1 - [1 - G(x,\xi)]^{a}\}^{b})^{2f}$$

Putting this relation into (6) and we obtain

$$(1 - \{1 - [1 - G(x,\xi)]^{a}\}^{b})^{2f+1} = \sum_{k=0}^{\infty} \binom{2f+1}{g} \{1 - [1 - G(x,\xi)]^{a}\}^{bg}$$

Reversing above equation in (6) and again we obtain

$$\{1 - [1 - G(x,\xi)]^{a}\}^{b(kg+1)-1} = \sum_{l=0}^{\infty} \binom{b(g+1)-1}{h} [1 - G(x,\xi)]^{ah}$$

Again procedure and we obtain the final binomial expansion

$$[1 - G(x,\xi)]^{a(h+1)-1} = \sum_{m=0}^{\{a(l+1)-1\}} \binom{\{a(h+1)-1\}}{i} G(x,\xi)^{i}$$

Finally equation (6) takes form

$$f_{TLEG-G}(x,\xi) = 2\alpha ab \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} (-1)^{f+g+h+i} \binom{\alpha-1}{f} \binom{2f+1}{k} \binom{\{b(g+1)-1\}}{h} \times \binom{\{a(h+1)-1\}}{i} g(x,\xi)G(x,\xi)^{i}$$

Or

$$f_{TLEG-G}(x,\xi) = 2\alpha ab \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, g(x,\xi)G(x,\xi)^{i} \qquad (7)$$

Because

$$W_{f,g,h,i} = (-1)^{f+g+h+i} \binom{\alpha-1}{f} \binom{2f+1}{k} \binom{\{b(g+1)-1\}}{h} \binom{\{a(h+1)-1\}}{i}$$

**Quantile Function:** A useful function so-called quantile function used to obtain random number generation that is also used for simulation, a function obtained by inverting the equation (5)

$$Q(x) = G(x,\xi)^{-1} \left[ 1 - \left\{ 1 - \left( 1 - \sqrt{1 - u^{\frac{1}{\alpha}}} \right)^{\frac{1}{b}} \right\}^{\frac{1}{a}} \right]$$

Where: $u \sim uniform(0,1)$

From $Q(x)$ one can obtain Median, Quartiles which are used to find skewness and kurtosis.

**r^th and Incomplete r^th moments :** for random variable $x$ the rth moments of the TLEG-G class of distribution can be obtained as

$$\bar{\mu_r} = \int x^{r} f(x) dx$$

$$\mu_r^{/} = 2\alpha ab \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h} \int_{-\infty}^{\infty} x^{r} g(x,\xi)G(x,\xi)^{i}\, dx$$

From above equation its nothing but the probability weighted moments of the base line distribution in other words the $r^{th}$ moments can be written as

$$\mu_r^{/} = 2\alpha ab \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,i}\, \beta_{r,i}$$

Similarly mean of the distribution is

$$\mu_1^{/} = 2\alpha ab \sum_{f,g,h=0}^{\infty} \sum_{m=0}^{a(h+1)-1} w_{f,g,h}\ \beta_{1,i} \quad (8)$$

In the same way incomplete $r^{th}$ moments of the TLEG-G class of distributions can be obtained as

$$m_r(x) = 2\alpha ab \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h} \int_{-\infty}^{x} x^r g(x,\xi) G(x,\xi)^i\, dx$$

Above equation can be written as because $\int_{-\infty}^{x} x^r g(x,\xi) G(x,\xi)^i\, dx = \phi_r(x)$

$$m_r(x) = 2\alpha ab \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h}\ \phi_r(x)$$

and $1^{st}$ incomplete $r^{th}$ moments is

$$m_1(x) = 2\alpha ab \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, \phi_1(x) \quad (9)$$

This incomplete moment is used to obtain Lorenz curve, Mean waiting time etc.

**Some Reliability Properties:** Now we drive some expressions of basic reliability properties of the distribution

Reliability or survival function:

$$R_{TLEG-G}(x) = 1 - F_{TLEG-G}(x,\xi) = 1 - [1 - (1 - \{1 - [1 - G(x,\xi)]^a\}^b)^2]^\alpha$$

Failure Rate $\phi(x) = \frac{f(x)}{S(x)}$

$$\phi_{TLEG-G}(x) = \frac{2\alpha abg(x,\xi)[1 - G(x,\xi)]^{a-1}\{1 - [1 - G(x,\xi)]^a\}^{b-1} \times (1 - \{1 - [1 - G(x,\xi)]^a\}^b)[1 - (1 - \{1 - [1 - G(x,\xi)]^a\}^b)^2]^{\alpha-1}}{1 - [1 - (1 - \{1 - [1 - G(x,\xi)]^a\}^b)^2]^\alpha}$$

Cumulative Hazard function $\omega(x) = -\ln S(x)$

$$\omega_{TLEG-G}(x) = -\ln(1 - [1 - (1 - \{1 - [1 - G(x,\xi)]^a\}^b)^2]^\alpha)$$

Reverse Hazard function $\varphi(x) = \frac{f(x)}{F(x)}$

$$\varphi_{TLEG-G}(x) = \frac{2\alpha abg(x,\xi)[1 - G(x,\xi)]^{a-1}\{1 - [1 - G(x,\xi)]^a\}^{b-1}(1 - \{1 - [1 - G(x,\xi)]^a\}^b)}{[1 - (1 - \{1 - [1 - G(x,\xi)]^a\}^b)^2]}$$

Odd function $O(x) = \frac{F(x)}{S(x)}$

$$O_{TLEG-G}(x) = \frac{[1 - (1 - \{1 - [1 - G(x,\xi)]^a\}^b)^2]^\alpha}{1 - [1 - (1 - \{1 - [1 - G(x,\xi)]^a\}^b)^2]^\alpha}$$

