Research Journal of Mathematical and Statistical Sciences ______________________________ISSN 2320-6047
Vol. **8(2),** 17-20, May **(2020)** *Res. J. Mathematical and Statistical Sci.*

Short Review Paper

A new fixed point theorem in modular metric spaces

Rita Pal
Bhilai institute of technology, Bhilai, India
ritapal001@gmail.com

Available online at: www.iscamaths.com , www.isca.in , www.isca.me
Received 31st December 2019, revised 5th April 2020, accepted 10th May 2020

Abstract

In this article, a brief introduction of Modular metric spaces with some fixed point theorems are given from the modular theory of Chistyakov. After that some new concepts are discussed, which are related with the existence of fixed point from the Reich contraction mapping in the modular metric spaces.

Keywords: Fixed point, Reich contraction mapping, modular metric spaces.

Introduction

The concept of a modular on a vector space was given by Nakano in 1950[1]. Further it is clarified by Musielak and orlicz in 1959[2]. The fixed point theory in modular function spaces was introduced by Khamsi, Kozlowski and Reich in 1990[3]. Vyacheslav chistyakov introduced the concept of a metric modular on a set in 2006[4]. It is encouraged partly by the classical linear modular on function spaces applied by Nanko[1] and others in 1950's. The concept of a metric modular on an arbitrary set was introduced in 2006 by Chistyakov[4] as a generalization of these ideas.

Vyacheslav Chsityakov introduced the concept of modular metrics spaces which have a physical interpretation via F-modular in 2008[5]. Also further he developed the theory of modular metric spaces in 2010[6,7]. On the other hand we can say precisely that the concept of fixed point theory in modular metric spaces was introduced by Abdou and Khamsi[8].

The main purpose of this paper is to present the fixed point results for contractive mappings in the generalized modular metric spaces.

Preliminaries

I recall the definitions of modular metric spaces, the concept of convergence and other result that will be needed in the sequel.

Definition 1: Let X be a linear space on R. If a function $m:$ X → [0,∞] satisfies the following conditions, we call that m is a modular on a vector space X[9,10].

m (0) = 0⇔ $x = 0$
$m\,(ax) = m(x)\ for\ every\ a\ \in R\ with\ |a| = 1$
$m(ax + by)\ \le \mathrm{m(x)} + m(y)\ if\ a, b \ge 0, a + b = 1$

If it holds the (4) property instead of (3) then m is said to be convex.

$m(ax + by)\ \le \mathrm{a}m(x) +\ bm(y)\ if\ a, b \ge 0, a + b = 1$

Thus the modular space $X_m$is defined, such that m a modular on X is:

$X_m = \{\, x\ \in X : m(ax) \to 0\ as\ a\ \to 0\}$

Let $\{x_n\}$, $n \in N$ be a sequence in X_m and $\in X_m$. If $\lim_{n\to\infty} W_\lambda\,(x_n$- $x)$ =0 then $\{x_n\}$ is said to converge to x. Here m is said to satisfy the Δ_2- conditions if there exists $k \neq 0$ such that $m(2x) \le k\, m(x))$ for any $\in X_m$. Also m is said to satisfy the Fatou property (FP) if $m(x - y) \le \lim_{n\to\infty} m(\,x_n - y)$, whenever $\{x_n\}$, m-converges to x for any $x, y\ , x_n\ \in X_m$.

Let (X, m) be Modular vector space. Define $W\colon (0\ , +\infty) \times X \times X \to [\ 0, +\infty]$ by $W_\lambda\,(x, y) = m\left(\frac{x-y}{\lambda}\right)$

Then the following conditions hold.
If $W_\lambda(x, y) =\ 0$ for some λ > 0 and any $x, y\ \in X\ , x = y$;
$W_\lambda(x, y) = W_\lambda\,(y, x)$ for any λ > 0 and $x, y\ \in X$;
If m satisfy the property of Fatou then $\{x_n\}$ such that $\{{}^{x_n}/_{\lambda}\}$, $m {\to} {}^{x}/_{\lambda}$, for any λ > 0
$m\left(\frac{x-y}{\lambda}\right)\ \le \lim_{n\to\infty} inf\, m\left(\frac{x_n-y}{\lambda}\right)) \le \lim_{n\to\infty} sup\, m\left(\frac{x_n-y}{\lambda}\right)$, for any $x, y\ , x_n\ \epsilon\, X_m$..

