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A class of outlier resistant two-sample scale tests

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Abstract

A class of two-sample scale tests which is resistant to outliers of one sample is suggested. The tests are distribution-free tests based on U-statistics being functions of median and extremes of subsamples respectively drawn from first and second samples. The null and asymptotic distributions of the class of tests are obtained. The asymptotic relative efficiencies of some members of the class with respect to various tests are calculated to analyze the large sample performance. Empirical power of the proposed class of tests for various sample and subsample sizes under various distributions is computed to investigate the small sample performance.

Keywords: Asymptotic relative efficiency (ARE), empirical power, kernel, subsample, two-sample, U-Statistic.

Introduction

Two-sample scale problem is a fundamental problem in statistical inference that arises while comparing two populations with respect to (wrt) their scale parameters. Suppose $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are two random samples from two absolutely continuous populations with distribution functions $F(x)$ and $G(x) = F\left(\frac{x}{\sigma}\right), \sigma > 0$. The problem of testing the null hypothesis $H_0: \sigma = 1$ against $H_1: \sigma > 1$ constitutes the two-sample scale problem.

F-test due to Fisher[1] is widely used two-sample scale test in parametric setup. Wilcoxon[2] proposed a nonparametric test for the two-sample scale problem that used ranks of relative arrangements of X and Y sample observations in a combined sample. Mann and Whitney[3] studied a U-statistic based on number of times Y preceeds X and established its relation with Wilcoxon[2] test. Lehmann[4] proposed a test based on generalized U-statistic, Mood[5] studied a test statistic based on ranks, Sukhatme[6] proposed test based on Mann-Whitney-Wilcoxon test and Sukhatme[7] modified this test. Deshpande and Kusum[8] extended the test due to Sukhatme[6] for two-sided alternatives and Kusum[9] proposed a modification to this test. Shetty and Pandit[10] defined a test statistic similar to that of Sukhatme[6] based on U-statistic being function of medians of subsamples of size 3 from each sample. Bhat et al.[11] generalized this test considering subsamples of size a_1 and a_2 respectively from first and second samples. Bhat and Shindhe[12] propose two equivalent classes of tests based on subsample maxima and minima. Taking the disparity of these classes of tests Bhat and Shindhe[13] establish a class of tests that has better performance in terms of Pitman ARE.

In this paper, we suggest a class of tests based on U-statistic depending on subsample median and maximum respectively from first and second samples. The class of tests is defined and its alternative form in terms of ordered ranks is obtained in section 2. The null and asymptotic distributions of the class are obtained in section 3. Section 4 deals with performance of the class of tests in terms of Pitman ARE and empirical power. Section 5 contains illustration of the tests and conclusions.

Suggested class of tests and its alternative form

In this section, we define a class of tests $V(b,d)$ based on a U-Statistic with its kernelbeing function of subsample median and maximum. Subsamples of size b and d are respectively chosen from first and second samples. The median of subsample of size b in the first sample and maximum of subsample of size d in the second sample is considered. The test statistic is a U-statistic that is resistant to $\left(\frac{b-1}{2}\right)$ outliers of the first sample. The class of tests is defined by

$$V(b,d) = \left(\binom{m}{b}\binom{n}{d}\right)^{-1} \sum_{V} \phi(X_1, X_2, \ldots, X_b; Y_1, Y_2, \ldots, Y_d), \qquad (1)$$

Where $\boldsymbol{V}$ denotes the sum over all combinations of X and Y observations,

$$\phi(x_1, x_2, \ldots, x_b; y_1, y_2, \ldots, y_d) = \begin{cases} 1 & if\ 0 < M_{X^+} < Y^+_{(d)},\ x_i, y_j > 0 \\ -1 & if\ Y^-_{(d)} < M_{X^-} < 0,\ x_i, y_j < 0, \\ 0 & Otherwise \end{cases} \qquad (2)$$

$M_{X^+} (M_{X^-})$ is the median of positive (negative) X observations, $Y^+_{(d)}$ $(Y^-_{(d)})$ is the maximum of positive (negative) Y observations ,$i = 1,2, \ldots b$,$j = 1,2, \ldots, d$, $1 \le d \le n$ and b is an odd positive integer.

The class of tests $V(b,d)$ is distribution-free and large values of the test statistics are significant for testing H_0 against H_1.