Mean Waiting Time $\psi(x) = x - \left\{\frac{1}{F(x)} \int_0^x x\, f(x) dx\right\}$

from equation (5) and (9) one can easily obtained Mean waiting time of the TLEG-G as

$$\psi_{TLEG-G}(x) = x - \left\{\frac{m_1(x)}{F_{TLEG-G}(x)}\right\}$$

**Lorenz Curve and Bonferroni Index[7]:** It is important to note here useful income inequality measures called Lorenz curve $L(x)$ and Bonferroni index $\left(B(x)\right)$

$$L(x) = \frac{\int_0^x xf(x)dx}{\mu}$$

This is a ratio of $1^{st}$ incomplete $r^{th}$ moment and the mean of the distribution, for TLEG-G distribution one can obtain by using (8) and (9)

$$L_{TLEG-G}(x) = \frac{m_1(x)}{\mu_1^{/}}$$

Similarly Bonferroni index is

$$B(x) = \frac{L(x)}{F(x)}$$

And for TLEG-G class of distributions it will be

$$B_{TLEG-G}(x) = \frac{m_1(x)}{\mu_1^{/}\{F_{TLEG-G}(x)\}}$$

## Parameter Estimation

Suppose we have $x_1, x_2, x_3, \ldots, x_n$ values of random sample drawn from the TLEG-G and then the log-likelihood function $L(x,\xi) = \log \prod_{i=1}^{n} f_{TLEG-G}(x,\xi)$ for estimating the parameter by maximum likelihood is

$$L(x,\xi) = n\log(2) + n\log(\alpha) + n\log(a) + n\log(b) + \sum_{i=1}^{n} \log g(x,\xi)$$

$$+(a-1)\sum_{i=1}^{n} \log[1 - G(x,\xi)] + (b-1)\sum_{i=1}^{n} \log\{1 - [1 - G(x,\xi)]^a\}$$

$$+\sum_{i=1}^{n}\log(1-\{1-[1-G(x,\xi)]^{a}\}^{b})+(\alpha-1)\sum_{i=1}^{n}\log[1-(1-\{1-[1-G(x,\xi)]^{a}\}^{b})^{2}]$$

To estimate the parameter space $\xi$ of TLEG-G we differentiae the function $L(x,\xi)$ w.r.t parameter to be estimated, assuming that $\tau=[1-G(x,\xi)]$

$$\frac{\partial L(x,\xi)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log[1-(1-\{1-\tau^{a}\}^{b})^{2}]$$

$$\frac{\partial L(x,\xi)}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log\tau-(b-1)\sum_{i=1}^{n}\frac{\tau^{a}\log\tau}{\{1-\tau^{a}\}}-\sum_{i=1}^{n}\frac{\{1-\tau^{a}\}^{b}\log\{1-\tau^{a}\}}{[1-\{1-\tau^{a}\}^{b}]}$$

$$-2b(\alpha-1)\sum_{i=1}^{n}\frac{(1-\{1-\tau^{a}\})\{1-\tau^{a}\}^{b-1}\tau^{a}\log\tau}{[1-(1-\{1-\tau^{a}\}^{b})^{2}]}$$

$$\frac{\partial L(x,\xi)}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\{1-\tau^{a}\}+\sum_{i=1}^{n}\frac{\{1-\tau^{a}\}^{b}\log\{1-\tau^{a}\}}{[1-\{1-\tau^{a}\}^{b}]}$$

$$+2(\alpha-1)\sum_{i=1}^{n}\frac{(1-\{1-\tau^{a}\}^{b})\{1-\tau^{a}\}^{b}\log\{1-\tau^{a}\}}{[1-(1-\{1-\tau^{a}\}^{b})^{2}]}$$

$$\frac{\partial L(x,\xi)}{\partial\xi}=\sum_{i=1}^{n}\frac{\partial/\partial\xi\; g(x,\xi)}{g(x,\xi)}-(a-1)\sum_{i=1}^{n}\frac{\partial/\partial\xi\; G(x,\xi)}{\tau}-(b-1)\sum_{i=1}^{n}\frac{a\tau^{a-1}\,\partial/\partial\xi\; G(x,\xi)}{\{1-\tau^{a}\}}$$

$$-ab\sum_{i=1}^{n}\frac{\{1-\tau^{a}\}^{b-1}\tau^{a}\;\partial/\partial\xi\; G(x,\xi)}{[1-\{1-\tau^{a}\}^{b}]}+2ab\sum_{i=1}^{n}\frac{(1-\{1-\tau^{a}\}^{b})\{1-\tau^{a}\}^{b-1}\tau^{a-1}\;\partial/\partial\xi\; G(x,\xi)}{[1-(1-\{1-\tau^{a}\}^{b})^{2}]}$$

After differentiating these equations, we put them equal to zero and then solve one by one but these are difficult to solve for this purpose different software are available like R and SAS etc with different algorithms.

**Topp-Leone-Exponentiated Generalized Dagum distribution or Topp-Leone-Extended Dagum distribution:** For continuous random variable $x$ the Dagum distribution has pdf and cdf respectively

$$g(x,\lambda,\beta,\delta)=\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}$$

and

$$G(x,\lambda,\beta,\delta)=\left(1+\lambda x^{-\delta}\right)^{-\beta}$$

it is important to note here, our base line model is Exponentiated generlaized-G class and Exponentiated generlaized Dagum or Extended Dagum distribution has already been provided by Silva et al[8]., we just extend a work given by Silva et al.[8] and suggest TLEGDa having pdf and cdf

$$f_{TLEGDa}(x,\lambda,\beta,\delta,\alpha,a,b)=2\alpha ab\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a-1}$$

$$\times\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b-1}\left[1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right]$$

$$\times\left[1-\left(1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right)^{2}\right]^{\alpha-1},x,\lambda,\beta,\delta,\alpha,a,b>0 \quad (10)$$

**Sub Models:** With parameteric changes TLEG-G takes different sub models – i. If $\lambda=1$ then Dagum distribution is called Burr-III distribution and we obtain TLEGB-III distribution (Topp-Leone Exponentiated generalized Burr-III ) New Model, ii. If $\beta=1$ then Dagum distribution is called Log-logistic distribution and we obtain TLEGLL (Topp-Leone-Exponentiated generalized log-Logistic) distribution. iii. If $a=1$ then we obtain Topp-Leone generalized Dagum distribution (New Model), iv. If $b=1$ the we obtain Topp-Leone Dagum distribution with additional shape parameter a, having cdf (New Model) $\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2a}\right]^{\alpha}$, v. If $\lambda=a=1$ then we obtain Topp-Leone genernalized Burr-III distribution (New Model), vi. If $\beta=a=1$ then we obtain Topp-Leone generalized log-Logistic distribution (New Model), vii. If $\lambda=b=1$ then we obtain Topp-Leone Burr-III with additional shape parameter a (New Model) $\left[1-\left\{1-\left(1+x^{-\delta}\right)^{-\beta}\right\}^{2a}\right]^{\alpha}$, viii. If $\beta=b=1$ then we obtain Topp-Leone Log-logistic with additional shape parameter a (New Model) having cdf $\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-1}\right\}^{2a}\right]^{\alpha}$, ix. If $a=b=1$ then we obtain Topp-Leone Dagum distribution provided by Rasheed[7]