Which implies
$W_\lambda(x, y) \le \lim_{n\to\infty} \inf W_\lambda\,(x_n\,, y) \le \lim_{n\to\infty} \sup W_\lambda\,(x_n\,, y)$.

Thus (X,W) satisfy all the properties of generalized modular metric spaces.

For some λ>0, the Fatou property is satisfied for a modular function W, if $\{x_n\}$ is defined in such that $\lim_{n\to\infty} W_\lambda(x_n, x) = 0$,

$W_\lambda(x,y) \le \lim_{n\to\infty} \inf W_\lambda(x_n, y) \le \lim_{n\to\infty} \sup W_\lambda(x_n, y)$, for any $y \in X_w$.

Definition 2: Assume that X be an abstract set, a function W: $(0, +\infty)\times X \times X \to [0, \infty]$, is defined by W $(\lambda, x, y) = W_\lambda(x,y)$ then W: $(0,\infty) \times X \times X \to [0,\infty]$ is said to be a regular modular metric on X if the following conditions are satisfied[9,10].

For any $x, y \in X$, $W_\lambda(x,y) = 0$ for all λ>0 if and only if $x = y$;
$W_\lambda(x,y) = W_\lambda(y,x)$ for all λ > 0;
$W_{\lambda+\mu}(x,y) = W_\lambda(x,z) + W_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$

If it satisfies the following inequality

$$W_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu} W_\lambda(x,z) + \frac{\mu}{\lambda+\mu} W_\mu(z,y)$$

W is convex, for $\lambda, \mu > 0$ and a,b,c $\in X$.
A set is introduced
$C(W, X, x) = \{\{x_n\} \subset X;\ \lim_{n\to\infty} W_\lambda(x_n, x) = 0\}$ for $x \in X$, where X is assumed an abstract set and a function $W: X \times X \to [0,\infty]$.

Definition 3: The function $W: X \times X \to [0,\infty]$ is said to be define a generalized metrics on X if the following axioms are satisfied[11].
For every $(x,y) \epsilon X \times X$, we have $W(x,y) = 0$ *then* $x = y$;
For every $(x,y) \epsilon X \times X, W(x,y) = W(y,x)$,
There exist c>0 such that If $(x,y) \epsilon X \times X$ and $\{x_n\} \in C(W, X, x)$, $W(x,y) \le c \lim_{n\to\infty} \sup W(x_n, y)$

Thus (X, W) is a generalized metric space.

Remark: It is clear that, if the set $C(W, X, x) = \emptyset$ for every $x \in X$, then (X, W) is a generalized metric space ⇔ Condition 1 and 2 of Definition 3 are satisfied.

Definition-4: Let (X, W) be a generalized metric space, the function W: $(0, +\infty) \times X \times X \to [0, +\infty]$ is such that

$$W_\lambda(x,y) = \frac{W(x,y)}{\lambda}$$

Also if $\{x_n\} \in C(W, X, x)$ for some $x \in X$ then we have $\lim_{n\to\infty} W_\lambda(x_n, x) = 0$ for any λ > 0 then it satisfies the following conditions.
$W_\lambda(x,y) = 0$ for some λ > 0 and $x, y \in X$ then $x = y$;
$W_\lambda(x,y) = W_\lambda(y,x)$ for any λ > 0 and $x, y \in X$
There exist $c > 0$ such that if $(x,y) \epsilon X \times X$, and $\{x_n\} \in C(W_\lambda, X, x)$,)
we have $W_\lambda(x,y) \le c \lim_{n\to\infty} \sup W_\lambda(x_n, y)$

Thus (X, W) is a generalized modular metric space.

Now some main definitions are given below which are used in further research work.

