# Topp-Leone compound Rayleigh distribution: properties and applications 

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#### Abstract

In this research, we introduce an extension form of compound Rayleigh distribution called Topp-Leone compound Rayleigh distribution; we also investigate its basic and important properties in which mean, variance, expression of coefficient of variation, rth moments, incomplete moments, skewness, Moors kurtosis, reliability properties, inequality measures, probability weighted moments, Reny entropy are include. We obtain the maximum likelihood estimates for the parameters of proposed distribution. In the end three applications with real data sets are presented and notice that new model is better than its baseline distribution.


Keywords: Topp-Leone, compound Rayleigh, maximum likelihood estimation, hazard rate function.

## Introduction

According to the literature review, we have come to know that compound Rayleigh (C.R) distribution is commonly used for the purpose of reliability analysis particularly in biology for medicine experiment. No of Statisticians/Mathematicians used this model to obtain its salient features and applications in daily life. Here we mention few of them, Aboushal ${ }^{1}$, Shajee et al. ${ }^{2}$, Barot and Patel ${ }^{3}$ are preformed all most same work with little bit difference or technique. All of these applied Bayes approach to estimate the unknown parameters of the distribution, reliability and hazard rate function. Abushal based his work on new censoring scheme so called first censoring scheme given by Wu and Kus ${ }^{4}$. While Shajee et al. associated their work with record values data. Barot and Patel applied both classical as well as Bayes approach to estimate the parameters. They obtained Bayes estimators under BLGLF and BGELF and conclude that the perfomance of BGELF is considered better as compare to other BLGLF. Similarly Abd-Elmougod and Mahamoud ${ }^{5}$, used adaptive type-II progressive hybrid censoring samples for constant partially life tests. Both approximate and Bootstrap confidence intervals are obtained and compared. Rayed and Othman ${ }^{6}$ studied this distribution with different angle they presented Beta compound Rayleigh distribution; they also obtained its mathematical properties of the distribution. Reyad et $\mathrm{al}^{7}$. introduced an extension in compound Rayleigh distribution called Kumaraswamy compound Rayleigh distribution ( KwCR ) and also obtained its mathematical properties. Cdf and pdf of the distribution are
$G(x ; \alpha, \beta)=1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}, \quad x>0, \alpha, \beta>0$
Where $\alpha$ is a scale parameter while $\beta$ is shape parameter, pdf of equation (1) is
$g(x ; \alpha, \beta)=2 \alpha \beta^{\alpha} x\left(\beta+x^{2}\right)^{-(\alpha+1)}$

## Topp-Leone compound Rayleigh Distribution

Topp and Leone ${ }^{8}$ introduced a bounded J-Shaped distribution because of closed form of its cdf it is preferred over Beta distribution. Nadarajah and $\operatorname{Kotz}^{9}$ provided its cdf and pdf as
$F_{T L}(x)=x^{\theta}(2-x)^{\theta}$
and
$f_{T L}(x)=2 \theta x^{\theta-1}(1-x)(2-x)^{\theta-1} \quad, 0 \leq x \leq 1$
With shape parameter $\theta>0$.
Using the cdf of TL distributin Al-Shomrani et al ${ }^{10}$. proposed a new family of distribution. The cdf and pdf of new generalized model given by Al Shomrani et al ${ }^{10}$. Are
$F_{T L G}=(G(x))^{\theta}(2-G(x))^{\theta} \quad x \in R, \theta>0$
With density
$f_{T L G}=2 \theta g(x) \bar{G}(x)(\bar{G}(x))^{\theta-1}(2-G(x))^{\theta-1}$
Where $g(x)$ is the pdf of baseline distribution and $G(x)$ is cdf of that distribution. More reduced form of (3) is
$F_{T L G}(x)=\left[1-(\bar{G}(x))^{2}\right]^{\theta}$
And modified form of (4) is
$f_{T L G}=2 \theta g(x) \bar{G}(x)\left\{1-(\bar{G}(x))^{2}\right\}^{\theta-1}$
Because $\bar{G}(x)=1-G(x)$