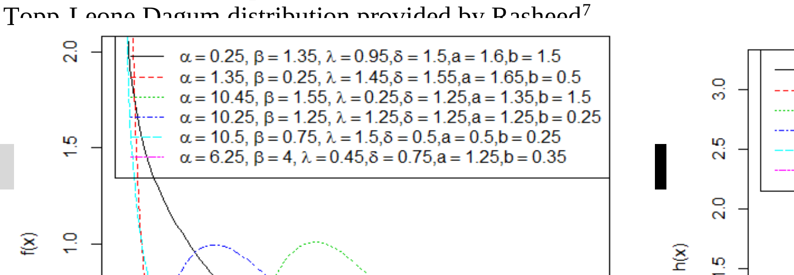

**Figure-1:** pdf and failure rate's plot of TLEGDa.

## Pdf Expansion of TLEG-Da Model

After some basic binomial expansions metioned above we obtain the TLEGDa model as

$$f_{TLEGDa}(x,\xi) = 2\alpha ab\,\lambda\beta\delta \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, x^{-\delta-1}\left(1 + \lambda x^{-\delta}\right)^{-[\beta(i+1)+1]}$$

$x, \alpha, \lambda, \beta, \delta, a, b > 0$

**rth and incomplete rth mometnts:** for continuous random variable $x$ rth moments of TLEGDa can be obtained as

$$\mu_r^{/} = 2\alpha ab\,\lambda\beta\delta \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h} \int_0^{\infty} x^{r-\delta-1}\left(1 + \lambda x^{-\delta}\right)^{-[\beta(i+1)+1]} dx$$

Let suppose $z = \lambda x^{-\delta}$ then limits will be change from $\infty\ to\ 0$ and $dx = \frac{dz}{-\lambda\delta x^{-\delta-1}}$ after substituting the relation in above equation then we get

$$\mu_r^{/} = 2\alpha ab\beta \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h} \int_0^{\infty} \left(\frac{z}{\lambda}\right)^{-\frac{r}{\delta}} (1 + z)^{-[\beta(i+1)+1]} dz$$

After some basic and necessery simplifications we get

$$\mu_r^{/} = 2\alpha ab\beta\lambda^{\frac{r}{\delta}} \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h} \int_0^{\infty} \frac{z^{1-\frac{r}{\delta}-1}}{(1+z)^{\beta(i+1)+1-\frac{r}{\delta}+\frac{r}{\delta}}} dz$$

finally

$$\mu_r^{/} = 2\alpha ab\beta\lambda^{\frac{r}{\delta}} \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1 - \frac{r}{\delta}, \beta(i+1) + \frac{r}{\delta}\right),\ \ r = 1,2,3 \ldots$$

where $B(l,m) = \int_0^{\infty} \frac{x^{l-1}}{(1+x)^{l+m}} dx$

Similarly Mean of the distribution can be obtined as

$$\mu_1^{/} = 2\alpha ab\beta\lambda^{\frac{1}{\delta}} \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1 - \frac{1}{\delta}, \beta(i+1) + \frac{1}{\delta}\right) \quad (12)$$

In the same way we obtain Incomplete rth moments of the distribution by integrating the function from 0 to sepcified value of $x$ , rest of the procdure is same as rth moments

$$m_r^{/} = 2\alpha ab\,\lambda\beta\delta \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h} \int_0^{x} x^{r-\delta-1}\left(1 + \lambda x^{-\delta}\right)^{-[\beta(i+1)+1]} dx$$

Let suppose $z = \lambda x^{-\delta}$ then limits will be change from $x\ to\ 0$ and $dx = \frac{dz}{-\lambda\delta x^{-\delta-1}}$

After some basic and necessery simplifications we get

$$m_r^{/} = 2\alpha ab\beta\lambda^{\frac{r}{\delta}} \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h} \int_0^{z} \frac{z^{1-\frac{r}{\delta}-1}}{(1+z)^{\beta(i+1)+1-\frac{r}{\delta}+\frac{r}{\delta}}} dz$$

finally

$$m_r^{/} = 2\alpha ab\beta\lambda^{\frac{r}{\delta}} \sum_{f,g,h=0}^{\infty} \sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1 - \frac{r}{\delta}, \beta(i+1) + \frac{r}{\delta}, z\right),\ \ r = 1,2,3 \ldots$$

where $B(l,m,z)$ represent incomplete Beta function.

and first incomplete moment of TLEGDa is

$$m_1^{/} = 2\alpha ab\beta\lambda^{\frac{1}{\delta}}\sum_{f,g,h=0}^{\infty}\sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1-\frac{1}{\delta},\beta(i+1)+\frac{1}{\delta},z\right) \quad (13)$$

## Reliability Properties of TLEGDa distribution

In this section we provide some expressions relatied to reliabilty properties, for random variable $x$ TLEGDa model has following properties

Reliability function or Survival function $R(x) = 1 - F_{TLEGDa}(x)$

$$R_{TLEGDa}(x) = 1 - \left[1-\left(1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right)^{2}\right]^{\alpha}$$

Failure function $\phi(x) = \frac{f(x)}{R(x)}$ using equation

$$\phi_{TLEGDa}(x) = \frac{\begin{array}{c}2\alpha ab\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a-1}\\ \times\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b-1}\left[1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right]\\ \times\left[1-\left(1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right)^{2}\right]^{\alpha-1}\end{array}}{1-[1-(1-\{1-[1-(1+\lambda x^{-\delta})^{-\beta}]^{a}\}^{b})^{2}]^{\alpha}}$$

Cumulative Hazard function $\omega(x)$, by definition of function we have $\omega(x) = -\ln R(x)$

$$\omega_{TLEGDa}(x) = -\ln\left\{1-\left[1-\left(1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right)^{2}\right]^{\alpha}\right\}$$

