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Topp-Leone Dagum Distribution: Properties and its Applications

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Abstract

In this work, we acquaint four parametric Topp-Leone Dagum distribution using the Topp-Leone-I (Type-I Topp-Leone) G class of distribution. We obtain some basic statistical properties, incomplete rth moments, mean deviation from mean, reliability and income inequality measures of the distribution, from graphical point of view we provide plots both its density and decreasing hazard function with assumed parametric values. We find Renyi and Tsallis entropies as well. We examine both parameter estimation methods, probability weighted moments and maximum likelihood. In the end we suggest four applications where this distribution is considered as best fitted model to the sub models of Dagum distribution and other related models in which Burr, Log-Dagum, generalized Dagum and Lomax distributions are include.

Keywords: Topp-Leone generalized family, Dagum distribution, Hazard rate function, Maximum likelihood estimate.

Introduction

Dagum distribution introduced by Camilo Dagum[1] and fit it to the empirical income and wealth data. The distribution has three types called type-I, II and III having three and four parameters respectively. Dagum presented type-II and III in 1980. Type-I or Dagum (Da) is the member of Burr system[2] of distribution. If in Burr-III distribution a scale parameter is added it becomes Dagum distribution, from equation (1) a scale parameter called λ. It is sometimes called generalized logistic-Burr distribution. Domma provided the Log-Dagum[3] (LDa) distribution and studied the changes accrue in Kurtosis. Authors who studied its properties and parameter estimations are including Kleiber and Kotz[4], Kleiber[5], Shazad and Asghar[6], Khan[7] and Dey et al[8]. Khan also presented generalized Dagum (GDa) and inverse Dagum models in his unpublished M.Phil thesis while Dey et al[8]. independently derived its properties. Domma et al.[9] studied this distribution for reliability analysis. Oluyede and Ye[10] developed weighted Dagum distribution.

Now-a-days different basic distributions are being used as baseline distributions with generalized family of distributions to get more flexible model and also explore more properties than baseline distribution. For this purpose Dagum distribution got much attention to the researchers. Domma and Condino[11] attached this distribution with Beta generalized distribution provided by Eugene et al.[12] and Jones[13] pioneer of generalized family of distributions. Oluyede and Rajasooriya[14] announced Mc-Dagum (McDa) models also explore mathematical properties of introduced model. Huang and Oluyede[15] introduced Exponentiated Kumaraswamy Dagum. In the same way Tahir et al.[16] introduced The Weibull-Dagum distribution or WD for short using the Weibull-G[17] family of distribution proposed by Bourguignon et al., Silva et al.[18] suggested Extended Dagum distribution, Nasiru et al[19]. provided Exponentiated generalized exponential Dagum distribution.

Although a lot of work has been done in Dagum distribution and it has been used as baseline model with different generalized class of distribution any how we fill some gap in its application therefore we introduce Topp-Leone Dagum distribution or TLDa and compare its utility. The pdf and cdf of Dagum (Da).

$$g(x) = \lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}, x>0 \quad (1)$$

$\lambda,\beta,\delta>0$, With three parameters β and δ are shape while λ is scale parameter respectively.

And

$$G(x) = \left(1+\lambda x^{-\delta}\right)^{-\beta} \quad (2)$$

Sub Models: Dagum distribution has different sub models when values of parameters change: i. If $\lambda = 1$ then (1) becomes the Burr III distribution[15] (Mentioned in introduction), ii. If $\beta = 1$ then (1) becomes the Fisk or Loglogistic distribution[15]. iii. If $\lambda = 1$ then (7) reduces to Topp-Leone Burr III distribution, iv. If $\beta = 1$ then (7) reduces to Topp-Leone Loglogistic distribution.

Topp-Leone Dagum Distribution

Using cdf and pdf of the Topp-Leone or TL distribution Al-Shomrani et al[20] recommended a new generalized family of distribution, the pdf and cdf of new proposed model is

$$f_{TLG}(x) = 2\alpha g(x)\bar{G}(x)\left(G(x)\right)^{\alpha-1}\left(2-G(x)\right)^{\alpha-1} \quad (3)$$

Having cdf

$$F_{TLG}(x) = \left(G(x)\right)^{\alpha}\left(2-G(x)\right)^{\alpha} \quad x\epsilon R\,, \alpha > 0 \tag{4}$$

It is clear from the above equations $g(x)$ is the pdf of baseline distribution while $G(x)$ is cdf of and $\bar{G}(x) = 1 - G(x)$ is survival function of that distribution. Eq (3) and (4) can easily be simplified as

$$f_{TLG}(x) = 2\alpha g(x)\bar{G}(x)\{1-(\bar{G}(x))^2\}^{\alpha-1} \tag{5}$$

And

$$F_{TLG}(x) = [1-(\bar{G}(x))^2]^{\alpha} \tag{6}$$

To obtain the density and cdf of four parametric Topp-Leone Dagum or TLDa distribution we insert (1) and $\bar{G}(x)$ of (2) in (5) and to get cdf we put in (6)

$$f_{TLDa}(x) = 2\alpha\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\} \tag{7}$$

$$\times\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha-1}$$

While

$$F_{TLDa}(x) = \left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha} \tag{8}$$

$x, \lambda, \beta, \delta, \alpha > 0$ With additional shape parameter α.

Mixture Density: From (7) expanding by Binomial series

$$\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha-1} = \sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2j}$$

Replacing above result in (7) one can get

$$f_{TLDa}(x) = \sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\, 2\alpha\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2j+1}$$

Again

$$\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2j+1} = \sum_{k=0}^{2j+1}\binom{2j+1}{k}(-1)^k\left(1+\lambda x^{-\delta}\right)^{-\beta k}$$

Inserting above relation in (7)

$$f_{TLDa}(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}\underbrace{\binom{\alpha-1}{j}\binom{2j+1}{k}(-1)^{j+k}2\alpha\beta}_{W_{j,k}}\;\underbrace{\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-[\beta(k+1)+1]}}_{Dagum\ Density}$$

Therefore TLDa density is consists of sum of infinite series and Dagum density. Finally (7) takes the form

$$f_{TLDa}(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}W_{j,k}\,\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-[\beta(k+1)+1]} \tag{9}$$

To study the behaviour of TLDa density, here we present plots with some assumed values of the parameters