The object of this research is to develop new three parametric model called Topp-Leone compound Rayleigh distribution or (TLCR for short) for this purpose we insert $\bar{G}(x)$ of (1) in (5), that is the cdf of TLCR model.

$$
\begin{equation*}
F_{T L C R}(x)=\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta} \tag{7}
\end{equation*}
$$

Similarly density of eq (7) can be obtained by inserting $\bar{G}(x)$ of eq (1) and eq (2) in eq (6)
$f_{T L C R}(x)=4 \alpha \theta \beta^{2 \alpha}\left(\beta+x^{2}\right)^{-(2 \alpha+1)}\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta-1}$
With two shape parameters $\beta$ and $\theta$ while $\alpha$ is a scale parameter.


Figure-1: Plots of TLCR density.


Figure-2: Plots of $\mathrm{h}(\mathrm{x})$ function.
Expansion of TLCR Pdf: To find more results we need an expansion form of the TLCR distribution, from (8) we consider.
$\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta-1}=\sum_{j=0}^{\infty} \frac{(-1)^{j} \Gamma \theta}{j!\Gamma \theta-j} \beta^{2 \alpha j}\left(\beta+x^{2}\right)^{-2 \alpha j}(9)$ Inserting (9) in (8) TLCR density takes the form
$f_{T L C R}(x)=2 \beta^{2 \alpha(1+j)} \sum_{j=0}^{\infty} \frac{(-1)^{j} \Gamma \theta}{j!\Gamma \theta-j} 2 \alpha \theta x\left(\beta+x^{2}\right)^{-[2 \alpha(1+j)+1]}$
Finally above equation can be written as
$f_{T L C R}(x)=2 \beta^{2 \alpha(1+j)} \sum_{j=0}^{\infty} W_{j} x\left(\beta+x^{2}\right)^{-[2 \alpha(1+j)+1]}$

## Quantile function

Quantile function or $q(f)$ can easily be obtained by inverting equation (7), this function helps to generate random numbers and other important measures.

$$
\begin{equation*}
x_{q}=\left[\left\{\frac{1}{\beta^{\alpha}}\left(1-u^{\frac{1}{\theta}}\right)^{1 / 2}\right\}^{-1 / \alpha}-\beta\right]^{\frac{1}{2}} \tag{11}
\end{equation*}
$$