Reverse Hazard function $\varphi(x) = \frac{f(x)}{F(x)}$

$$\varphi_{TLEGDa}(x) = \frac{\begin{array}{c}2\alpha ab\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a-1}\\ \times\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b-1}\left[1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right]\end{array}}{[1-(1-\{1-[1-(1+\lambda x^{-\delta})^{-\beta}]^{a}\}^{b})^{2}]}$$

Odd function, by odd function we mean the ratio of cdf and survival function of the distriution

$$O_{TLEGDa} = \frac{\left[1-\left(1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right)^{2}\right]^{\alpha}}{1-[1-(1-\{1-[1-(1+\lambda x^{-\delta})^{-\beta}]^{a}\}^{b})^{2}]^{\alpha}}$$

Mean Waiting Time of the distribution is obtained as $\psi_{TLEGDa}(x) = x - \left\{\frac{m_1(x)}{F_{TLEGDa}(x)}\right\}$

Using equation (11) and (13) we get

$$\psi_{TLEGDa}(x) = x - \frac{2\alpha ab\beta\lambda^{\frac{1}{\delta}}\sum_{f,g,h=0}^{\infty}\sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1-\frac{1}{\delta},\beta(i+1)+\frac{1}{\delta},z\right)}{[1-(1-\{1-[1-(1+\lambda x^{-\delta})^{-\beta}]^{a}\}^{b})^{2}]^{\alpha}}$$

## Lorenz Curve and Bonferroni Index

Using equation (12) and (13)

$$L_{TLEGDa}(x) = \frac{2\alpha ab\beta\lambda^{\frac{1}{\delta}}\sum_{f,g,h=0}^{\infty}\sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1-\frac{1}{\delta},\beta(i+1)+\frac{1}{\delta},z\right)}{2\alpha ab\beta\lambda^{\frac{1}{\delta}}\sum_{f,g,h=0}^{\infty}\sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1-\frac{1}{\delta},\beta(i+1)+\frac{1}{\delta}\right)}$$

similarly from equation (11), (12) and (13) we have

$$B_{TLEGDa}(x) = \frac{2\alpha ab\beta\lambda^{\frac{1}{\delta}}\sum_{f,g,h=0}^{\infty}\sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1-\frac{1}{\delta},\beta(i+1)+\frac{1}{\delta},z\right)}{2\alpha ab\beta\lambda^{\frac{1}{\delta}}\sum_{f,g,h=0}^{\infty}\sum_{i=0}^{a(h+1)-1} w_{f,g,h}\, B\left(1-\frac{1}{\delta},\beta(i+1)+\frac{1}{\delta}\right) \times [1-(1-\{1-[1-(1+\lambda x^{-\delta})^{-\beta}]^{a}\}^{b})^{2}]^{\alpha}}$$

## Quantile function

Quanitile function is obtained by inverting the cdf of the distribution and here we have

$$Q(x) = \left[\frac{1}{\lambda}\left(\left[1-\left\{1-\left(1-\sqrt{1-u^{\frac{1}{\alpha}}}\right)^{\frac{1}{2}}\right\}^{\frac{1}{a}}\right]^{\frac{1}{\beta}}-1\right)\right]^{-\frac{1}{\delta}}$$

replacing the suitable values of $u$ one can obtaine different measures.

## Parameter Estimation By ML Method

There are no of methods used to estimate the parmater of the distribuion but ML method is most commonly used, anyhow we have log-likelihood function of TLEGDa is

$$\begin{aligned} L(x,\xi) = {} & n\log(2) + n\log(\alpha) + n\log(a) + n\log(b) + n\log(\lambda) + n\log(\delta) + n\log(\beta) \\ & -(\delta+1)\sum_{i=1}^{n}\log x_i - (\beta+1)\sum_{i=1}^{n}\log\left(1+\lambda x_i^{-\delta}\right) \\ & + (a-1)\sum_{i=1}^{n}\log\left[1-\left(1+\lambda x_i^{-\delta}\right)^{-\beta}\right] \end{aligned}$$

$$+(b-1)\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1+\lambda x_i^{-\delta}\right)^{-\beta}\right]^{a}\right\} + \sum_{i=1}^{n}\log\left[1-\left\{1-\left[1-\left(1+\lambda x_i^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right]$$

$$+(\alpha-1)\sum_{i=1}^{n}\log\left[1-\left(1-\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}^{b}\right)^{2}\right]$$

Now differentiating the above equation w.r.t parameter space one by one.
assuming that

$$w=\left(1+\lambda x^{-\delta}\right),\bar{w}=\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right],\bar{\bar{w}} = \left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{a}\right\}$$

$$\frac{\partial L(x,\xi)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log[1-(1-\bar{\bar{w}}^{b})^{2}]$$

$$\frac{\partial L(x,\xi)}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log\bar{w}-(b-1)\sum_{i=1}^{n}\frac{\bar{w}^{a}\log\bar{w}}{\bar{\bar{w}}} + b\sum_{i=1}^{n}\frac{\bar{\bar{w}}^{b-1}\bar{w}^{a}\log\bar{w}}{[1-\bar{\bar{w}}^{b}]}$$

$$+2b(\alpha-1)\sum_{i=1}^{n}\frac{(1-\bar{\bar{w}}^{b})\bar{\bar{w}}^{b-1}\bar{w}^{a}\log\bar{w}}{[1-(1-\bar{\bar{w}}^{b})^{2}]}$$

$$\frac{\partial L(x,\xi)}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\bar{\bar{w}}+\sum_{i=1}^{n}\frac{\bar{\bar{w}}^{b}\log\bar{\bar{w}}}{[1-\bar{\bar{w}}^{b}]}+2(\alpha-1)\sum_{i=1}^{n}\frac{(1-\bar{\bar{w}}^{b})\bar{\bar{w}}^{b}\log\bar{\bar{w}}}{[1-(1-\bar{\bar{w}}^{b})^{2}]}$$

$$\frac{\partial L(x,\xi)}{\partial\lambda}=\frac{n}{\lambda}-(\beta+1)\sum_{i=1}^{n}\frac{x_i^{-\delta}}{w}+\beta(a-1)\sum_{i=1}^{n}\frac{w^{-\beta-1}x_i^{-\delta}}{\bar{w}}-a(b-1)\beta\sum_{i=1}^{n}\frac{\bar{w}^{a-1}w^{-\beta-1}x_i^{-\delta}}{\{1-\bar{\bar{w}}\}}$$