Following Bhat[14], taking $m = m^+ + m^-$ and $n = n^+ + n^-$, where m^+ (n^+) are number of positive X (Y) observations, $X^+_{(1)} < X^+_{(2)} < \cdots < X^+_{(m^+)}\left(Y^+_{(1)} < Y^+_{(2)} < \cdots < Y^+_{(n^+)}\right)$ are ordered positive observations, m^- (n^-) arenumber of negative X (Y) observations and $X^-_{(1)} < X^-_{(2)} < \cdots < X^-_{(m^-)}(Y^-_{(1)} < Y^-_{(2)} < \cdots < Y^-_{(n^-)})$ are ordered negative observations, we obtain an alternative form of the test statistic defined in (1) in terms of ordered ranks and is given by

$$V^*(b,d) = \binom{m}{b}\binom{n}{d}V(b,d)$$
$$= \sum_{i=1}^{m^+}\sum_{k=0}^{d-1}\binom{i-1}{p}\binom{m^+-i}{p}\binom{R^+_{(i)}-i}{d-k-1}\binom{n-R^+_{(i)}+i}{k+1}$$
$$-\sum_{j=1}^{n^-}\sum_{k=0}^{p}\binom{j-1}{d-1}\binom{S^-_{(j)}-j}{p-k}\binom{m^- - S^-_{(j)}+j}{p+k+1}, \quad (3)$$

where, $R^+_{(i)}\left(S^-_{(j)}\right)$ is rank of $X_i^{th}\left(Y_j^{th}\right)$ observation in the joint rankings of $X^+_1, \ldots, X^+_{m^+}, Y^+_1, \ldots, Y^+_{n^+}(X^-_1, \ldots, X^-_{m^-}, Y^-_1, \ldots, Y^-_{n^-})$ and$p = \frac{b-1}{2}$.

Distributions of $V(b,d)$

In this section, we obtain the null mean, the asymptotic distribution of $V(b,d)$ and the null distribution of $V^*(b,d)$.

The expectation of $V(b,d)$ is given by

$$E[V(b,d)] = P\left(0 < M_{X^+} < Y^+_{(d)}\right) - P\left(Y^-_{(d)} < M_{X^-} < 0\right)$$
$$= 1 - \frac{b!}{(p!)^2}\int_0^\infty (2G(x)-1)^d(2F(x) - 1)^p\left(2\bar{F}(x)\right)^p d\left(2F(x)\right) - \frac{b!}{(p!)^2}\int_0^\infty \left(2G(x)\right)^d\left(2F(x)\right)^p\left(1 - 2F(x)\right)^p d\left(2F(x)\right).$$

Under H_0, the null mean is
$$\mu_0 = 1 - 2\frac{b!}{(p!)^2}B(d+p+1, p+1). \quad (4)$$

Since $V(b,d)$ is two-sample U-statistic with a square integrable kernel, as $n \to \infty$ it has asymptotic normal distribution as its limiting distribution such that $0 < \lambda = \lim_{n\to\infty}\frac{m}{N} < 1$ with mean zero and variance σ^2 due to Lehmann[4].
Here,
$$\sigma^2 = \frac{b^2\zeta_{10}}{\lambda} + \frac{d^2\zeta_{01}}{(1-\lambda)} = \frac{d^2\zeta_{01}}{\lambda(1-\lambda)} \quad (5)$$

where,

$$\zeta_{10} = Cov[\phi(X_1, X_2, \ldots, X_b; Y_1, Y_2, \ldots, Y_d), \phi(X_1, X_{b+1}, \ldots, X_{2b-1}; Y_{d+1}, \ldots, Y_{2d})],$$

$$\zeta_{01} = Cov[\phi(X_1, X_2, \ldots, X_b; Y_1, Y_2, \ldots, Y_d), \phi(X_{b+1}, \ldots, X_{2b}; Y_1, Y_{d+1}, \ldots, Y_{2d-1})]$$

$$= \int_0^\infty P_1^2\, dF(y) + \int_{-\infty}^0 P_2^2 dF(y) - 2\int_0^\infty P_1\, dF(y)\int_{-\infty}^0 P_2\, dF(y) - \mu_0^2,$$

$$P_1 = \sum_{i=p+1}^{b}\binom{b}{i}(2F(y)-1)^{d+i-1}\left(2\bar{F}(y)\right)^{b-i} + (d-1)\sum_{i=p+1}^{b}\binom{b}{i}B(b-i+1, d+i-1)\sum_{k=b-i+1}^{b+d-1}\binom{b+d-1}{k}(2F(y)-1)^{b+d-k-1}\left(2\bar{F}(y)\right)^k,$$

$$P_2 = \sum_{i=0}^{p}\binom{b}{i}\left(2F(y)\right)^{d+i-1}\left(1-2F(y)\right)^{b-i} + (d-1)\sum_{i=0}^{p}\binom{b}{i}B(b-i+1, d+i-1)\sum_{k=b-i+1}^{b+d-1}\binom{b+d-1}{k}\left(2F(y)\right)^{b+d-k-1}\left(1-2F(y)\right)^k$$

and $b^2\zeta_{10} = d^2\zeta_{01}$.