Where $u \sim$ uniform distribution $u(0,1)$.
One can obtain $1^{\text {st }}\left(Q_{1}\right), 2^{\text {nd }}\left(Q_{2}\right)$ and $3^{\text {rd }}\left(Q_{3}\right)$ quartiles of the distribution from eq (11)
$x_{0.25}=Q_{1}=\left[\left\{\frac{1}{\beta^{\alpha}}\left(1-0.25^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right\}^{-\frac{1}{\alpha}}-\beta\right]^{\frac{1}{2}}$,
$x_{0.5}=Q_{2}=\left[\left\{\frac{1}{\beta^{\alpha}}\left(1-0.5^{\frac{1}{\theta}}\right)^{1 / 2}\right\}^{-1 / \alpha}-\beta\right]^{\frac{1}{2}}$,
$x_{0.75}=Q_{2}=\left[\left\{\frac{1}{\beta^{\alpha}}\left(1-0.75^{\frac{1}{\theta}}\right)^{1 / 2}\right\}^{-1 / \alpha}-\beta\right]^{\frac{1}{2}}$
Similarly Q.D. (Quartile deviation) and coefficient of Skewness (S.K) by quartile method.
$Q . D=\frac{x_{0.75}-x_{0.25}}{2} \quad, \quad S . K=\frac{x_{0.75}+x_{0.25}-2 x_{0.5}}{x_{0.75}-x_{0.25}}$
and Kurtosis given by Moors ${ }^{11}$ is
$M=\frac{x_{3 / 8}-x_{1 / 8}+x_{7 / 8}-x_{5 / 8}}{x_{6 / 8}-x_{2 / 8}}$
$\mathbf{r}^{\text {th }}$ Moments: The $\mathrm{r}^{\text {th }}$ moments of TLCR distribution of random variable x is defined as
$\mu_{r}^{\prime}=2 \beta^{2 \alpha(1+j)} \sum_{j=0}^{\infty} W_{j} \int_{0}^{\infty} x^{r+1}\left(\beta+x^{2}\right)^{-[2 \alpha(1+j)+1]} d x$
$=\frac{2}{\beta} \sum_{j=0}^{\infty} W_{j} \int_{0}^{\infty} x^{r+1}\left(1+x^{2} / \beta\right)^{-[2 \alpha(1+j)+1]} d x$
Substituting $y=\left(1+x^{2} / \beta\right)^{-1}$, then we obtain
$\mu_{r}^{\prime}=\beta^{\frac{r}{2}} \sum_{j=0}^{\infty} W_{j} \int_{0}^{1} y^{2 \alpha(1+j)-\frac{r}{2}-1}(1-y)^{\frac{r}{2}+1-1} d y$
Finally
$\mu_{r}^{\prime}=\beta^{\frac{r}{2}} \sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{r}{2}, \frac{r}{2}+1\right)$
Where $B(a, b)=\int_{0}^{1} x^{a-1}(1-x)^{b-1} d x$ is the beta function of $1^{\text {st }}$ kind. After putting the values of $r$ we obtain the mean and variance of TLCR distribution as
$\mu_{1}^{\prime}=\beta^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{1}{2}, \frac{3}{2}\right)$
Variance of the distribution
$\mu_{2}=\beta\left[\sum_{j=0}^{\infty} W_{j} B(2 \alpha(1+j)-1,2)-\left\{\sum_{j=0}^{\infty} W_{j} B(2 \alpha(1+\right.\right.$ j) $\left.\left.\left.-\frac{1}{2}, \frac{3}{2}\right)\right\}^{2}\right]$
and Coefficient of variation is easily obtained from (13) and (14)
$C . V=\frac{\sqrt{\beta\left[\sum_{j=0}^{\infty} W_{j} B(2 \alpha(1+j)-1,2)-\left\{\sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{1}{2}, \frac{3}{2}\right)\right\}^{2}\right]}}{\beta^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{1}{2}, \frac{3}{2}\right)}$

## Incomplete Moments

The $r^{t h}$ incomplete moments of the distribution denoted by $m_{r}(y)$ are obtained as

$$
\begin{aligned}
m_{r}(y)=\int_{0}^{y} x^{r} f & (x) d x \\
& =\frac{2}{\beta} \sum_{j=0}^{\infty} W_{j} \int_{0}^{y} x^{r+1}(1 \\
& \left.+x^{2} / \beta\right)^{-[2 \alpha(1+j)+1]} d x
\end{aligned}
$$

Let $z=\left(1+x^{2} / \beta\right)^{-1}$ then we get
$m_{r}(y)=\sum_{j=0}^{\infty} W_{j} \beta^{\frac{r}{2}} \int_{\left(1+\frac{y^{2}}{\beta}\right)^{-1} z^{2 \alpha(1+j)-\frac{r}{2}-1}(1-z)^{\frac{r}{2}} d z, ~}^{1}$

By binomial theorem we obtain
$(1-z)^{\frac{r}{2}}=\sum_{k=0}^{\frac{r}{2}}\binom{\frac{r}{2}}{k}(-1)^{\mathrm{k}} \mathrm{z}^{\mathrm{k}}$
Replacing above result in equation (15) and we get
$=\beta^{\frac{r}{2}} \sum_{k=0}^{\frac{r}{2}} \sum_{j=0}^{\infty} W_{j}\binom{\frac{r}{2}}{k}(-1)^{\mathrm{k}} \int_{\left(1+\frac{y^{2}}{\beta}\right)^{-1} z^{2 \alpha(1+j)-\frac{r}{2}+k-1} d z}^{1}$
After simple integration, the final result is
$m_{r}(y)=\beta^{\frac{r}{2}} \sum_{k=0}^{\frac{r}{2}} \sum_{j=0}^{\infty} W_{j}\binom{\frac{r}{2}}{k}(-1)^{\mathrm{k}}\left[\frac{1-\left(1+\frac{y^{2}}{\beta}\right)^{\frac{r}{2}-[2 \alpha(1+j)-k]}}{2 \alpha(1+j)-\frac{r}{2}+k}\right]$