$$+ab\beta\sum_{i=1}^{n}\frac{\{\bar{\bar{w}}\}^{b-1}[\bar{w}]^{a-1}w^{-\beta-1}x_i^{-\delta}}{[1-\bar{\bar{w}}^{b}]} - 2ab\beta(\alpha-1)\sum_{i=1}^{n}\frac{(1-\{\bar{\bar{w}}\}^{b})\bar{\bar{w}}^{b-1}\bar{w}^{a-1}w^{-\beta-1}x_i^{-\delta}}{[1-(1-\{\bar{\bar{w}}\}^{b})^{2}]}$$

$$\frac{\partial L(x,\xi)}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\log w+(a-1)\sum_{i=1}^{n}\frac{w^{-\beta}\log w}{\bar{w}}-a(b-1)\sum_{i=1}^{n}\frac{[\bar{w}]^{a-1}w^{-\beta}\log w}{\bar{\bar{w}}}$$

$$+ab\sum_{i=1}^{n}\frac{\bar{\bar{w}}^{b-1}\bar{w}^{a-1}w^{-\beta}\log w}{[1-\bar{\bar{w}}^{b}]}-2ab(\alpha-1)\sum_{i=1}^{n}\frac{(1-\{\bar{\bar{w}}\}^{b})\{\bar{\bar{w}}\}^{b-1}[\bar{w}]^{a-1}w^{-\beta}\log w}{[1-(1-\{\bar{\bar{w}}\}^{b})^{2}]}$$

$$\frac{\partial L(x,\xi)}{\partial\delta}=\frac{n}{\delta}-\sum_{i=1}^{n}\log x_i+\lambda(\beta+1)\sum_{i=1}^{n}\frac{x_i^{-\delta}\log x_i}{w}+\lambda\beta(a-1)\sum_{i=1}^{n}\frac{w^{-\beta-1}x_i^{-\delta}\log x_i}{\bar{w}}$$

$$-a\beta\lambda(b-1)\sum_{i=1}^{n}\frac{[\bar{w}]^{a-1}w^{-\beta-1}x_i^{-\delta}\log x_i}{\bar{\bar{w}}} - a\beta\lambda b\sum_{i=1}^{n}\frac{\{\bar{\bar{w}}\}^{b-1}[\bar{w}]^{a-1}w^{-\beta-1}x_i^{-\delta}\log x_i}{[1-\bar{\bar{w}}^{b}]}$$

$$+2\lambda\beta ab(\alpha-1)\sum_{i=1}^{n}\frac{(1-\bar{\bar{w}}^{b})\{\bar{\bar{w}}\}^{b-1}[\bar{w}]^{a-1}w^{-\beta-1}x_i^{-\delta}\log x_i}{[1-(1-\{\bar{\bar{w}}\}^{b})^{2}]}$$

**Applications:** Now we compare utility of new proposed model with sub models of TLEGDa and other models. Sub models of TLEGDa are Dagum (Da), Extended Dagum (EGDa), Kumaraswamy Dagum (KDa), Exponentiated Kumarswamy Dagum (EKDa) and other models in which Kumaraswamy Weibull Poisson (KWP), Type-II Topp-Leone Inverse Rayleigh (TLIR) and Zubair Weibull (ZW) distributions are include. This anaylisis is carried out with the help of R language, a language has very useful in statistical theory with lot of packages[9]. This analysis is perofmed with adequacy model package with BFGS method. The package provides us different statistical tests and compariosn measues, here we use minimum log-likelihood value, AD (Anderson Darlling test), C (Cramer-von Misses statistics) and S (Kolmogorov-Smirnov test). Now we describe some detaile about data sets used in this analysis.

The probability density functions of distributions used in this analysis: Kumaraswamy Dagum distribution (KDa)[10]

$$f(x)=ab\lambda\beta\delta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta a-1}\{1-(1+\lambda x^{-\delta})^{-\beta a}\}^{b-1}$$

Extended Dagum distribution (EGDa)[8]

$$f(x)=ab\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{a-1}\times\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{a}\right]^{b-1}$$

Kumaraswamy Weibull Poisson distribution (KWP)[11]

$$f(x)=\frac{\lambda abc\beta^{\alpha}}{e^{\lambda}-1}x^{\alpha-1}\left[1-e^{-(\beta x)^{\alpha}}\right]^{a-1}\left\{1-\left[1-e^{-(\beta x)^{\alpha}}\right]^{a}\right\}^{b-1}\times e^{\left[\lambda\left\{1-\left[1-e^{-(\beta x)^{\alpha}}\right]^{a}\right\}^{b}-(\beta x)^{\alpha}\right]}$$

Type-II Topp Leone Inverse Rayleigh distribution (TLIR)[12]

$$f(x) = 4\lambda\alpha^2 x^{-3} e^{-2\left(\frac{\alpha}{x}\right)^2}\left\{1 - e^{-2\left(\frac{\alpha}{x}\right)^2}\right\}^{\lambda-1}$$

Zubair Weibull (ZW)[13]

$$f(x) = \frac{2\alpha\lambda\beta x^{\lambda-1} e^{-\beta x^\lambda}\left(1 - e^{-\beta x^\lambda}\right) e^{\alpha\left(1-e^{-\beta x^\lambda}\right)^2}}{e^\alpha - 1}$$

Exponentiated Kumaraswamy Dagum distribution (EKDa)[15]

$$f(x) = \alpha b\beta\delta a\lambda x^{-\delta-1}\left(1 + \lambda x^{-\delta}\right)^{-\beta-1}\left\{1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta a}\right\}^{\alpha-1}$$
$$\times\left[1 - \left\{1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta a}\right\}^{\alpha}\right]^{b-1}$$

Exponentiated generalized Topp-Leone Dagum distribution (EGTLDa)

$$f(x) 2\alpha\lambda\beta\delta ab x^{-\delta-1}\left(1 + \lambda x^{-\delta}\right)^{-\beta-1}\left\{1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right\}$$
$$\times\left[1 - \left\{1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha-1}\left(1 - \left[1 - \left\{1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha}\right)^{\alpha-1}$$
$$\times\left\{1 - \left(1 - \left[1 - \left\{1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha}\right)^{a}\right\}^{b-1}$$