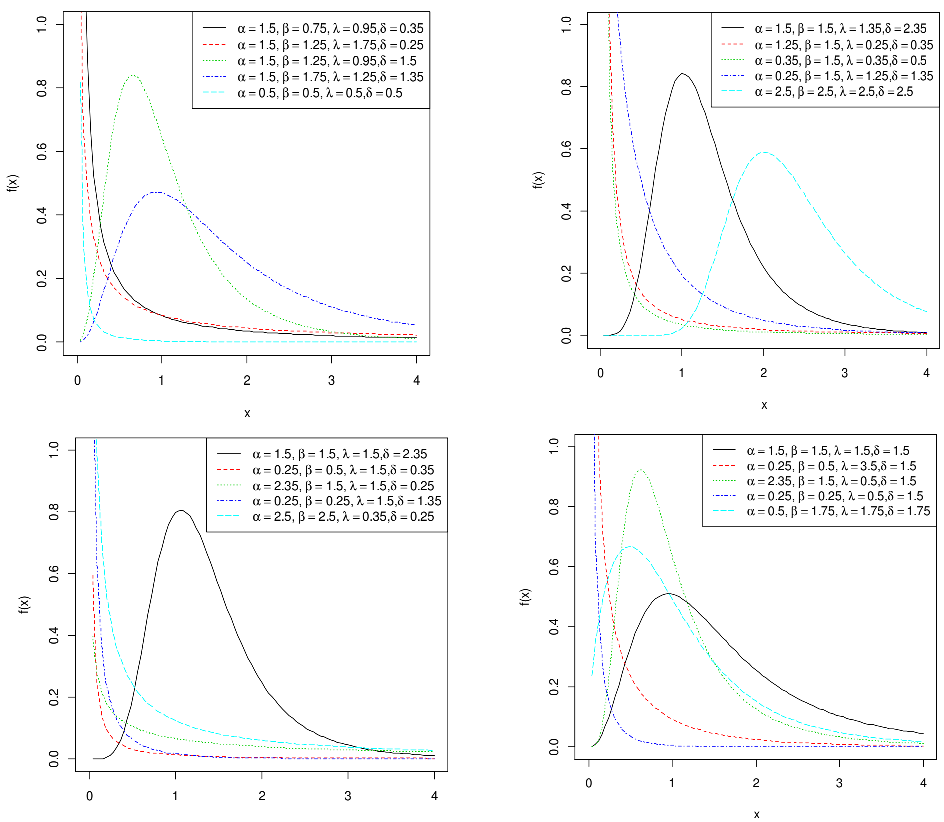

Figure-1: Plots of TLDa density with assumed values of parameters.

Random no generator, Quartiles, Skewness and Kurtosis

An important function so called quantile function is used as random number generator, its key feature is to obtain quartiles and further from these measures one can obtain skewness and kurtosis, inverting (8) we obtain

$$x_q = \left[\frac{1}{\lambda}\left(\left\{1-\sqrt{(1-u^{1/\alpha})}\right\}^{-1/\beta}-1\right)\right]^{-1/\delta}$$

$u \sim$ uniform distribution ranging from 0 to 1

From above equation, 1st, 2nd and 3rd quartiles can easily be obtained

$$Q_1 = x_{0.25} = \left[\frac{1}{\lambda}\left(\left\{1-\sqrt{(1-0.25^{1/\alpha})}\right\}^{-1/\beta}-1\right)\right]^{-1/\delta},$$

$$Q_2 = x_{0.5} = \left[\frac{1}{\lambda}\left(\left\{1-\sqrt{(1-0.5^{1/\alpha})}\right\}^{-1/\beta}-1\right)\right]^{-1/\delta},$$

$$Q_3 = x_{0.75} = \left[\frac{1}{\lambda}\left(\left\{1-\sqrt{(1-0.75^{1/\alpha})}\right\}^{-1/\beta}-1\right)\right]^{-1/\delta}$$

Further quartile deviation (Q.D), Coefficient of skewness (Sk) and Kurtosis (K) by Bowlyes's[21] and Moors[22] method respectively can be obtained

$$Q.D = \frac{x_{0.75}-x_{0.25}}{2}, \quad Sk = \frac{x_{0.75}+x_{0.25}-2x_{0.5}}{x_{0.75}-x_{0.25}}, K = \frac{x_{3/8}-x_{1/8}+x_{7/8}-x_{5/8}}{x_{6/8}-x_{2/8}}$$

r^{th} and inverse r^{th} moments

r^{th} moments are useful measures to find mean, variance , skeweness and kurtosis by relation of mean moments for TLDa distribution (9) it is obtained as

$$\mu_r^{/} = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda\delta\int_0^{\infty} x^{r-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-[\beta(k+1)+1]}dx \qquad (10)$$

Let $y=\lambda x^{-\delta}$ $\Rightarrow dx = \frac{dy}{-\lambda\delta x^{-\delta-1}}$, $y=\frac{\lambda}{x^{\delta}}$ Limits will change from ∞ to 0

After some basic simplification (10) takes the form

$$\mu_r^{/} = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{\frac{r}{\delta}}\int_0^{\infty} y^{-\frac{r}{\delta}}(1+y)^{-[\beta(k+1)+1]}dy$$

$$\mu_r^{/} = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{\frac{r}{\delta}}\,B\left(1-\frac{r}{\delta},\beta(k+1)+\frac{r}{\delta}\right), \; r=1,2,3,\ldots \qquad (11)$$

Where $B(a,b)=\int_0^{\infty}\frac{x^{a-1}}{(1+x)^{a+b}}\,dx$ is 2nd kind Beta function.

Inserting the $r=1$ we obtain Mean of the distribution

$$\mu_1^{/} = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{\frac{1}{\delta}}\,B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta}\right) \qquad (12)$$

Table-1: Values of above mentioned measures of TLDa distribution.

Parameter values	Q_1	Q_2	Q_3	*I.Q.R*	*Q.D*	*C.Q.D*
$\lambda=\beta=1$ $\delta=\alpha=1$	1.66667	3	7	5.3333	2.66665	0.53333
$\lambda=\beta=\delta=1$ $\alpha=2$	1.28571	1.6666	2.2	0.914286	0.457143	0.262295
$\lambda=\beta=\delta=1$ $\alpha=3$	1.18181	1.4	1.66667	0.484852	0.242426	0.170213
$\lambda=\beta=\delta=1$ $\alpha=4$	1.333	1.2857	1.461538	0.128208	0.064104	0.045873
$\lambda=\beta=1,$ $\alpha=\delta=-1$	1.66667	3	7	5.3333	2.66665	0.53333
$\lambda=\beta=2,$ $\alpha=\delta=-1$	1.11111	1.2	1.272727	0.161627	0.080813	0.06780
$\lambda=\beta=3,$ $\alpha=\delta=-1$	1.04651	1.08	1.105263	0.058751	0.029375	0.02730
$\lambda=\beta=4,$ $\alpha=\delta=-1$	1.0256	1.0434	1.056604	0.03096	0.01548	0.01486
$\lambda=\beta=-1$ $\alpha=\delta=1$	1.66667	3	7	5.3333	2.66665	0.53333
$\lambda=\beta=-1$ $\alpha=\delta=2$	0.642857	0.8333	1.1	0.45714	0.22857	0.26229
$\lambda=\beta=-1$ $\alpha=\delta=3$	0.393939	0.4666	0.55555	0.16161	0.80805	0.17020
$\lambda=\beta=-1$ $\alpha=\delta=4$	0.28333	0.321429	0.365385	0.08205	0.04102	0.12648

Note: I.Q.R = Inter quartile range, Q.D = Quartile deviation and C.Q.D = Coefficient of quartile deviation.

Table-2: Skewness and Kurtosis of TLDa model with assumed parametric values.