The null distribution of $V^*(b,d)$ for specified values of m, n, m^+, n^+, b and $d$is obtained by generating 10000 random samples from uniform distribution employing Monte-Carlo simulation technique and is presented in Figure-1.

It is observed from the Figure-1 that, the null distribution of $V^*(b,d)$ tends to normal distribution for smaller values of b, d and moderately higher values of m and n.

Performance of the class of tests

In this section, we discuss about the performance of $V(b,d)$ in terms of large and small samples. The large sample performance is studied in terms of Pitman ARE while small sample performance in terms of empirical power.

The performance of the suggested class of tests is compared with tests due to Mood[5] (M), Siegel and Tukey[15] (ST), Deshpande and Kusum[8] (T_1), Kusum[9] (T_2), Shetty and Bhat[16] $(A(3,k))$, Bhat et al.[11] $(U(a_1, a_2))$, Bhat and Shindhe[12] $(B_h(c_1, c_2))$ and Bhat and Shindhe[13] $(S(r_1, r_2))$.

Pitman ARE of $V(b,d)$wrt$B_h(c_1, c_2)$ is given by

$$ARE\left(V(b,d), B_h(c_1, c_2)\right) = \frac{e(V(b,d))}{e(B_h(c_1, c_2))} \quad (6)$$

Where $e\left(V(b,d)\right)$ is given by,

$$e\left(V(b,d)\right) = \frac{\frac{d^2(b!)^2}{(p!)^4}[I_1 - I_2]^2}{\sigma^2}, \quad (7)$$