## Mean Deviation from Mean

The amount of variation in a random variable can be measured by Mean deviation, for continuous random variable it is defined as
$\delta_{1}(x)=\int_{0}^{\infty}|x-\mu| f(x) d x$
$\mu=E(x)$ equation (13), after simplification (17) takes the form
$\delta_{1}(x)=2\left\{\mu F(\mu)-\int_{0}^{\mu} x f(x) d x\right\}$
$\int_{0}^{\mu} x f(x) d x$ It is nothing but the incomplete moment then by equation (16) just replacing $y^{2}=\mu^{2}$ and putting $r=1$

$$
\begin{align*}
& \int_{0}^{\mu} x f(x) d x= \\
& \beta^{\frac{1}{2}} \sum_{k=0}^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j}\binom{\frac{1}{2}}{k}(-1)^{\mathrm{k}}\left[\frac{1-\left(1+\frac{\mu^{2}}{\beta}\right)^{\frac{1}{2}-[2 \alpha(1+j)-k]}}{2 \alpha(1+j)-\frac{1}{2}+k}\right] \tag{19}
\end{align*}
$$

Inserting (19) in (18) and we obtain the mean deviation of TLCR distribution
$\delta_{1}(x)=$
$2\left\{\mu F(\mu)-\beta^{\frac{1}{2}} \sum_{k=0}^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j}\binom{\frac{1}{2}}{k}(-1)^{\mathrm{k}}\left[\frac{1-\left(1+\frac{\mu^{2}}{\beta}\right)^{\frac{1}{2}-[2 \alpha(1+j)-k]}}{2 \alpha(1+j)-\frac{1}{2}+k}\right]\right\}$

## Reliability Properties

Reliability is a waste field, take a lot of time to study but for introduction here we obtain some important functions which are commonly used such as Reliability or survival function $s(x)$, Hazard rate function $h(x)$, Reverse Hazard rate function $r(x)$, Cumulative Hazard rate function $H(x)$, Mean residual life and Mean waiting time are include.
$S(x)=1-F(x)=1-\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta}$
$h(x)=\frac{f(x)}{S(x)}=\frac{4 \alpha \theta \beta^{2 \alpha}\left(\beta+x^{2}\right)^{-(2 \alpha+1)}\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta-1}}{1-\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta}}$
$r(x)=\frac{f(x)}{F(x)}=\frac{4 \alpha \theta \beta^{2 \alpha}\left(\beta+x^{2}\right)^{-(2 \alpha+1)}}{\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]}$
$H(x)=-\ln S(x)=-\ln \left[1-\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta}\right]$

## Mean Residual Life

The function denoted by $m(x)$ can be obtained as
$m(x)=\frac{1}{s(x)} \int_{x}^{\infty} x f(x) d x-x$
Considering for TLCR distribution
$\int_{x}^{\infty} x f(x) d x=\frac{2}{\beta} \sum_{j=0}^{\infty} W_{j} \int_{x}^{\infty} x^{2}\left(1+x^{2} / \beta\right)^{-[2 \alpha(1+j)+1]} d x$
Assuming $y=\left(1+x^{2} / \beta\right)^{-1}$ then above equation takes the form
$=\sum_{j=0}^{\infty} W_{j} \beta^{\frac{1}{2}} \int_{0}^{\left(1+\frac{x^{2}}{\beta}\right)^{-1}} y^{2 \alpha(1+j)-\frac{1}{2}}(1-y)^{\frac{1}{2}} d y=$
$\sum_{j=0}^{\infty} W_{j} \beta^{\frac{1}{2} B}{ }_{\left(1+\frac{x^{2}}{\beta}\right)^{-1}}\left(2 \alpha(1+j)-\frac{1}{2}, \frac{3}{2}\right)$
Where $B_{\left(1+x^{2} / \beta\right)^{-1}}$ is the incomplete beta function, after inserting (21) and above equation in (22) we obtained the MRL of the TLCR distribution.
$\left.m(x)=\frac{\sum_{j=0}^{\infty} W_{j} \beta^{\frac{1}{2}} B}{1-\left[1+\frac{x^{2}}{\beta}\right)^{-1}\left(2 \alpha(1+j)-\frac{1}{2^{\prime}} \frac{3}{2}\right)} \beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta}-x$