Parameter values	Sk	K
$\lambda=\beta=1$ $\delta=\alpha=1$	0.5	2.171429
$\lambda=\beta=\delta=1,$ $\alpha=2$	0.16667	1.066045
$\lambda=\beta=\delta=1,$ $\alpha=3$	0.1	1.022941
$\lambda=\beta=\delta=1,$ $\alpha=4$	0.071429	1.011593
$\lambda=\beta=1,$ $\alpha=\delta=-1$	0.5	2.17149
$\lambda=\beta=2,$ $\alpha=\delta=-1$	-0.1	1.022941
$\lambda=\beta=3,$ $\alpha=\delta=-1$	-1.03063	1.045832
$\lambda=\beta=4,$ $\alpha=\delta=-1$	-1.33574	1.05454
$\lambda=\beta=-1$ $\alpha=\delta=1$	-0.30769	2.17149
$\lambda=\beta=-1$ $\alpha=\delta=2$	0.16667	31.066045
$\lambda=\beta=-1$ $\alpha=\delta=3$	0.1	1.022941
$\lambda=\beta=-1$ $\alpha=\delta=4$	0.071429	1.011593

By the relation between rth and mean moments

$$Variance=\mu_2=\mu_2^{/}-\left(\mu_1^{/}\right)^2$$

Then Coefficient of variation or C.V is

$$C.V=\frac{\sqrt{\mu_2}}{\mu_1^{/}}\times 100$$

Further moments can easily be obtained. In the same way inverse rth moments of the distribution can be obtained, these moments are also useful measures specially used to find Harmonic mean

$$\mu_{-r}^{/}=\int x^{-r}\,f(x)dx$$

For TLDa distribution we use (9) and get

$$\mu_{-r}^{/}=\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}W_{j,k}\,\lambda\delta\int_0^{\infty}x^{-r-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-[\beta(k+1)+1]}dx$$

The rest of procedure is same as the rth moments inverse rth moments finally take the form

$$\mu_{-r}^{/}=\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}W_{j,k}\,\lambda^{-\frac{r}{\delta}}B\left(1+\frac{r}{\delta},\beta(k+1)-\frac{r}{\delta}\right)\quad,\qquad r=1,2,3,\ldots$$

Then Harmonic Mean of distribution is

$$\frac{1}{\mu_{-1}^{/}}=\frac{1}{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}W_{j,k}\,\lambda^{-\frac{1}{\delta}}B\left(1+\frac{1}{\delta},\beta(k+1)-\frac{1}{\delta}\right)}$$

Incomplete rth Moments

These moments are same as the rth moments the difference between them is that we integrate the function from lower limit to specified value of random variable x instead of lower limit to upper limit for TLDa distribution these are obtained as

$$m(x)=\int_0^x x^r f_{TLDa}(x)dx$$

$$m(x)=\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}W_{j,k}\int_0^x x^{r-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-[\beta(k+1)+1]}dx \quad (13)$$

Let $y=\lambda x^{-\delta}\ \Rightarrow dx=\frac{dy}{-\lambda\delta x^{-\delta-1}}$ Limits will change from ∞ to y, rest of the procedure is same as rth moments, after some simplification we get the result

$$m_r(x)=\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}W_{j,k}\,\lambda^{-\frac{r}{\delta}}B\left(1-\frac{r}{\delta},\beta(k+1)+\frac{r}{\delta};y\right)\qquad r=1,2,3,\ldots$$

Where $B(a,b;y)=\int_y^{\infty}\frac{y^{a-1}}{(1+y)^{a+b}}dy$ is the incomplete beta function of kind II, then first incomplete moment of the distribution is

$$m_1(x)=\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}W_{j,k}\,\lambda^{-\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta};y\right) \quad (14)$$

This moment is used in finding Mean Waiting Time (MWT), Lorenz curve and average deviation (Mean deviation).

Average Deviation from Mean: Average amount of absolute deviations of values from their mean is known as Average deviation or Mean deviation and obtained as

$$\pi(x)=\int_0^{\infty}|x-\mu|f(x)dx$$

$\mu=E(x)$ after simplification above eq takes the form

$$\pi(x)=2\left\{\mu F(\mu)-\int_0^{\mu}xf(x)dx\right\} \qquad (15)$$

Consider from equation (15)

$$\int_0^{\mu} xf(x)dx = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{-\frac{r}{\delta}} \times B\left(1-\frac{r}{\delta}\,,\beta(k+1)+\frac{r}{\delta}\;;y\right)$$

But $y = \frac{\lambda}{\mu^{\delta}}$

Finally (15) can be obtained as

$$\pi(x) = 2\left\{\mu F(\mu) - \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{-\frac{r}{\delta}}\,B\left(1-\frac{r}{\delta}\,,\beta(k+1)+\frac{r}{\delta}\;;y\right)\right\} \quad (16)$$

$F(\mu)$ can be obtained from eq (8).

Reliability Properties

This section deals with some basic reliability properties of TLDa model which are frequently used in probability theory and engineering, these measures are

Reliability function or survival function $S(x) = 1 - F(x)$

$$S_{TLDa}(x) = 1 - \left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha}$$

Hazard function $h(x) = \frac{f(x)}{S(x)}$

$$h_{TLDa}(x) = \frac{2\alpha\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\} \times \left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha-1}}{1-[1-\{1-(1+\lambda x^{-\delta})^{-\beta}\}^2]^{\alpha}}$$

Cumulative Hazard function $H(x) = -\ln S(x)$

$$H_{TLDa}(x) = -\ln\left\{1-\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha}\right\}$$

Reverse Hazard function $r(x) = \frac{f(x)}{F(x)}$

$$r_{TLDa}(x) = \frac{2\alpha\lambda\beta\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}}{[1-\{1-(1+\lambda x^{-\delta})^{-\beta}\}^2]^{\alpha}}$$

Mean Waiting Time (MWT) $\varphi(x) = x - \left\{\frac{1}{F(x)}\int_0^x x\,f(x)dx\right\}$

From equation (14) we get

$$\int_0^{x} x\,f(x)dx = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{-\frac{1}{\delta}}\,B\left(1-\frac{1}{\delta}\,,\beta(k+1)+\frac{1}{\delta}\;;y\right)$$

$$\varphi_{TLDa}(\mathrm{x}) = x - \frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{-\frac{1}{\delta}}\,B\left(1-\frac{1}{\delta}\,,\beta(k+1)+\frac{1}{\delta}\;;y\right)}{[1-\{1-(1+\lambda x^{-\delta})^{-\beta}\}^2]^{\alpha}}$$

Mean Residual Life $\phi(x) = \frac{1}{S(x)}\int_x^{\infty} x\,f(x)dx - x$

Consider for TLDa and use equation (9)

$$\int_x^{\infty} x\,f(x)dx = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda\delta\int_x^{\infty} x^{-\delta}\left(1+\lambda x^{-\delta}\right)^{-[\beta(k+1)+1]}dx$$

Supposing

$$y = \lambda x^{-\delta} \;\Rightarrow dx = \frac{dy}{-\lambda\delta x^{-\delta-1}}$$

Then after some simplification we get the final result as

$$\int_x^{\infty} x\,f(x)dx = \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{\frac{1}{\delta}} B_y\left(1-\frac{1}{\delta}\,,\beta(k+1)+\frac{1}{\delta}\right)$$

Here $B_y(a,b) = \int_0^y \frac{y^{a-1}}{(1+y)^{a+b}}dy$ is incomplete Beta function of 2nd kind.