**Table-1:** Discription of data with numeric values.

| Serial | Discription | Numeric values |
|---|---|---|
| Data 1 | This waiting time data were recorded by Professor Jim Irish consists of 65 observations, for more detail see Bhatti et al[14]. | 83,51,87,60,28,95,8,27,15,10,18,16,29,54,91,8,17,55,10,35,47,77,36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18,169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9,12 |
| Data 2 | The data set of March precipitation (in inches) referes to Hinkley[15] | 0.77,1.74,0.81,1.2,1.95,1.2,0.47,1.43,3.37,2.2,3,3.09, 1.51,2.1,0.52,1.62,1.31,0.32,0.59,26,0.81,2.81,1.87, 1.18, 1.35,4.75, 2.48, 0.96, 1.89, 0.9, 2.05 |
| Data 3 | It referes to active repair times in (hours) for more detail see[15]. | 0.50,0.60,0.60,0.70,0.70,0.70,0.80,0.80,1.00,1.00,1.00,1.00,1.10,1.30,1.50,1.50,1.50,1.50,2.00,2.00,2.20,2.50,2.70,3.00,3.00,3.30,4.00,4.00,4.50,4.70,5.00,5.40,5.40,7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50 |

**Table-2:** Parameter Estimations with Standard error of Data set 1.

| Model | $\hat{a}$ $S.E(\hat{a})$ | $\hat{b}$ $S.E(\hat{b})$ | $\hat{\lambda}$ $S.E(\hat{\lambda})$ | $\hat{\alpha}$ $S.E(\hat{\alpha})$ | $\hat{\beta}$ $S.E(\hat{\beta})$ | $\hat{\delta}$ $S.E(\hat{\delta})$ |
|---|---|---|---|---|---|---|
| TLEGDa | 2.1898675 (0.12327158) | 8.0287019 (0.0927811) | 2.6996423 (0.216516) | 0.1073145 (0.0334789) | 11.3804088 (0.075466) | 0.8801073 (0.11593962) |
| EGDa | 0.1452883 (0.01363577) | 10.4482487 (2.4033736) | 4.5532186 (0.027872) | - | 5.2676586 (----------) | 6.9048983 (0.00509269) |
| Da | - | - | 0.3924673 (0.392467) | - | 125.5542277 (1162.93963) | 1.3289706 (1.3289706) |
| KDa | 8.1511652 (25.8311519) | 8.8121958 (15.043416) | 0.2607081 (0.837564) | - | 7.1192468 (22.5606845) | 0.5229351 (0.2989451) |
| TLIR | - | - | 0.5422765 (0.080945) | 9.1664580 (0.9005459) | - | - |
| ZW | - | - | 0.2915531 (0.1146908) | 39.2045444 (67.840443) | 1.8051128 (1.3601083) | - |
| EKDa | 1.1706031 (5.0962075) | 18.4402714 (49.912025) | 12.794821 (52.33129) | 27.4255259 (86.945706) | 1.1613177 (5.0558362) | 0.3538674 0.4750394 |

**Table-3:** Statistics and log-likelihood (value).

| Model | Value | C | AD | S (P-Value) |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| TLEGDa | 293.4942 | 0.1069305 | 0.7891918 | 0.0922 (**0.6485)** |
| EGDa | 298.7448 | 0.164099 | 1.137428 | 0.128 (0.2454) |
| Da | 295.5127 | 0.1813837 | 1.243386 | 0.0995 (0.551) |
| KDa | 294.054 | 0.1128216 | 0.8276962 | 0.0951 (0.6095) |
| TLIR | 296.7181 | 0.2378582 | 1.559056 | 0.137 (0.1811) |
| ZW | 294.105 | 0.1198179 | 0.8742479 | 0.0969 (0.5856) |
| EKDa | 294.1032 | 0.116739 | 0.8545638 | 0.0951 (0.6087) |

**Table-4:** Parameter Estimations with Standard error of Data set 2.

| Model | $\hat{a}$ $S.E(\hat{a})$ | $\hat{b}$ $S.E(\hat{b})$ | $\hat{\lambda}$ $S.E(\hat{\lambda})$ | $\hat{\alpha}$ $S.E(\hat{\alpha})$ | $\hat{\beta}$ $S.E(\hat{\beta})$ | $\hat{\delta}$ $S.E(\hat{\delta})$ |
|---|---|---|---|---|---|---|
| TLEGDa | 0.10940879 (0.03289567) | 0.09326596 (0.0718821) | 9.5431507 (0.034719) | 1.71459182 (1.8813700) | 5.13863316 (0.03517496) | 2.96675386 (0.02477059) |
| EGDa | 0.3862395 (0.03388113) | 0.1169443 (0.0210875) | 7.3691362 (0.006443) | - | 5.0757298 (0.03247206) | 3.4810699 (0.03307943) |
| Da | - | - | 0.0097763 (0.015589) | - | 113.1515859 (181.207953) | 1.44949327 ( 0.1934504) |
| KDa | 0.5244585 (4.4573099) | 0.5725128 (0.5860665) | 2.1571139 (3.454879) | - | 1.4414265 (12.2517114) | 3.2757164 (2.8511135) |
| KWP | 17.7271508 (25.3649276) | 0.4461020 (0.7309756) | 4.9068150 (2.622594) | 0.4430413 (0.3107752) | 6.3651051 (14.0665151) | - |
| TLIR | - | - | 0.6199580 (0.138032) | 0.5604957 (0.0774318) | - | - |
| ZW | - | - | 0.9193759 (0.1102197) | -5.8685222 (2.0787358) | 0.2842754 (0.07249346) | - |
| EKDa | 1.7464535 (0.01918836) | 0.1190987 (0.0237279) | 5.9364200 (0.197976) | 0.3605843 (0.1968257) | 3.0470661 (0.01713354) | 3.6427910 (0.01685727) |