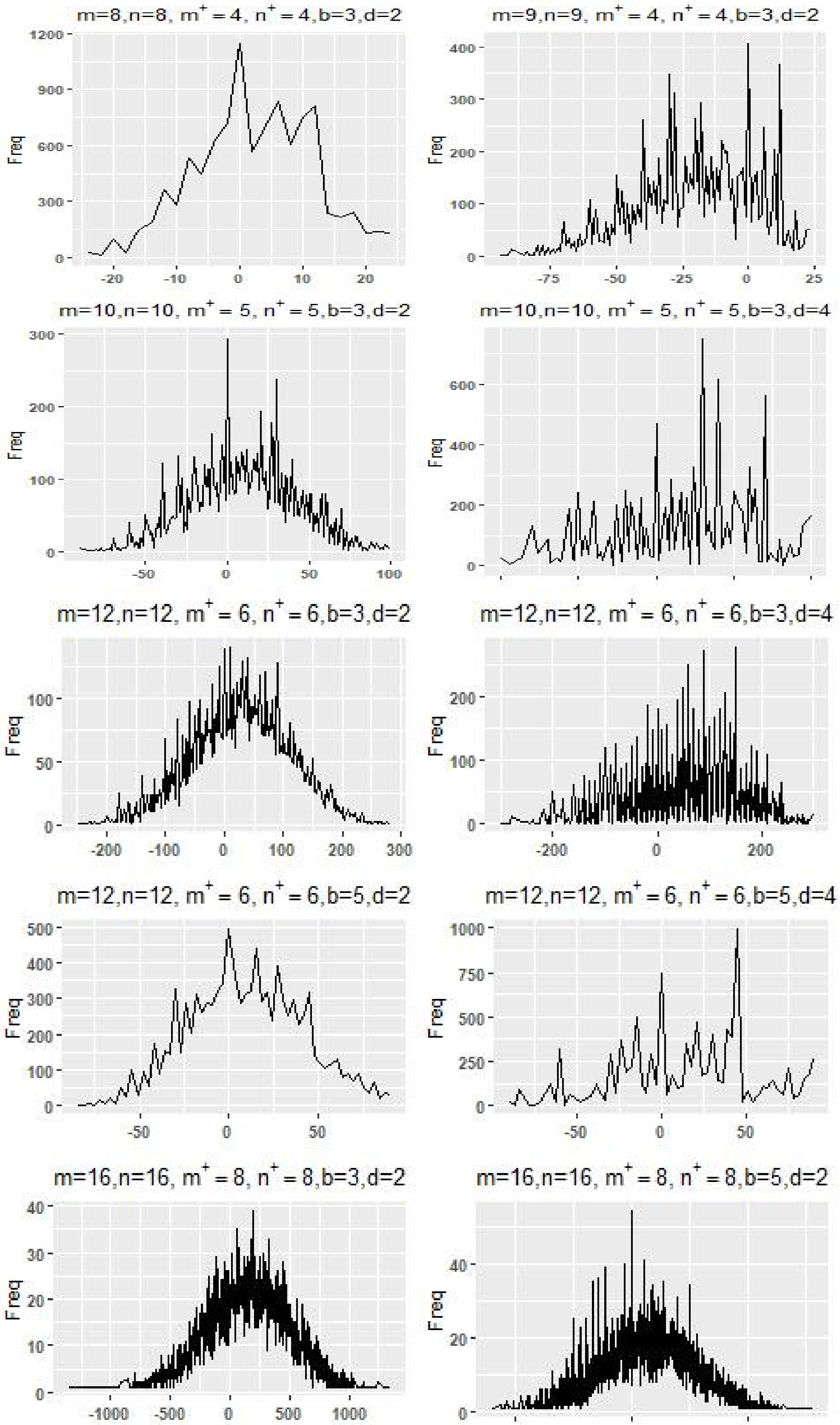

Figure-1: Null distribution of $V^*(b, d)$.

where,

$$I_1 = \int_0^\infty x(2F(x)-1)^{d+p-1}\left(2\bar{F}(x)\right)^p\left(2f(x)\right)^2 dx,$$
$$I_2 = \int_{-\infty}^0 x\left(2F(x)\right)^{d+p-1}\left(1-2F(x)\right)^p\left(2f(x)\right)^2 dx,$$

and $e\left(B_h(c_1,c_2)\right)$ is given in Bhat and Shindhe[12]. The ARE of $V(b,d)$wrtany test T is given by

$$ARE(V(b,d),T) = ARE\left(V(b,d),B_h(c_1,c_2)\right) * ARE(B_h(c_1,c_2),T) \quad (8)$$

The efficacy values for $V(b,d)$ are computed using (7) and are presented in Table-1 of appendix. The values of Pitman ARE of $V(b,d)$ wrt$B_h(c_1,c_2)$, $M, ST, T_1$and T_2, $A(3,k)$ and $U(a_1,a_2)$ respectively are furnished in Tables-2, 3, 4, 5, 6 and 7.

From Table-1, we observe that, for fixed b and increasing d the efficacy values of $V(b,d)$ increase under uniform and exponential distributions. Table-2 shows that, the proposed class of tests outperforms $B_h(c_1,c_2)$ for smaller values of b,d and $c^*(c_1+c_2)$ under uniform, triangular, logistic and Laplace distributions. Also, $ARE\left(V(b,d),S(r_1,r_2)\right) = \frac{1}{2}ARE\left(V(b,d),B_h(c_1,c_2)\right)$ since $ARE\left(B_h(c_1,c_2),S(r_1,r_2)\right) = \frac{1}{2}$.

It is observed from Table-3 and 4 that, $V(b,d)$ is better than M and ST tests for all distributions under consideration. For given b and increasing values of d, ARE of $V(b,d)$ wrt M and ST increases respectively under exponential and uniform distributions.

From Table-5, we observe that $V(b,d)$, outperforms T_1 and T_2 tests under normal, Laplace and Cauchy distributions. The ARE of $V(b,d)$ wrt $T_1$and T_2 tests is decreasing with increasing subsample sizes.

Table-6 shows that, the proposed class of tests is better than $A(3,k)$ when the values of b,d,k are small and the ARE is found to decrease as these values increase when the observations are from uniform, normal and logistic distributions. However, under Laplace distribution the proposed class of tests outperform $A(3,k)$ when ever $k \geq 5$. From Table-7, it is observed that the ARE of proposed class of tests wrt $U(a_1,a_2)$ is increasing with increasing values of $d, a^*(a_1+a_2)$ for a given b and it is decreasing with increasing values of b, a^* for a given d. Also, the proposed class of tests is better than $U(a_1,a_2)$ for light tailed distributions than for medium and heavy tailed distributions.