Finally The Mean Residual Life of TLDa distribution is

$$\phi_{TLDa}(x) = \frac{1}{S(x)}\int_x^{\infty} x\,f(x)dx - x$$

$$= \frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{\frac{1}{\delta}} B_y\left(1-\frac{1}{\delta}\,,\beta(k+1)+\frac{1}{\delta}\right)}{(1-[1-\{1-(1+\lambda x^{-\delta})^{-\beta}\}^2]^{\alpha})} - x$$

Income Inequality Measures

The real application of Dagum model is in Economics, therefore it's necessary to discuss some inequality measures here.

These are also used in demographic research which make TLDa distribution more flexible and make its field vaste. The following measures are presented here

Gini Index: Italian Statistician Corrado Gini[23] (1912) introduced this inequality

$$G = \frac{1}{\mu}\int_0^{\infty}\{F(x)(1-F(x)\}dx \quad (17)$$

Where μ and $F(x)$ are mean and cumulative distribution function respectively.

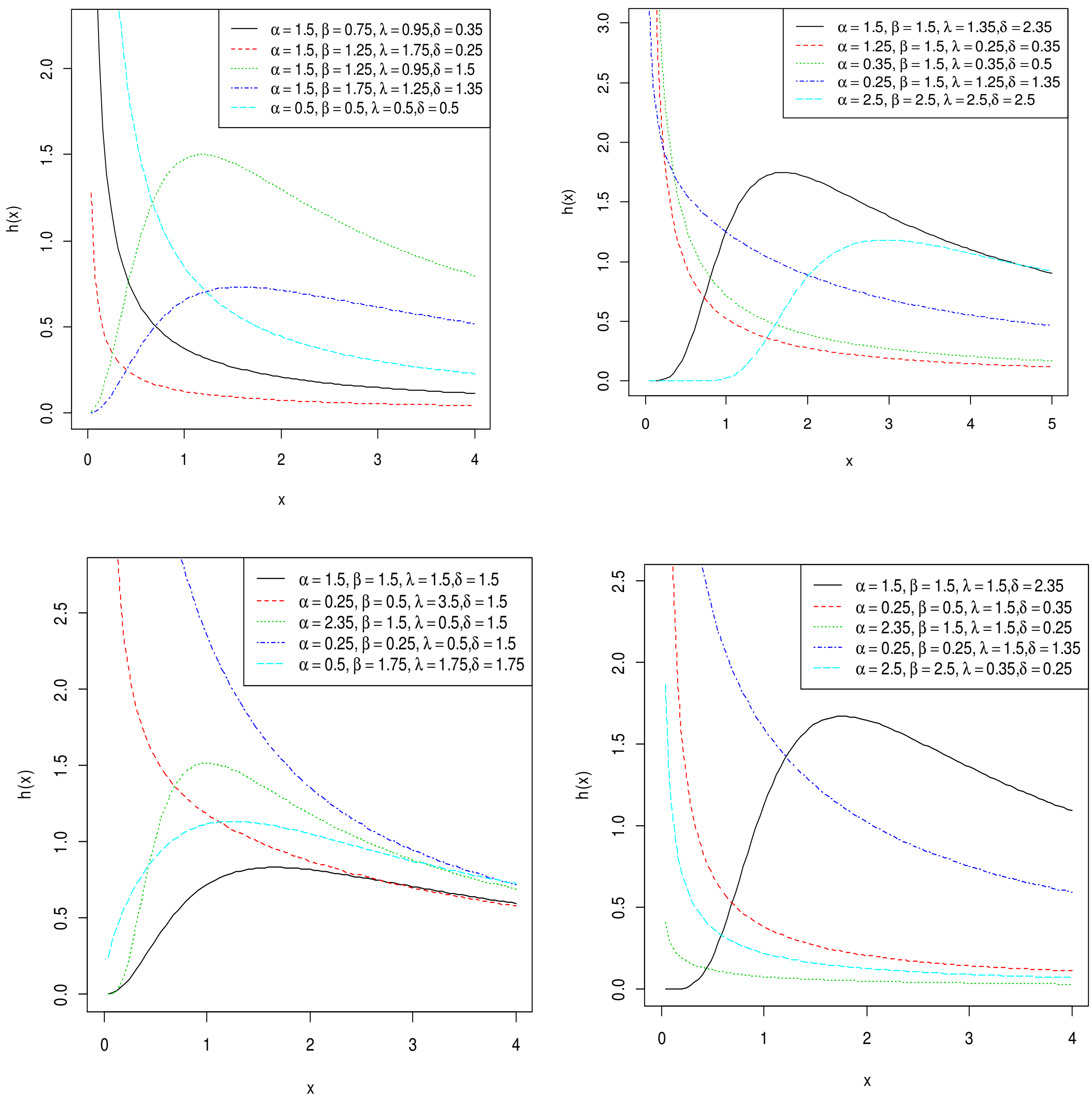

Figure-2: Plots of Hazard Rate function with assumed values of parameters.

Now consider for TLDa distribution

$$\{F(x)(1-F(x)\} = \left\{ \left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^\alpha - \left(1-\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^\alpha\right)\right\}$$

$$= \left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^\alpha - \left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{2\alpha}$$

By Binomial expansion we have the results, expand two times each term then we get

$$\sum_{j=0}^{\infty}\sum_{k=0}^{2j}(-1)^{j+k}\binom{\alpha}{j}\binom{2j}{k}\left(1+\lambda x^{-\delta}\right)^{-\beta k} - \sum_{m=0}^{\infty}\sum_{l=0}^{2m}(-1)^{l+m}\binom{2\alpha}{m}\binom{2m}{l}\left(1+\lambda x^{-\delta}\right)^{-\beta l}$$

$$\sum_{j=}^{\infty}\sum_{k=0}^{2j} W_{j,k}\int_0^{\infty}(1+\lambda x^{-\delta})^{-\beta k}dx-\sum_{m=0}^{\infty}\sum_{l=0}^{2m} W_{m.l}\int_0^{\infty}(1+\lambda x^{-\delta})^{-\beta l}dx$$

Supposing

$$y=\lambda x^{-\delta}\Rightarrow dx=\frac{dy}{-\lambda\delta x^{-\delta-1}}$$

After some basic simplification and using equation (12) we get G of TLDa distribution.

$$G_{TLDa}=\frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j} W_{j,k}\,\lambda^{\frac{1}{\delta}}\delta^{-1}B\left(-\frac{1}{\delta},\beta k+\frac{1}{\delta}\right)-\sum_{m=0}^{\infty}\sum_{l=0}^{2m} W_{m,l}\,\lambda^{\frac{1}{\delta}}\delta^{-1}B\left(-\frac{1}{\delta},\beta l+\frac{1}{\delta}\right)}{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{j,k}\,\lambda^{\frac{1}{\delta}}\,B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta}\right)}$$