Definition 5: Suppose $(X_w$, W) be a generalized modular metricspace and $\{x_n\}_{n \in N}$ be a sequence in X_w, a subset M of X_w then

The sequence $\{x_n\}_{n \in N}$ is called a W- convergent to $x \in X_w \Leftrightarrow W_\lambda(x_n,) \to 0$ as n →∞; for some λ > 0.
A sequence $\{x_n\}_{n \in N}$ is called a W- cauchy if $W_\lambda(x_m, x_n) \to 0$ and m, n →∞ for some λ > 0.
It is said to be a W- complete if for any W- cauchy sequence { x_n } in m, such that $\lim_{n,m\to\infty} W_\lambda(x_n, x_m) \to 0$ there exist a point $x \in M$ such that $\lim_{n\to\infty} W_\lambda(x_n, x) = 0$ for some λ > 0.
It is called W- bounded if $\delta_{w,\lambda}$ (M)= Sup $\{W_\lambda(x,y)$: x,y ∈ M} <∞ for some λ >0.

Remark: For modular function spaces W satisfies Δ_2 – condition ⇔ $\lim_{n\to\infty} W_\lambda(x_n, x) = 0$,
⇒ $\lim_{n\to\infty} W_\lambda(x_n, x) = 0$, for some λ >0

Main Result: The Banach contraction principle in generalized modular metric space

Now the following sections, are presented as an extension of the banach contraction principle for the setting of generalised modular metric spaces.

Definition 6: Let $(X_w$,W) be a generalized modular metric space and f: $X_w \to X_w$ be a mapping, then f is called a W-contraction mapping if there exists k ∈ (0, 1) such that

$W_\lambda(f(x), f(y)) \le kW_\lambda(x,y)$ for any $(x,y) \in X_w \times X_w$.
x is said to be a fixed point of fif f$(x) = x$.

Now, in the following proposition 1, we understand the W-limit and its uniqueness in the generalized modular metric spaces.

Proposition-1: Let $(X_w$,W) be a generalized modular metric space. Let a sequence $\{x_n\}$ in X_w and pair (x,y) ∈ $X_w \times X_w$ is defined in such a way that $W_\lambda(x_n, x) \to 0$ and $W_\lambda(x_n, y) \to 0$ when n → ∞ for some λ > 0. Then x =y.

Proof: By taking property 3 from the definition of generalized modular metric space, we get, there exist $c > 0$ such that if $(x,y) \in X_w \times X_w$, and a sequence $\{x_n\} \in C(W_\lambda, X_w, x)$,) for some λ > 0, we have

$$W_\lambda(x,y) \le c \lim_{n\to\infty} \sup W_\lambda(x_n, y) = 0$$

⇒ from the property 1, of the definition of generalized modular metric space.
$W_\lambda(x,y) = 0 \Rightarrow x = y$

Let (X_w,W) be a generalized modular metric space and a map f : $X_w \to X_w$, then the orbit of $x$is defined by o(x) = { x, f(x), $f^2(x)$ ....} for any $x \in$ M
set $S_{w,\lambda}(x)$ = Sup { $W_\lambda(f^n(x), f^t(x))$: n ,t ∈ N } , Where λ > 0

Theorem-1: Let (X_w,W) be a generalized modular metric space. Assume that the following conditions are hold:
X_w is W- complete
Map f: $X_w \to X_w$be a W contraction mapping of some k∈ (0,1)
There exists $x_0 \in X_w$ such that $\delta_{w,\lambda}(x_0) < \infty$

Then {$f^n(x_0)$}, W- converges to fixed point of f, for some s ∈ X_w. If $W_\lambda(x$, s) <∞ for $x \in X_w$ then {f^n(x)}, W – converges to s. Also, if $s' \in X_w$ be another fixed point of f such that W_λ(s,s') <∞ then s = s'

Proof: Let $x_0 \in X_w$ be such that $\delta_{w,\lambda}(x_0) < \infty$
Then $W_\lambda(f^{n+p}(x_0), f^n(x_0)) \le k^n W_\lambda(f^p(x_0), x_0) \le k^n \delta_{w,\lambda}(x_0) < \infty$
for any n, p∈ N. Since k< 1, {($f^n(x_0)$ } is W – cauchy.
As X_w is W- complete, then there exists s∈ X_w such that

$$\lim_{n\to\infty} W_\lambda(f^n(x_0), s) = 0$$

since $W_\lambda(f^n(x_0), f(s)) \le k\ W_\lambda(f^{n-1}(x_0), s)$; n= 1,2 …….