With regard to small sample performance, empirical power of the proposed class of tests is presented in Table-8 of appendix. The empirical power is computed using the rank expression of $V(b,d)$ given in (3) using Monte-Carlo simulation technique. It is observed from the table that, the empirical power is better when larger subsamples are chosen from moderately larger samples. The small sample performance of the proposed class of tests is better for light tailed distributions and it decreases for thicker tailed distributions for $\sigma > 1.2$. Empirical power under normal distribution is better than other distributions for $\sigma \leq 1.2$.

Conclusion

In this section, we illustrate the application of $V(b,d)$ and furnish concluding remarks. We consider the following example given in Deshpande et al.[17].

Following are the median adjusted annual rainfalls (in cm) at two different stations over a period of 14 and 12 years respectively.

Station 1: 8.155, -13.045, -8.165, 31.155, 9.355, 39.455, -17.655, 3.675, -5.365, -3.675, -22.565, 35.245,5.395, -6.645

Station 2: -18.61, -39.78, -35, 5.42, -5.42, 15.69, -43.38, -17.65, 14.38, 28.47, 25.85, 15.34

Test	*P –value*
$V^*(3,2)$	0.0674
$V^*(3,5)$	0.0350
$V^*(5,2)$	0.1384
$V^*(5,5)$	0.0809
$V^*(7,5)$	0.1142
$U^*(3,3)$	0.1334
M	0.2189
F	0.3261

Here, $U^*(3,3) = \binom{14}{3}\binom{12}{3}U(3,3)$ and is resistant to outliers present in both samples.

The proposed class of tests has lower p-values than F, M and $U^*(3,3)$ tests. Also, p-values are smaller for the members of the class with small subsamples from the first sample and larger subsamples from the second sample.

It can be noted that, there are 3 possible outliers 31.155, 39.455 and 35.245 in the sample pertaining to station 1. Since the proposed class of tests is resistant to outliers in the first sample, it achieves lower p-values than F, M and $U^*(3,3)$ tests showing that $V(b,d)$ is effective when outliers are present in the first sample.

Table-1: Efficacy of $V(b,d)$ for different values of b,d and various distributions.

b	d	Uniform	Triangular	Exponential	Normal	Logistic	Laplace	Cauchy
3	2	3.9773	2.1951	0.3905	2.1495	1.8991	1.3535	0.9659
3	3	4.1250	2.1641	0.4136	2.0770	1.8191	1.2985	0.8984
3	4	4.3794	2.1544	0.4270	2.0127	1.7420	1.2460	0.8258
3	5	4.6847	2.1527	0.4343	1.9533	1.6689	1.1965	0.7563
3	6	5.0184	2.1575	0.4377	1.9006	1.6031	1.1520	0.6942
3	7	5.3690	2.1615	0.4385	1.8495	1.5405	1.1096	0.6381
5	2	3.4483	2.0475	0.3654	2.0620	1.8437	1.3113	0.9753
5	3	3.5065	2.0251	0.3858	2.0160	1.7928	1.2763	0.9306
5	4	3.6296	2.0140	0.3998	1.9716	1.7393	1.2398	0.8775
5	5	3.7884	2.0113	0.4093	1.9313	1.6881	1.2053	0.8241
7	2	3.1780	1.9570	0.3490	2.0002	1.8008	1.2794	0.9756
7	3	3.2055	1.9400	0.3668	1.9681	1.7654	1.2549	0.9438
7	4	3.2766	1.9308	0.3800	1.9364	1.7270	1.2288	0.9043
7	5	3.3741	1.9270	0.3900	1.9062	1.6884	1.2027	0.8621

Table-2: ARE of $V(b,d)$ wrt $B_h(c_1,c_2)$.