Lorenz Curve: M.O. Lorenz[24] introduced the another inequality measure that is

$$L(x)=\frac{1}{\mu}\int_0^x x\,f(x)dx$$

From (12) and (14) we have $L(x)$ of TLDa distribution

$$L(x)=\frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta};y\right)}{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta}\right)} \tag{18}$$

Bonferroni Index: Credit goes to Bonferroni[25] for introduction of that inequality. It is obtained by quotient of Lorenz curve and cdf.

$$B(x)=\frac{L(x)}{F(x)}$$

Using (18) and (8) we get

$$B_{TLDa}(x)=\frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta};y\right)}{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta}\right)[1-\{1-(1+\lambda x^{-\delta})^{-\beta}\}^2]^{\alpha}}$$

Generalized Entropy (GE): Cowell and Shorrocks[26] provide it, that is

$$GE(\omega,\theta)=\frac{1}{\theta(\theta-1)\mu^{\theta}}\int_0^{\infty}x^{\theta}f(x)dx-1$$

Where μ is the mean of the distribution.

We know that for TLDa distribution basically its θth origin moments

$$\int_0^{\infty}x^{\theta}f(x)dx=\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{\frac{\theta}{\delta}}B\left(1-\frac{\theta}{\delta},\beta(k+1)+\frac{\theta}{\delta}\right)$$

Finally GE for TLDa model is written as

$$GE(\omega,\theta)=\frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{\frac{\theta}{\delta}}B\left(1-\frac{\theta}{\delta},\beta(k+1)+\frac{\theta}{\delta}\right)}{\theta(\theta-1)\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta}\right)\right\}^{\theta}}$$

Pietra Index: Pietra[27] introduced this index, it is a ratio of Mean deviation from mean and twice of mean of the distribution.

$$P(x)=\frac{\pi(x)}{2\mu}$$

Then for TLDa distribution

$$P(x)=\frac{\left\{\mu F(\mu)-\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{-\frac{r}{\delta}}B\left(1-\frac{r}{\delta},\beta(k+1)+\frac{r}{\delta};y\right)\right\}}{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\lambda^{\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta}\right)}$$

Zenga Index: Zenga[28] gave this inequality measure

$$Z=1-\frac{\mu^{-}_{(x)}}{\mu^{+}_{(x)}}$$

Where

$$\mu^{-}_{(x)}=\frac{1}{F(x)}\int_0^x xf(x)dx,\qquad \mu^{+}_{(x)}=\frac{1}{1-F(x)}\int_x^{\infty}xf(x)dx$$

For TLDa distribution obtained from Mean waiting time we have

$$\mu^{-}_{(x)}=\frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{-\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta};y\right)}{[1-\{1-(1+\lambda x^{-\delta})^{-\beta}\}^2]^{\alpha}}$$

Similarly from Mean Residual life the following equation is obtained.

$$\mu^{+}_{(x)}=\frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{\frac{1}{\delta}}B_y\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta}\right)}{(1-[1-\{1-(1+\lambda x^{-\delta})^{-\beta}\}^2]^{\alpha})}$$

Finally Zenga index for TLDa distribution is

$$Z=1-\left[\frac{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{-\frac{1}{\delta}}B\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta};y\right)}{[1-\{1-(1+\lambda x^{-\delta})^{-\beta}\}^2]^{\alpha}}\times\frac{\left(1-\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^2\right]^{\alpha}\right)}{\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} W_{j,k}\,\lambda^{\frac{1}{\delta}}B_y\left(1-\frac{1}{\delta},\beta(k+1)+\frac{1}{\delta}\right)}\right]$$

Entropy: Entropy is an indicatory measure of variation of uncertainty that is used in science, probability and other fields; here we drive expressions of two entropies so called Rényi entropy and Tsallis entropy.

Rényi entropy[29]: For continues random variable it is defined as

$$I_R(\theta)=\frac{1}{1-\theta}\log\{\int f^{\theta}(x)dx\} \quad , \quad \theta\neq 1$$

Consider for TLDa distribution

$$\int f^{\theta}(x)dx=(2\alpha\beta\delta\lambda)^{\theta}x^{-\theta(\delta+1)}\left(1+\lambda x^{-\delta}\right)^{-\theta(\beta+1)}$$
$$\times\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{\theta}\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2}\right]^{\theta(\alpha-1)}$$

Consider the factor

$$\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2}\right]^{\theta(\alpha-1)}$$
$$=\sum_{j=0}^{\infty}\binom{\theta(\alpha-1)}{j}(-1)^{j}\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2j}$$

Again

$$\left\{\mathbf{1}-\left(\mathbf{1}+\boldsymbol{\lambda x^{-\delta}}\right)^{-\boldsymbol{\beta}}\right\}^{\mathbf{2j}}=\sum_{k=0}^{2j}\binom{\mathbf{2j}}{\boldsymbol{k}}(-\mathbf{1})^{\boldsymbol{k}}\left(\mathbf{1}+\boldsymbol{\lambda x^{-\delta}}\right)^{-\boldsymbol{\beta k}}$$

$$\int \boldsymbol{f^{\theta}(x)dx}$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{2j}\binom{\boldsymbol{\theta(\alpha-1)}}{\boldsymbol{j}}\binom{\mathbf{2j}}{\boldsymbol{k}}(-\mathbf{1})^{\boldsymbol{j+k}}(\mathbf{2}\boldsymbol{\alpha\beta\delta\lambda})^{\boldsymbol{\theta}}\int_{\mathbf{0}}^{\infty}\boldsymbol{x^{-\theta(\delta+1)}}$$
$$\times\left(1+\lambda x^{-\delta}\right)^{-[\theta(\beta+1)+\beta k]}dx$$

Supposing

$$y=\lambda x^{-\delta}\Rightarrow\frac{dy}{dx}=-\lambda\delta x^{-\delta-1}$$

Then After some basic and necessary calculation we get the result

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{2j}\binom{\theta(\alpha-1)}{j}\binom{2j}{k}(-1)^{j+k}(2\alpha\beta)^{\theta}\delta^{\theta-1}\lambda^{\frac{(1-\theta)}{\delta}}B\left(\theta\left(1+\frac{1}{\delta}\right)\right.$$
$$\left.-\frac{1}{\delta},\theta(\beta+1)+\beta k-\left(\theta\left(1+\frac{1}{\delta}\right)-\frac{1}{\delta}\right)\right)$$

Then Rényi entropy for TLDa distribution is

$$I_R(\theta)=\frac{1}{1-\theta}\log\left\{\begin{array}{c}\sum_{j=0}^{\infty}\sum_{k=0}^{2j}\binom{\theta(\alpha-1)}{j}\binom{2j}{k}(-1)^{j+k}(2\alpha\beta)^{\theta}\delta^{\theta-1}\lambda^{\frac{(1-\theta)}{\delta}}\\ \times B\left(\theta\left(1+\frac{1}{\delta}\right)-\frac{1}{\delta},\theta(\beta+1)+\beta k-\left(\theta\left(1+\frac{1}{\delta}\right)-\frac{1}{\delta}\right)\right)\end{array}\right\}$$