We have $\lim_{n\to\infty} W_\lambda(f^n(x_0), f(s)) = 0$
Proposition 1 ⇒ f(s) = s
⇒ s is a fixed point of f
Let$x \in X_w$ such that W_λ (s) <∞

Then $W_\lambda(f^n(x), s) = W_\lambda(f^n(x), f^n(s)) \le k^n W_\lambda(x, s)$
For any n ≥ 1. Since K < 1, we get$\lim_{n\to\infty} W_\lambda(f^n(x), s) = 0$.

Thus {$f^n(x)$}W -converges to s.

Now suppose we have s and s' are two fixed point of f such that W_λ (s,s') <∞

W_λ (s,s') = W_λ(f (s), f (s$'$)) ≤ k W_λ (s, s') since f is W-contraction and k< 1
W_λ (s, s') = 0 since W_λ (s,s') <∞ for some λ > 0

So the condition 1 of the definition 4 from generalized modular metric space.
⇒s = s'

Now in the following section, Reich contraction are presented as an extension of the banach contraction principle for the modular metric spaces.

Reich contraction in modular metric spaces

Definition-7: Suppose (X,W) be a modular metric spaces and M be a nonempty subset of X_w then the selfmap f of M is called a Reich contraction only if it satisfies lim $sup_{s\to t+}$k (s) < 1 for any $0 \le t < \infty$, and there exists k: (0 ,+ ∞) → [0, 1) is defined by
$W_1(f(a), f(b)) \le k(W_1(a,b))\, W_1(a,b)$ for any distinct element a,b ∈ M .

A point a is said to be a fixed point of f if f $(a) = a$
Proposition-2: Suppose (X, W) be a modular metric space and W is convex and regular. Suppose that Wholds the Δ_2 – type condition. Let { x_n } be in X_w is defined by

$W_1(x_{n+1}, x_n) \le k\, \alpha^n$, n= 1,2 ……..
for k≠ 0 is an arbitrary constant and $0 < \alpha < 1$. Then we say that { x_n } is cauchy for W[12].

Theorem 2: Suppose (X,W) be a modular metric space and M be a nonempty subset of X_w. Assume that the following conditions are hold:

W is convex and regular.
X_w is W- complete.
A self-map f of M be a Reich contraction

Then point x is said to be a fixed point of f if f $(x) = x$ shows a unique fixed point for$x \in$ M and f^n (z), W- converges to x for any z ∈M.

Proof: By the definition 7 of Reich contraction for any different$a, b \in M$

$W_1(f(a), f(b)) \le k(W_1, (a,b))$ (a,b).

Since lim $sup_{s\to t+}$k (s) < 1 for any $0 \le t < \infty$ and there exist k: (0, + ∞) → (0, 1)

⇒ f has one fixed point, W is regular.

Now for the fixed point existence, set the point $x_0 \in X_w$

Case-I: Suppose for some n ∈ N,$f^n(x_0)$ is a fixed point of f, then it is trivial.