b	d	c^*	Uniform	Triangular	Logistic	Laplace
3	2	5	1.5713	5.6568	15.6286	20.6007
		6	1.1299	4.2970	12.3851	16.1787
		7	0.8811	3.5490	10.7055	13.8496
3	3	5	1.6296	5.6568	14.9704	19.7631
		6	1.1719	4.2970	11.8634	15.5209
		7	0.9138	3.5490	10.2547	13.2865
3	5	5	1.8507	5.5477	13.7344	18.2107
		6	1.3309	4.2141	10.8840	14.3017
		7	1.0378	3.4806	9.4080	12.2428
5	2	5	1.3623	5.2765	15.1724	19.9580
		6	0.9796	4.0080	12.0236	15.6739
		7	0.7639	3.3104	10.3931	13.4175
5	3	5	1.3853	5.2188	14.7540	19.4252
		6	0.9962	3.9642	11.6920	15.2555
		7	0.7768	3.2742	10.1064	13.0593
5	5	5	1.4967	5.1833	13.8925	18.3439
		6	1.0763	3.9373	11.0093	14.4064
		7	0.8393	3.2520	9.5163	12.3324
7	2	5	1.2555	5.0433	14.8201	19.4721
		6	0.9028	3.8309	11.7443	15.2924
		7	0.7040	3.1641	10.1517	13.0909
7	3	5	1.2664	4.9994	14.5282	19.1000
		6	0.9107	3.7976	11.5130	15.0002
		7	0.7101	3.1366	9.9518	12.8408
7	5	5	1.3330	4.9659	13.8947	18.3047
		6	0.9586	3.7722	11.0110	14.3756
		7	0.7475	3.1156	9.5178	12.3061

$c^* = c_1 + c_2$.

Table-3: ARE of $V(b,d)$ wrt M.

b	d	Exponential	Normal	Logistic	Laplace	Cauchy
3	2	2.8112	1.4143	2.0379	1.5581	2.0947
3	3	2.9780	1.3666	1.9520	1.4948	1.9484
3	4	3.0740	1.3243	1.8693	1.4344	1.7909
3	5	3.1264	1.2852	1.7909	1.3773	1.6403
5	2	2.6305	1.3567	1.9784	1.5095	2.1152
5	3	2.7775	1.3265	1.9238	1.4692	2.0182
5	4	2.8781	1.2972	1.8664	1.4272	1.9031
5	5	2.9467	1.2708	1.8115	1.3874	1.7873
7	2	2.5127	1.3161	1.9324	1.4728	2.1158
7	3	2.6405	1.2950	1.8944	1.4446	2.0468
7	4	2.7360	1.2741	1.8532	1.4146	1.9611
7	5	2.8074	1.2542	1.8118	1.3845	1.8698

Table-4: ARE of $V(b,d)$ wrt ST.

b	d	Uniform	Normal	Logistic	Laplace	Cauchy
3	2	1.3259	1.7680	1.8142	1.8117	1.9612
3	3	1.3751	1.7083	1.7378	1.7381	1.8242
3	4	1.4599	1.6555	1.6641	1.6678	1.6767
3	5	1.5617	1.6066	1.5943	1.6015	1.5357
5	2	1.1495	1.6960	1.7612	1.7552	1.9803
5	3	1.1689	1.6582	1.7127	1.7083	1.8895
5	4	1.2099	1.6216	1.6615	1.6595	1.7817
5	5	1.2629	1.5885	1.6127	1.6132	1.6733
7	2	1.0594	1.6452	1.7203	1.7125	1.9809
7	3	1.0686	1.6188	1.6865	1.6797	1.9163
7	4	1.0923	1.5927	1.6498	1.6448	1.8361
7	5	1.1248	1.5678	1.6129	1.6098	1.7506

Table-5: ARE of $V(b,d)$ wrt T_1 and T_2.

b	d	T_1			T_2		
		Normal	Laplace	Cauchy	Normal	Laplace	Cauchy
3	2	1.7693	1.8033	1.9635	1.2796	1.4815	2.3766
3	3	1.7096	1.7299	1.8264	1.2365	1.4213	2.2106
3	4	1.6567	1.6600	1.6787	1.1982	1.3639	2.0319
3	5	1.6078	1.5941	1.5376	1.1628	1.3097	1.8611
5	2	1.6972	1.7470	1.9828	1.2275	1.4353	2.3999
5	3	1.6594	1.7004	1.8918	1.2002	1.3970	2.2898
5	4	1.6228	1.6518	1.7839	1.1737	1.3571	2.1592
5	5	1.5897	1.6057	1.6754	1.1497	1.3192	2.0278
7	2	1.6464	1.7045	1.9833	1.1907	1.4004	2.4006
7	3	1.6200	1.6719	1.9186	1.1716	1.3736	2.3223
7	4	1.5939	1.6371	1.8383	1.1528	1.3450	2.2251
7	5	1.5690	1.6023	1.7527	1.1348	1.3164	2.1214

Table-6: ARE of $V(b,d)$ wrt $A(3,k)$.