Tsallis Entropy: Havrada and Charvat [30] provided it after sometimes Tsallis[31] extended its application.

$$I_T(\theta)=\frac{1}{1-\theta}\left\{\int f(x)^{\theta}dx\right\}$$

Same results of Renyi entropy are inserted and we get Tsallis entropy

$$I_T(\theta)=\frac{1}{1-\theta}\left\{\begin{array}{c}\sum_{j=0}^{\infty}\sum_{k=0}^{2j}\binom{\theta(\alpha-1)}{j}\binom{2j}{k}(-1)^{j+k}(2\alpha\beta)^{\theta}\delta^{\theta-1}\lambda^{\frac{(1-\theta)}{\delta}}\\ \times B\left(\theta\left(1+\frac{1}{\delta}\right)-\frac{1}{\delta},\theta(\beta+1)+\beta k-\left(\theta\left(1+\frac{1}{\delta}\right)-\frac{1}{\delta}\right)\right)\end{array}\right\}$$

Probability Weighted Moments

These moments are suggested by Landwehr et al. and Greenwood et al.[32] and used to find estimation of parameters as well as L-Moments of the distribution, that is

$$\xi_r=\int x\left(F(x)\right)^{r}f(x)dx$$

For TLDa distribution

$$\xi_r=2\alpha\lambda\beta\delta\int_{0}^{\infty}x^{-\delta}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}$$

$$\times\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2}\right]^{\alpha(r+1)-1}dx$$

By Binomial expansion

$$\left[1-\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2}\right]^{\alpha(r+1)-1}=\sum_{j=0}^{\infty}\binom{\alpha(r+1)-1}{j}(-1)^{j}\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2j}$$

Again using binomial expansion the above result can be written as

$$\left\{1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}^{2j+1}\sum_{k=0}^{2j+1}\binom{2j+1}{k}(-1)^{k}\left(1+\lambda x^{-\delta}\right)^{-\beta k}$$

Replacing these results and we obtain

$$\xi_r=2\alpha\beta\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}\binom{\alpha(r+1)-1}{j}\binom{2j+1}{k}(-1)^{j+k}\lambda\delta\int_{0}^{\infty}x^{-\delta}(1$$
$$+\lambda x^{-\delta})^{-[\beta(k+1)+1]}dx$$

Supposing that

$$y=\lambda x^{-\delta}\Rightarrow\frac{dy}{dx}=-\lambda\delta x^{-\delta-1}$$

And after some basic simplification we get the final result as

$$\xi_r = 2\alpha\beta \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} W_{j,k} \lambda^{\frac{1}{\delta}} B\left(1 - \frac{1}{\delta}, \beta(k+1) + 1 + \frac{1}{\delta}\right)$$

In statistics different software's are available which calculate the numerical values of these moments

Parameter estimation by Maximum Likelihood Method

Suppose random variable $x \backsim TLDa(\alpha, \lambda, \beta, \delta)$ and we draw a random sample of size n from that population, Let $x_1, x_2, \ldots, x_n$ are n observations then Log Likelihood function

$L(\Theta) = \log \prod_{i=1}^{n} f(x)$ *is*

$$L(\Theta) = n\log(2) + n\log(\alpha) + n\log(\beta) + n\log(\lambda) + n\log(\delta) - (\delta+1)\sum_{i=1}^{n} \log x_i - (\beta+1) \times \sum_{i=1}^{n} \log\left(1 + \lambda x_i^{-\delta}\right) + \sum_{i=1}^{n} \log\left\{1 - \left(1 + \lambda x_i^{-\delta}\right)^{-\beta}\right\} + (\alpha - 1)\sum_{i=1}^{n} \log\left[1 - \left\{1 - \left(1 + \lambda x_i^{-\delta}\right)^{-\beta}\right\}^2\right]$$

Now differentiating the function w.r.t parameters and obtain ML estimates.

Assuming $Z_i = \left(1 + \lambda x_i^{-\delta}\right)$

$$\frac{\partial L(\Theta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[1 - \left\{1 - Z_i^{-\beta}\right\}^2\right]$$

$$\frac{\partial L(\Theta)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log Z_i + \sum_{i=1}^{n} \frac{Z_i^{-\beta} \log Z_i}{\left(1 - Z_i^{-\beta}\right)} - 2(\alpha - 1)\sum_{i=1}^{n} \frac{Z_i^{-\beta}\left(1 - Z_i^{-\beta}\right) \log Z_i}{\left\{1 - \left(1 - Z_i^{-\beta}\right)^2\right\}}$$

$$\frac{\partial L(\Theta)}{\partial \lambda} = \frac{n}{\lambda} - (\beta+1)\sum_{i=1}^{n} \frac{x_i^{-\delta}}{Z_i} + \sum_{i=1}^{n} \frac{x_i^{-\delta} \beta Z_i^{-\beta-1}}{\left(1 - Z_i^{-\beta}\right)} - 2(\alpha - 1)\sum_{i=1}^{n} \frac{x_i^{-\delta} \beta Z_i^{-\beta-1}\left(1 - Z_i^{-\beta}\right)}{\left[1 - \left\{1 - Z_i^{-\beta}\right\}^2\right]}$$

$$\frac{\partial L(\Theta)}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n} \log x_i + (\beta+1)\lambda \sum_{i=1}^{n} \frac{x_i^{-\delta} \log x_i}{Z_i} - \lambda\beta \sum_{i=1}^{n} \frac{x_i^{-\delta} \log x_i \, Z_i^{-\beta-1}}{\left(1 - Z_i^{-\beta}\right)} + 2\lambda\beta(\alpha - 1)\sum_{i=1}^{n} \frac{x_i^{-\delta}\left(1 - Z_i^{-\beta}\right) Z_i^{-\beta-1} \log x_i}{\left[1 - \left\{1 - Z_i^{-\beta}\right\}^2\right]}$$

These equations cannot be solved numerically for this purpose different statistical softwares are available like R with different algorithms. From above equations, one can obtain Fisher information matrix and further $100(1-\alpha)\%$ confidence interval of parameters of the distribution.

Four Applications

Now we suggest four data where TLDa distribution can be applied to compare utility of this distribution with other distributions we use Dagum (Da), its sub models (Burr-III (B.III), Log-Logistic (LL) or Fisk) and other related models in which Log-Dagum (LDa), Generalized Dagum (GDa), Lomax (Lom) and Burr are include. Probability density functions of these distributions are listed below

$f_{B.III}(x) = \beta\delta x^{-\delta-1}\left(1 + x^{-\delta}\right)^{-\beta-1}$, $f_{LL}(x) = \lambda\delta x^{-\delta-1}\left(1 + \lambda x^{-\delta}\right)^{-2}$

$f_{GDa}(x) = \beta\alpha\lambda\delta x^{-\delta-1}\left(1 + \lambda x^{-\delta}\right)^{-\alpha\beta-1}$,
$f_{LDa}(x) = \lambda\delta\beta e^{-\delta x}\left(1 + \lambda e^{-\delta x}\right)^{-\beta-1}$

$f_{Lom}(x) = \frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}$, $f_{Burr}(x) = \frac{\alpha\beta x^{\alpha-1}}{(1 + x^{\alpha})^{\beta+1}}$

In this analysis we see that TLDa model is best fitted model among above mentioned distributions, description of data are

Data-1: The data shows successive failures for the air conditioning system for more detail see Proschan[33]. This data also studied by Huang and Oluyede[16].