## Mean Waiting Time

To the given interval of time an item/ individual fails to its performance is measured by Mean waiting time, function is defined as
$\bar{\mu}(x, \theta)=x-\left\{\frac{1}{F(x)} \int_{0}^{x} x f(x) d x\right\}$
$\int_{0}^{x} x f(x) d x=\frac{\beta^{\frac{1}{2}} \sum_{k=0}^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j}\binom{\frac{1}{2}}{k}(-1)^{\mathrm{k}}\left[\left[1-\left(1+\frac{y^{2}}{\beta}\right)^{\frac{1}{2}-[2 \alpha(1+j)-k]}\right]\right.}{2 \alpha(1+j)-\frac{1}{2}+k}$

Inserting above equation and (7) in (23) we obtain the Mean Waiting Time of TLCR distribution.
$\bar{\mu}(x, \theta)$
$=x-\frac{\beta^{\frac{1}{2}} \sum_{k=0}^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j}\binom{\frac{1}{2}}{k}(-1)^{\mathrm{k}}\left[1-\left(1+\frac{y^{2}}{\beta}\right)^{\frac{1}{2}-[2 \alpha(1+j)-k]}\right]}{\left\{2 \alpha(1+j)-\frac{1}{2}+k\right\}\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]^{\theta}}$

## Inequality Measures

To obtain income inequality measures or inequality measures is an important aspect of TLCR model. These unique concepts make this model useful. Apart from economical point of view these measures are helpful in demographical research and insurance companies also use them. Here we discuss some useful measures.

Lorenz Curve: In honour of M. O. Lorenz ${ }^{12}$, this inequality measure called Lorenz curve which is defined as
$L(p)=\frac{1}{\mu} \int_{0}^{x} x f(x) d x$
Inserting (24) and (13) we obtain the $L(p)$ of the TLCR distribution
$L(p)=\frac{\sum_{k=0}^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j}\left(\frac{1}{2}\right)(-1)^{\mathrm{k}}\left[1-\left(1+\frac{y^{2}}{\beta}\right)^{\frac{1}{2}-[2 \alpha(1+j)-k]}\right]}{\left(2 \alpha(1+j)-\frac{1}{2}+k\right)\left\{\sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{1}{2}, \frac{3}{2}\right)\right\}}$

Bonferroni Curve: By simple ratio of Lorenz curve and cdf of the distribution one can be obtained Bonferroni index, introduced by Bonferroni ${ }^{13}$.
$B C(p)=\frac{L(p)}{F(x)}$
Equation (26) and (7) provide us $B C(p)$ of the TLCR distribution
$B C(p)=$
$\frac{1}{\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right] \theta}\left\{\frac{\sum_{k=0}^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j}\left(\frac{1}{2}\right)(-1)^{\mathrm{k}}\left[1-\left(1+\frac{y^{2}}{\beta}\right)^{\frac{1}{2}-[2 \alpha(1+j)-k]}\right]}{\left(2 \alpha(1+j)-\frac{1}{2}+k\right)\left\{\sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{1}{2}, \frac{3}{2}\right)\right\}}\right\}$
Pietra Index: Pietra ${ }^{14}$ introduced an income inequality measure which is defined as.
$P_{x}=\frac{\delta_{1}(x)}{2 \mu}=\frac{\left\{\mu F(\mu)-\beta^{\frac{1}{2}} \sum_{k=0}^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j}\left(\frac{1}{2}\right)(-1)^{\mathrm{k}}\left[1-\left(1+\frac{\mu^{2}}{\beta}\right)^{\frac{1}{2}-[2 \alpha(1+j)-k]}\right]\right\}}{\left(2 \alpha(1+j)-\frac{1}{2}+k\right) \beta^{\frac{1}{2}} \sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{1}{2}, \frac{3}{2}\right)}$
Equation (20) and (13) provide us the above result.