b	d	k	Uniform	Normal	Logistic	Laplace
3	2	2	1.7045	1.8109	1.8321	0.9076
		5	1.4482	1.6796	1.7241	1.6748
		7	1.1764	1.5371	1.6111	1.6406
3	3	2	1.7678	1.7498	1.7549	0.8707
		5	1.5020	1.6230	1.6515	1.6067
		7	1.2201	1.4853	1.5433	1.5739
3	5	2	1.8768	1.6957	1.6805	0.8355
		5	1.5946	1.5727	1.5814	1.5418
		7	1.2953	1.4393	1.4778	1.5103
5	2	2	2.0077	1.6456	1.6101	0.8023
		5	1.7058	1.5263	1.5151	1.4805
		7	1.3856	1.3968	1.4159	1.4503
5	3	2	1.4778	1.7372	1.7786	0.8792
		5	1.2556	1.6112	1.6738	1.6225
		7	1.0199	1.4745	1.5641	1.5894
5	5	2	1.5027	1.6985	1.7296	0.8558
		5	1.2768	1.5753	1.6276	1.5792
		7	1.0371	1.4417	1.5210	1.5470
7	2	2	1.5555	1.6610	1.6779	0.8313
		5	1.3216	1.5406	1.5790	1.5341
		7	1.0735	1.4099	1.4755	1.5028
7	3	2	1.6236	1.6271	1.6286	0.8081
		5	1.3794	1.5091	1.5326	1.4913
		7	1.1205	1.3811	1.4322	1.4609
7	5	2	1.3620	1.6851	1.7373	0.8578
		5	1.1572	1.5630	1.6349	1.5830
		7	0.9400	1.4303	1.5278	1.5507

Table-7: ARE of $V(b, d)$ wrt $U(a_1, a_2)$.

b	d	a^*	Uniform	Triangular	Exponential	Normal	Logistic	Laplace
3	2	6	1.1278	1.0482	1.1646	1.0193	1.0072	1.0092
		8	1.2315	1.1036	1.2012	1.0573	1.0376	1.0409
		10	1.3051	1.1443	1.2295	1.0864	1.0613	1.0655
3	3	6	1.1696	1.0335	1.2337	0.9849	0.9647	0.9682
		8	1.2772	1.0880	1.2725	1.0217	0.9939	0.9986
		10	1.3536	1.1282	1.3025	1.0497	1.0166	1.0222
3	5	6	1.3283	1.0280	1.2952	0.9262	0.8851	0.8922
		8	1.4505	1.0823	1.3359	0.9608	0.9118	0.9202
		10	1.5372	1.1222	1.3674	0.9872	0.9327	0.9419
5	2	6	0.9778	0.9778	1.0897	0.9778	0.9778	0.9778
		8	1.0677	1.0294	1.1240	1.0143	1.0073	1.0085
		10	1.1315	1.0673	1.1505	1.0421	1.0303	1.0323
5	3	6	0.9943	0.9671	1.1507	0.9560	0.9508	0.9517
		8	1.0857	1.0181	1.1868	0.9917	0.9795	0.9815
		10	1.1506	1.0557	1.2148	1.0189	1.0019	1.0047
5	5	6	1.0742	0.9605	1.2207	0.9158	0.8953	0.8987
		8	1.1730	1.0112	1.2591	0.9500	0.9223	0.9269
		10	1.2431	1.0485	1.2888	0.9761	0.9434	0.9488
7	2	6	0.9011	0.9345	1.0410	0.9485	0.9551	0.9540
		8	0.9840	0.9839	1.0736	0.9839	0.9839	0.9839
		10	1.0428	1.0202	1.0990	1.0109	1.0064	1.0071
7	3	6	0.9089	0.9264	1.0939	0.9332	0.9362	0.9357
		8	0.9925	0.9754	1.1282	0.9681	0.9645	0.9651
		10	1.0519	1.0113	1.1549	0.9947	0.9866	0.9879
7	5	6	0.9567	0.9202	1.1631	0.9039	0.8954	0.8968
		8	1.0447	0.9688	1.1996	0.9377	0.9225	0.9249
		10	1.1072	1.0045	1.2279	0.9634	0.9436	0.9468

$a^* = a_1 + a_2$.