Data-2: The data consists of 40 observations of survival times of patients, data originally studied by Harter and Moore[34], after that it also been used by Bhatti et al.[35].

Data-3: The third data are taken from Murthy et al.[36] which represent the time between failures of 30 repairable items.

Data-4: This data obtained from Bhaumik et al[37]. having vinyl chloride observations obtained from clean up gradient ground water monitoring wells in mg/L.

For data analysis, we use R-Language which is freely available having lot of packages. We use Adequacy Model package using goodness of fit function with "L-BFGS-B" algorithm, the doi of the script is http://cran.r-project.org/web/packages/Adequacy Model/AdequacyModel.pdf.

The function provide us some important statistics to compare the goodness of fit of probabilistic models and these are W (Cramer-von Misses), A (Anderson Darling), K-S (Kolmogorov Smirnov test) with its p-value. Function also gives us some comparison criteria. Here we use $\hat{l}$(loglikelihood), AIC, CAIC and BIC. The criterion of good fitted model is that values of these comparison measures should be smaller as compare to others. Detail of our analysis is presented with the help of tables. Comparison measures and statistics for air conditioning system data are presented with Table-3 and 4, similarly Table-5 and 6 are used for survival time data. While for third data we use Table-7 and 8, in the same way table 9 and 10 provide results of data four.

Table-3: Comparison measures for air conditioning system data.

Distribution	$\hat{l}$	AIC	BIC	CAIC	A	W	K-S P-value
TLDa	1035.93	2079.86	2092.806	2080.079	0.4639	0.0616	0.037 **0.9527**
Da	1037.11	2080.221	2089.93	2080.351	0.5969	0.0826	0.048 0.7799
LDa	1087.823	2181.646	2191.355	2181.776	6.7389	1.1258	0.1368 0.0017
Burr	1173.026	2350.053	2356.526	2350.118	4.2085	0.6239	0.3708 0.0000
GDa	1037.11	2082.221	2095.166	2082.439	0.5969	0.0826	0.048 0.78
Burr-III	1056.278	2116.555	2123.028	2116.62	2.7800	0.4187	0.0882 0.1073

Table-4: Estimates of parameters with their standard errors (S.E).

Distribution	$\hat{\alpha}$ $S.E(\hat{\alpha})$	$\hat{\beta}$ $S.E(\hat{\beta})$	$\hat{\lambda}$ $S.E(\hat{\lambda})$	$\hat{\delta}$ $S.E(\hat{\delta})$
TLDa	0.2163 (0.02060)	8.1916 (0.09837443)	29.3844 (0.12838306)	0.9964 (0.0276623)
Da	-	0.6942199 (0.210992)	1089.478457 (1604.6148)	1.6290878 (0.238282)
LDa	-	159.4752552 (1.19048600)	0.01471191 (0.0109825)	0.01679042 (0.0010717)
Burr	7.5779775 (12.4441)	0.03403953 (0.05585864)	-	-
GDa	0.4570172 (5.82207)	1.5190366 (19.3530793)	1089.314083 (1519.81627)	1.6290586 (0.2261152)
Burr-III	-	14.1638203 (1.66544922)	-	0.7989004 (0.0376154)

Table-5: Comparison measures and statistics of survival time data.

Distribution	$\hat{l}$	AIC	BIC	K-S P-Value
TLDa	202.0621	412.1243	418.8798	0.1257 **0.5125**
Da	205.4982	416.9964	422.063	0.147 0.3208
Lomax	217.4892	438.9785	442.3562	0.5253 0.000
Burr	272.5729	549.1458	552.5236	0.1504 0.2954
GDa	205.6923	419.3846	426.1401	0.1517 0.2858
Burr-III	205.7626	415.5252	418.903	0.3239 0.0003

Table-6: Estimates of parameters with their standard errors (S.E).

Distribution	$\hat{\alpha}$ $S.E(\hat{\alpha})$	$\hat{\beta}$ $S.E(\hat{\beta})$	$\hat{\lambda}$ $S.E(\hat{\lambda})$	$\hat{\delta}$ $S.E(\hat{\delta})$
TLDa	0.4584151 (0.53293)	51.1234499 (99.607504)	64.3069771 (107.02525)	1.6956256 (0.26382)
Da	-	60.27972 --	55.22917 --	1.97023 (0.20546)
Burr	3.0473754 (15.5200)	0.07552523 (0.3847211)	-	-
GDa	14.808564 --	14.808564 --	14.037509 --	1.954867 (0.20895)
Burr-III	-	3025.98461 (2483.4575)	-	1.952117 (0.21024)
Lomax	11756.55 (1875.77)	-	993820.40 (715.391)	-

Table-7: Comparison measure and statistics for failure time data.

Distribution	$\hat{l}$	A	W	K-S P-value
TLDa	39.72518	0.1186	0.0162	0.0741 **0.9965**
Da	39.88714	0.1215	0.0173	0.0811 0.9891
LDa	40.82267	0.2483	0.0327	0.1026 0.9104
Burr	40.90617	0.2733	0.0431	0.1162 0.813
GDa	39.88714	0.1215	0.0173	0.0811 0.9891
Burr-III	40.83342	0.2662	0.04142	0.1089 0.8691
Lomax	43.00604	0.1439	0.01891	0.1845 0.2589

Table-8: Estimates of parameters with their standard errors (S.E).

Distribution	$\hat{\alpha}$ $S.E(\hat{\alpha})$	$\hat{\beta}$ $S.E(\hat{\beta})$	$\hat{\lambda}$ $S.E(\hat{\lambda})$	$\hat{\delta}$ $S.E(\hat{\delta})$
TLDa	0.1638607 (0.0331554)	4.6290067 (0.26639302)	3.2817584 (0.17566)	2.2359527 (0.2586625)
Da	-	0.5161319 (0.3149202)	6.0684449 (9.384465)	3.0375585 (1.0463508)
LDa	-	180.56337137 (515.56094255)	0.02224977 (0.06378631)	1.30358066 (0.19532868)
Burr	2.3708725 (0.421826)	0.8098405 (0.1725167)	-	-
GDa	0.718397 (37.16078)	0.718397 (37.160784)	6.069411 (9.385111)	3.037649 (1.046279)
Burr-III	-	1.239423 (0.2375432)	-	2.050994 (0.3212846)
Lomax	10919.75 (2030.6642)	-	16844.06 (671.9463)	-

Table-9: Comparison measure and statistics of vinyl chloride data.