Case-II: Suppose $f^{n+1}(x_0) \ne f^n(x_0)$, for any n ∈ N

$W_1(f^{n+1}(x_0), f^n(x_0)) \le k\,(W_1(f^n(x_0), f^{n-1}(x_0)))\, W_1(f^n(x_0), f^{n-1}(x_0))$,

⇒ $W_1(f^{n+1}(x_0), f^n(x_0)) < W_1(f^n(x_0), f^{n-1}(x_0))$ for any n ∈ N

Thus { $W_1(T^{n+1}(x_0), T^n(x_0))$ } is convergent for any n ∈ N

Now set $t_0 = \lim_{n\to+\infty} W_1(T^{n+1}(x_0), T^n(x_0))$= in $f_{n\in N}$ ($T^{n+1}(x_0)$, $T^n(x_0)$)

Since lim $sup_{s\to t_{0+}}$k(s) <1, there exist $0 < \alpha < 1$ and $n_0 \in$ (1,∞) such that

$k\,(W_1(f^{n+1}(x_0),\ f^n(x_0)))\leq \alpha$

For any $n \geq n_0$. Then we have
$W_1(f^{n+1}(x_0),\ f^n(x_0)) \leq \prod_{k=n_0}^{k=n} k(W_1(f^{k+1}(x_0),\ f^k(x_0))$
$W_1(f^{n_0+1}(x_0),\ f^{n_0}(x_0)) \leq \alpha^{n-n_0} W_1(f^{n_0+1}(x_0),\ f^{n_0}(x_0))$

For any $n \geq n_0$, by the w- completeness of M and proposition 2,

$\Rightarrow\{f^n(x_0))$ is W- Cauchy,

$\Rightarrow\{f^n(x_0)\}$ W- converges to some $x \in M$.

Now to show: x is a fixed point of f.

We know that for any $n \geq 1$,

$w_2(x, f(x)) \leq W_1(x, f^n(x_0)) + W_1(f^n(x_0), T(x))$

$\leq W_1(x, f^n(x_0)) + k(W_1(f^{n-1}(x_0), x))\, W_1(f^{n-1}(x_0), x)$

$\leq W_1(x, f^n(x_0)) + W_1(f^{n-1}(x_0), x))$

also $\{f^n(x_0)\}$ W- converges to x,
$\Rightarrow W_2(x, f(x)) = 0$
$\Rightarrow f(x) = x$, since W is regular
Since uniqueness of the fixed point of f
$\Rightarrow \{f^n(z)\}$, W – converges to x for any $z \in M$.

Conclusion

Ageneralized modular metric space is established, from the conditions of generalized metric space and a Modular metric space, to prove the Banach contraction principle. This result has found many applications in computer science, physics, image processing engineering, economics, and telecommunication.

References

1 Nakano, H., (1950). Modulared semi–ordered linear spaces. *In Tokyo Math. Bookser.*, 1, Marcezen Co., Tokyo.

2 Musielak, J. and Orlicz, W. (1959). On modular spaces. *Studia Math.*, 18(1), 49-65.

3 Khamsi, M.A, Koziowski, W.K. and Reich, S. (1990). Fixed point theory in modular function spaces. *Nonlinear Anal.*, 14(11), 935-953.

4 Chistyakov, V.V. (2006). Metric modular and their application. *Akad. Nauk*, 406(2), 165-168.

5 Chistyakov, V.V. (2008). Modular metric spaces generated by F- modular. *Folia Math*, 15(1), 3-24.

6 Chistyakov, V.V. (2010). Modular metric spaces I: basic concepts. *Nonlinear Anal.*, 72(1), 1-14.

7 Chistyakov, V.V. (2010). Modular metric spaces, II: application to superposition operators. Nonlinear Analysis: Theory, Methods & Applications, 72(1), 15-30.

8 Abdou, A. A., & Khamsi, M. A. (2013). Fixed point results of point wise contractions in modular metric spaces. *Fixed Point Theory & Applications*, 2013(1), 163.

9 Khamsi, M. A. (1996). A convexity property in modular function spaces. *Mathematica Japonica*, 44, 269-280.

10 Kozlowski, W.M., (1988). Modular function spaces, monographs and Text books in pure and applied mathematics. 122, Marcel Dekker, Inc., New York .

11 Jleli, M., & Samet, B. (2015). A generalized metric space and related fixed point theorems. *Fixed point theory and Applications*, (61)2015, 1-14.

12 Abdou, A. A., & Khamsi, M. A. (2014). Fixed points of multivalued contraction mappings in modular metric spaces. *Fixed Point Theory & Applications*, 2014(1), 249.