Table-8: Empirical power of $V^*(b,d)$ for different values of m, n, m^+, n^+, b, d and various distributions for 10% level of significance.

m	n	m^+	n^+	b	d	Distribution	1.2	1.5	2	2.5	3	4	5
8	8	4	4	3	2	Uniform	0.1341	0.1747	0.2060	0.2344	0.2381	0.2595	0.2715
						Normal	0.1429	0.1758	0.2125	0.2270	0.2439	0.2563	0.2597
						Logistic	0.1471	0.1658	0.1923	0.2224	0.2279	0.2501	0.2564
						Laplace	0.1376	0.1594	0.1839	0.1946	0.2000	0.2242	0.2387
						Cauchy	0.1347	0.1433	0.1448	0.1581	0.1589	0.1793	0.1787
10	10	5	5	3	4	Uniform	0.1224	0.1709	0.2097	0.2320	0.2627	0.2904	0.3001
						Normal	0.1465	0.1785	0.2214	0.2463	0.2649	0.2934	0.3052
						Logistic	0.1409	0.1602	0.2109	0.2247	0.2450	0.2683	0.2789
						Laplace	0.1310	0.1561	0.1807	0.2048	0.2228	0.2459	0.2605
						Cauchy	0.1286	0.1392	0.1465	0.1548	0.1555	0.1800	0.1869
14	14	7	7	3	4	Uniform	0.1003	0.1615	0.1993	0.2429	0.2597	0.2995	0.3159
						Normal	0.1250	0.1655	0.2030	0.2380	0.2662	0.2974	0.2997
						Logistic	0.1241	0.1459	0.1947	0.2208	0.2439	0.2747	0.2792
						Laplace	0.1169	0.1345	0.1823	0.1902	0.2020	0.2289	0.2472
						Cauchy	0.1050	0.1160	0.1294	0.1458	0.1498	0.1533	0.1637
14	14	7	7	3	6	Uniform	0.1041	0.1609	0.2074	0.2268	0.2473	0.2743	0.2918
						Normal	0.1251	0.1589	0.1996	0.2165	0.2383	0.2445	0.2517
						Logistic	0.1230	0.1484	0.1787	0.2013	0.2120	0.2335	0.2432
						Laplace	0.1184	0.1385	0.1651	0.1852	0.2006	0.2083	0.2134
						Cauchy	0.1123	0.1151	0.1204	0.1291	0.1311	0.1404	0.1499
14	14	7	7	5	4	Uniform	0.1058	0.1485	0.1835	0.2160	0.2290	0.2599	0.2829
						Normal	0.1211	0.1505	0.1861	0.2074	0.2271	0.2580	0.2530
						Logistic	0.1135	0.1426	0.1700	0.2014	0.2142	0.2369	0.2431
						Laplace	0.1125	0.1345	0.1601	0.1700	0.1879	0.2066	0.2221
						Cauchy	0.1040	0.1108	0.1169	0.1256	0.1395	0.1343	0.1447
14	14	7	7	5	6	Uniform	0.1446	0.2004	0.2499	0.2783	0.3040	0.3233	0.3579
						Normal	0.1788	0.2077	0.2496	0.2772	0.3068	0.3332	0.3466
						Logistic	0.1752	0.1939	0.2338	0.2669	0.2814	0.3082	0.3209
						Laplace	0.1668	0.1790	0.2060	0.2359	0.2503	0.2869	0.2964
						Cauchy	0.1536	0.1660	0.1715	0.1836	0.1898	0.2062	0.2118

Based on the study, we conclude that the suggested class of tests, $V(b,d)$: i. is distribution-free and its large values are significant for testing H_0 against H_1. ii. is resistant to $\frac{b-1}{2}$ outliers in the first sample. iii. has asymptotic normal distribution as its limiting distribution with mean μ_0 and variance σ^2. iv. Outperforms M, ST, T_1 and T_2 tests for all values of b, d under all distributions considered. v. Outperforms $A(3,k)$ under uniform, normal and logistic distributions for smaller values of b, d and $k$and under Laplace distribution for smaller values of b, d and larger values of k. vi. Outperforms $U(a_1, a_2)$ under exponential distribution for all values of b, d and a^* considered whereas it is better than $U(a_1, a_2)$ under uniform, triangular, normal, logistic and Laplace distributions when b, d are small and $a^*$is large. vii. outperforms $B_h(c_1, c_2)$ under triangular, logistic and Laplace distributions for all values of b, d and c^* considered. It is better than $B_h(c_1, c_2)$ under uniform distribution when values of b, d and c^* are small. viii. has higher empirical power when observations are from light tailed distribution for moderately larger sample sizes and $\sigma > 1.2$ whereas the empirical power is lower when $\sigma \leq 1.2$. ix. achieves lower p-values than F, M and $U^*(3,3)$ tests when there are outliers in the first sample.

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