**Table-5:** Statistics and log-likelihood (value).

| Model | Value | AD | S (P-Value) |
|---|---|---|---|
| TLEGDa | 47.76569 | 0.1295212 | 0.0652 (**0.9994)** |
| EGDa | 48.06221 | 0.1460241 | 0.0684 (0.9987) |
| Da | 49.54965 | 0.5385124 | 0.1299 (0.6721) |
| KDa | 48.38629 | 0.2012338 | 0.0693 (0.9984) |
| KWP | 48.21763 | 0.2262544 | 0.0837 (0.9816) |
| TLIR | 50.8244 | 0.7952887 | 0.176 (0.2922) |
| ZW | 50.82302 | 0.5492447 | 0.0868 (0.9736) |
| EKDa | 48.04878 | 0.1666865 | 0.077 (0.9928) |

**Table-6:** Parameter Estimations with Standard error of Data set 3.

| Model | $\hat{a}$ $S.E(\hat{a})$ | $\hat{b}$ $S.E(\hat{b})$ | $\hat{\lambda}$ $S.E(\hat{\lambda})$ | $\hat{\alpha}$ $S.E(\hat{\alpha})$ | $\hat{\beta}$ $S.E(\hat{\beta})$ | $\hat{\delta}$ $S.E(\hat{\delta})$ |
|---|---|---|---|---|---|---|
| TLEGDa | 0.118685662 (0.03789155) | 7.10639376 (9.9556392) | 0.0047576 (0.007395) | 0.28444519 (0.4061710) | 1.656265322 (3.33077796) | 6.086262809 (0.60753576) |
| EGDa | 167.3720210 (---------) | 27.8150330 (39.472103) | 0.9070359 (2.858471) | - | 6.3117644 (15.7500611) | 0.1092972 (0.03470747) |
| Da | - | - | 0.0139766 (0.024770) | - | 113.3259644 (200.208281) | 1.21244755 (0.15250833) |
| KDa | 7.53443771 (31.2195603) | 0.94202608 (1.3029041) | 0.0247364 (0.060154) | - | 8.21813518 (34.0525139) | 1.26196677 (1.07597420) |
| KWP | 8.0957392 (0.06790490) | 0.1049629 (0.0676004) | 2.2426263 (1.881488) | 0.7792500 (0.0549027) | 3.5598735 (0.03952583) | - |
| TLIR | - | - | 0.4719454 (0.087287) | 0.6508361 (0.0846226) | - | - |
| ZW | - | - | 0.7769951 (0.0897729) | -2.6414990 (1.8775246) | 0.3381266 (0.12685941) | - |
| EKDa | 1.28145595 (7.80194039) | 3.26118064 (4.0442178) | 0.0038949 (0.004727) | 0.21200587 (0.1464713) | 1.74027580 (11.4608783) | 4.89498771 (2.95679751) |

**Table-7:** Statistics and log-likelihood (value).

| Model | Value | C | AD | S (P-Value) |
|---|---|---|---|---|
| TLEGDa | 88.38316 | 0.05638259 | 0.3449365 | 0.0909 (**0.8955)** |
| EGDa | 89.9912 | 0.06066177 | 0.4119573 | 0.107 (0.7493) |
| Da | 89.46881 | 0.06716324 | 0.419291 | 0.0948 (0.8645) |
| KDa | 89.48341 | 0.06805972 | 0.4240592 | 0.0961 (0.8538) |
| TLIR | 89.59898 | 0.08303663 | 0.5052194 | 0.1043 (0.7768) |
| ZW | 92.26685 | 0.09157705 | 0.6442503 | 0.1221 (0.5903) |
| EKDa | 89.09544 | 0.07185693 | 0.4378899 | 0.0939 (0.8726) |

Now we compare the TLEGDa distribution with EGTLDa model, from table below one can easily see that all comparision measures of TLEGDa model are smaller than EGTLDa it is an indication that TLEGDa is best fitted model to the dat sets mentioned above.

**Table-8 :** Comparision between TLEGDa and EGTLDa with three data sets.

| Data Set 1 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Model | Value | AIC | BIC | CAIC | HQIC | AD | C | S<br>P-V |
| TLEGDa | 293.4942 | 598.9883 | 611.9416 | 600.462 | 604.0913 | 0.7891918 | 0.1069305 | 0.0922<br>0.6485 |
| EGTLDa | 295.4768 | 602.9536 | 615.9069 | 604.4273 | 608.0566 | 1.287982 | 0.1898595 | 0.1054<br>0.4758 |
| Data Set 2 | | | | | | | | |
| TLEGDa | 47.76569 | 107.5314 | 116.1353 | 111.0314 | 110.3361 | 0.1295212 | 0.01732913 | 0.0652<br>0.9994 |
| EGTLDa | 48.01671 | 108.0334 | 116.6374 | 111.5334 | 110.8381 | 0.1642864 | 0.02234949 | 0.0729<br>0.9965 |
| Data Set 3 | | | | | | | | |
| TLEGDa | 88.38316 | 188.7663 | 198.8996 | 191.3118 | 192.4302 | 0.3449365 | 0.05638259 | 0.0909<br>0.8955 |
| EGTLDa | 88.6041 | 189.2082 | 199.3415 | 191.7537 | 192.8721 | 0.4144553 | 0.0679149 | 0.0922<br>0.8856 |