Distribution	$\hat{l}$	A	W	K-S P-value
TLDa	55.32876	0.21548	0.03177	0.0789 **0.984**
Da	55.84348	0.25910	0.03958	0.0871 0.9586
LL	55.94561	0.27523	0.03970	0.0868 0.9598
LDa	62.2745	1.04601	0.16416	0.1571 0.3708
Burr	56.07628	0.29268	0.04178	0.104 0.8557
GDa	55.84348	0.25910	0.03958	0.0871 0.9586
Burr-III	56.01333	0.28856	0.04097	0.0954 0.9165
Lomax	55.43944	0.27996	0.04327	0.0813 0.9781

Table-10: Estimates of parameters with their standard errors (S.E).

Distribution	$\hat{\alpha}$ $S.E(\hat{\alpha})$	$\hat{\beta}$ $S.E(\hat{\beta})$	$\hat{\lambda}$ $S.E(\hat{\lambda})$	$\hat{\delta}$ $S.E(\hat{\delta})$
TLDa	0.1358254 (0.02675219)	7.9959111 (0.17921131)	1.3099181 (0.16369742)	1.3354653 (0.17214147)
Da	-	0.7001792 (0.5359863)	2.2852870 (3.4955441)	1.7665991 (0.6388800)
LL	-	-	1.212517 (0.3681457)	1.531319 (0.2159162)
LDa	-	161.98203841 (294.701434)	0.01579063 (0.02875262)	0.85699684 (0.12741294)
Burr	1.5620879 (0.2479241)	0.9304646 (0.1791183)	-	-
GDa	0.9296443 (36.8734018)	0.7531449 (29.8723736)	2.2854097 (3.4954416)	1.7666211 (0.6388407)
Burr-III	-	1.101887 (0.2034348)	-	1.476837 (0.2227298)
Lomax	30.00606 (187.0808)	-	54.52313 (351.1274)	-

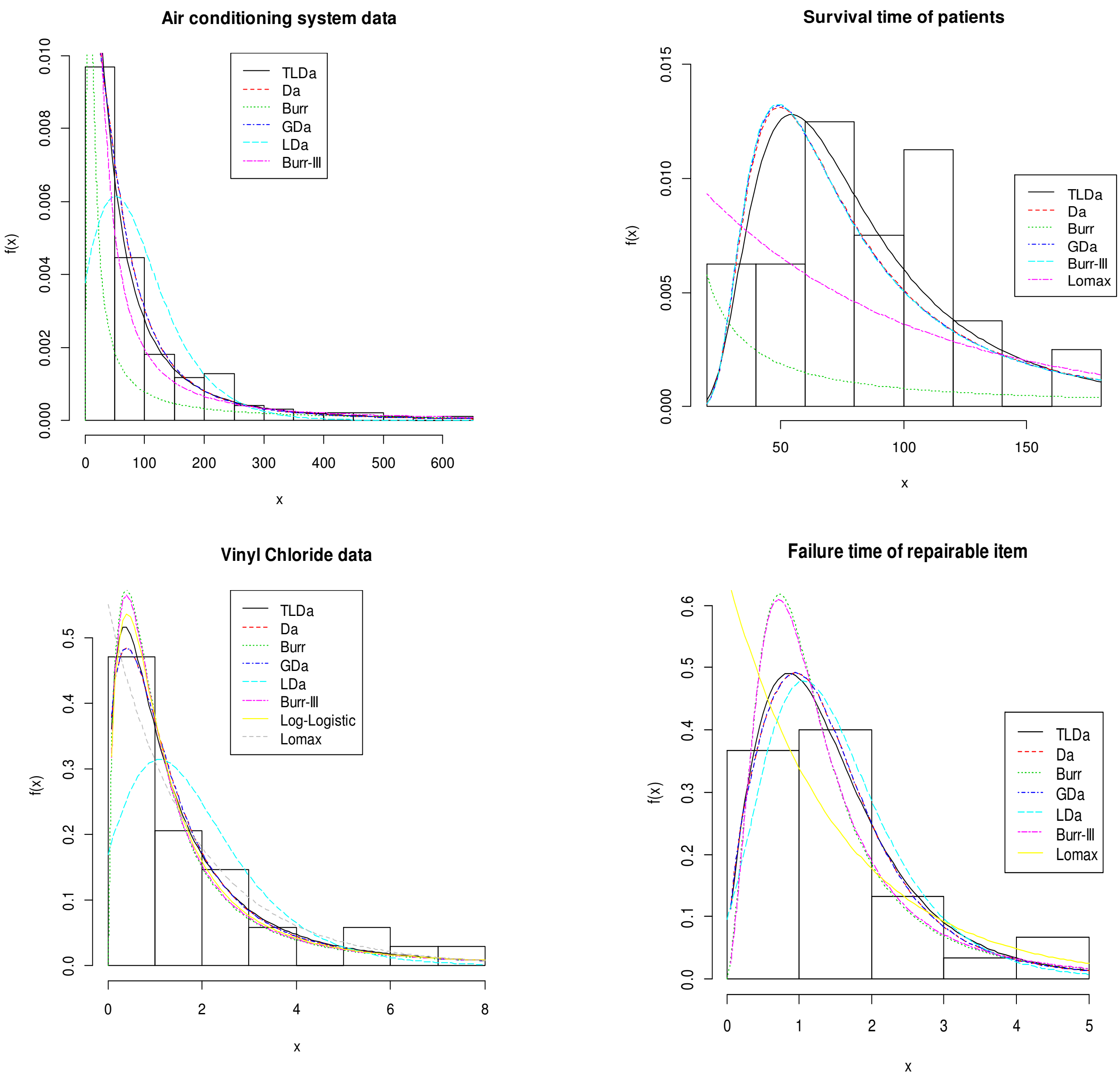

Figure-3: Plots of Estimated density functions.

Conclusion

In this research we presented four parametric Topp-Leone Dagum distribution, although lot of work has been done of this distribution but to full fill some gap we introduce this model. Firstly we introduced its some basic properties in which rth moments, inverse r^{th} momenst, incomplete moments and expression of mean, variance and CV are presented. From quantile function we obtained Quartiles, Moors Kurtosis and Skewness and their numerical values are also presented with assumed values of parameters, from these values of parameters we observe that skewneness shows positive or negative behaviour. In reliability properties we obtained Hazard, cumulative and reverse hazard functions, similarly expressions of mean waiting and mean residual life are also examined. We described plots of parameters of pdf and $h(x)$ of the distribution, from these plots one can easily say thay distribution has decreasing hazard function. Some income inequality measures in which Gini Coefficient, Lornz curve; Bonferroni index, Generlaized entropy, Pietra index and Zinga index are also part of this research. Renyi and Tsallis entropies are familiarized, two methods of parameter estimation are suggested one is maximum likelihood and other is probability weighted moments. In the end the utility is compared with sub models of Dagum distribution and other distributions. We conclude that TLDa is best among all other distributions, that's why it is suggested that use this model in real life especially reliability analysis instead of above mentioned distributions.

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