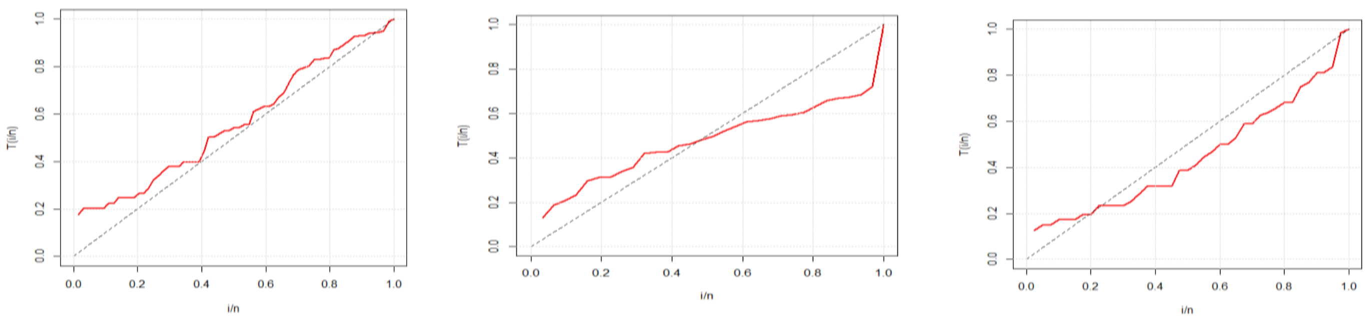


**Figure-2:** TTT Plots of dat set 1,2 and 3 respectively.

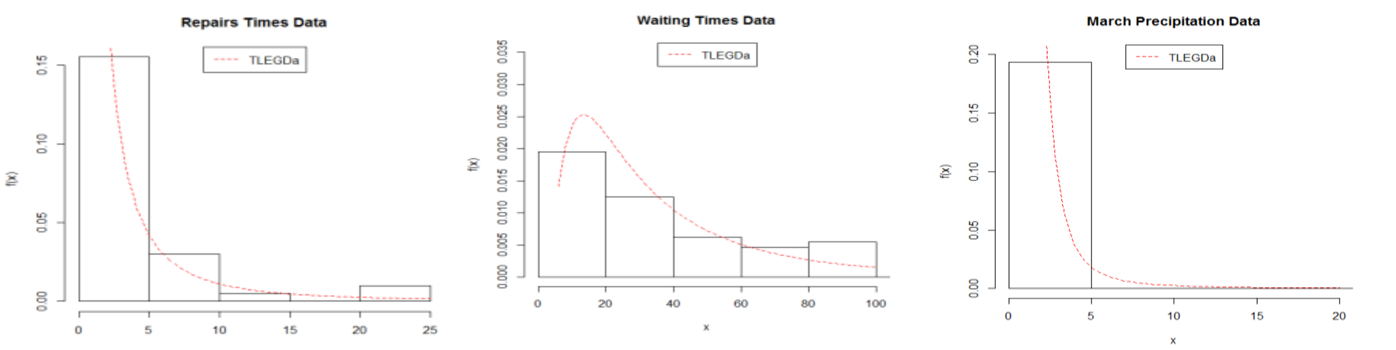


**Figure-3:** Plots of fitted densities.

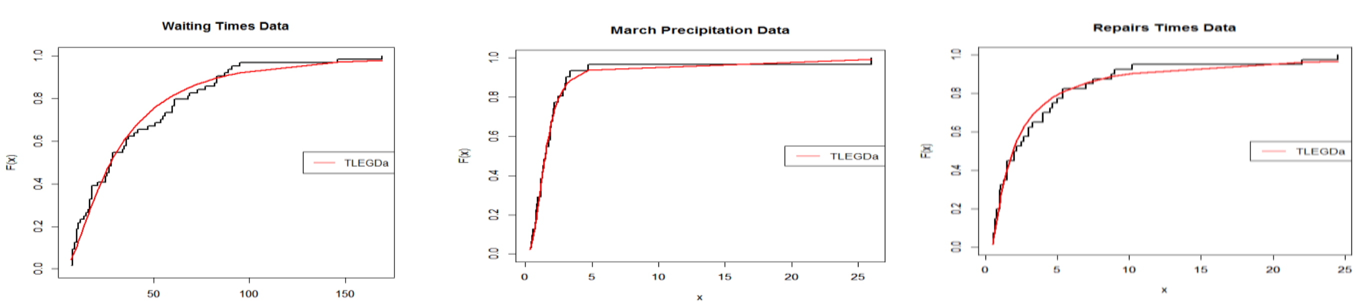


**Figure-4:** Plots of Estimated Cdfs.

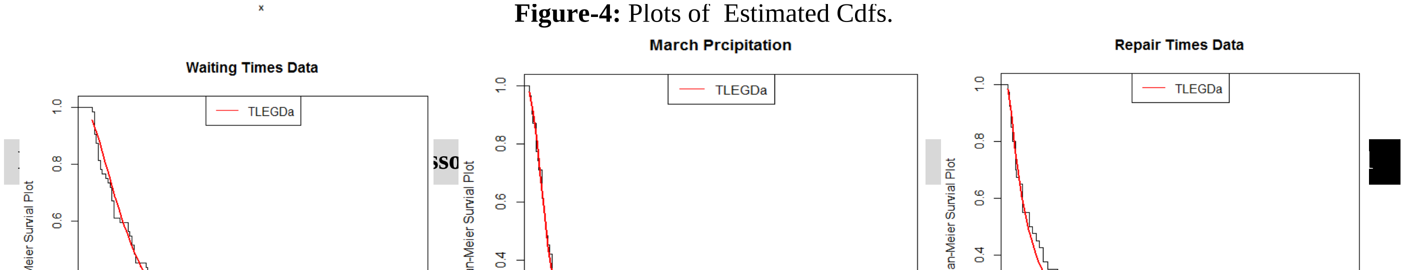

**Figure-5:** Kaplan-Meier Survival Plots.

## Conclusion

In this work we presented new class of continuous distribution so-called Topp-Leone-Exponentiated generalized G class of distributions (or TLEGDa), using Exponentiated genalaized G class has baseline model provided by Cordeiro et al. on the grounds of type-I Topp-Leone G class introduced by Al-Shomrani et al. Similar work named Exponentiated generalized Topp-Leone G (EGTL-G) class has already been proposed by Reyad et al.[4] Both distributions are almost same with little bit difference, if in EGTL-G the shape parameter $a = 2$ then both TLEG-G and EGTL-G will same. After introducing this G class we illustrate our work with the help of Dagum distribution the new model is called Topp-Leone-Exponentiated generalized Dagum or TLEGDa distribution, we obtained its basic properties in which rth moments, mean, incomplete rth moments, Quantile function, Lorenz and Bonferroni index and estimated its parameters by maximum likelihood method. We provided its density and hazard rate plots with assumed parametric values. In the end, we provided three applications where this model has minimum values of comparison measure that indicates that this new model is best among all its sub models and other models in which baseline distribution Extended Dagum, Dagum, Kumaraswamy Dagum, Kumarswamy Weibull Poisson, Zubair Weibull, Type II Topp Leone inverse Rayleigh and Exponentiated Kumarswamy Dagum. We also examined utility of proposed model with EGTL class using Dagum as baseline model, this utility proved importance or superiority of TLEGDa. It is hoped that this work is used extensively.

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