Generalized Entropy: Cowell and Shorrocks ${ }^{15}$ introduced the generalized entropy (GE) index
$G E_{F}(\psi, \lambda)=\frac{1}{\lambda(\lambda-1) \mu^{\lambda}} \int_{0}^{\infty} x^{\lambda} f(x) d x-1$
Where $\mu$ is the mean of the distribution and here we have the equation (13) and
$\int_{0}^{\infty} x^{\lambda} f(x) d x=\beta^{\frac{\lambda}{2}} \sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{\lambda}{2}, \frac{\lambda}{2}+1\right)$
Then using (28) and (13) we get the equation (27)
$G E_{F}(\psi, \lambda)=\frac{\sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{\lambda}{2}, \frac{\lambda}{2}+1\right)}{\lambda(\lambda-1)\left\{\sum_{j=0}^{\infty} W_{j} B\left(2 \alpha(1+j)-\frac{1}{2}, \frac{3}{2}\right)\right\}^{\lambda}}-1$
Rényi Entropy: By entropy we mean an amount of measure of variation of uncertainty in a random variable. Here we drive explicit expression for Rényi entropy which was introduced by Rény ${ }^{16}$. For continuous distribution it is defined as
$I_{R}(\gamma)=\frac{1}{1-\gamma} \log \left\{\int f^{\gamma}(x) d x\right\}$
Where $\gamma>0$ and $\gamma \neq 1$., Consider for TLCR distribution
$\int_{0}^{\infty} f_{T L C R}{ }^{\gamma}(x) d x$
$=2^{2 \alpha-1} \beta^{-\gamma}(\alpha \theta)^{\gamma} \sum_{j=0}^{\gamma(\theta-1)}(-1)^{j}\binom{\gamma(\theta-1}{j} \int_{0}^{\infty} x^{\gamma}(1+$
$\left.\frac{x^{2}}{\beta}\right)^{-[2 \alpha(\gamma+j)+\gamma]} d x$
Supposing the $y=\left(1+\frac{x^{2}}{\beta}\right)^{-1}$
$=$
$2^{2 \alpha-1} \beta^{\frac{3 \gamma-1}{2}}(\alpha \theta)^{\gamma} \sum_{j=0}^{\gamma(\theta-1)}(-1)^{j}\binom{\gamma(\theta-1}{j} \int_{0}^{1} y^{2 \alpha(\gamma+j)+\gamma-\frac{(\gamma+1)}{2}-1}(1-r$ we can obtain the moments.
$y)^{\frac{\gamma}{2}-\frac{1}{2}} d y$
Finally the above equation takes the form

$$
\begin{gathered}
=2^{2 \alpha-1} \beta^{\frac{3 \gamma-1}{2}}(\alpha \theta)^{\gamma} \sum_{j=0}^{\gamma(\theta-1)}(-1)^{j}\binom{\gamma(\theta-1}{j} B(2 \alpha(\gamma+j)+\gamma \\
\left.-\frac{(\gamma+1)}{2}, \frac{(\gamma+1)}{2}\right)
\end{gathered}
$$

Inserting the above equation in (29), we get

$$
\begin{gathered}
I_{R}(\gamma)=\frac{1}{1-\gamma} \log \left\{2^{2 \alpha-1} \beta^{\frac{3 \gamma-1}{2}}(\alpha \theta)^{\gamma} \sum_{j=0}^{\gamma(\theta-1)}(-1)^{j}\binom{\gamma(\theta-1}{j} B(2 \alpha(\gamma\right. \\
\left.\left.+j)+\gamma-\frac{(\gamma+1)}{2}, \frac{(\gamma+1)}{2}\right)\right\}
\end{gathered}
$$

## Probability Weighted Moments

Probabilities weighted moments are used to estimate the parameters of the distribution introduced by Landwehr et al. and Greenwood et al ${ }^{17}$ for continuous distribution, moments obtained as
$\beta_{r}=\int x\{F(x)\}^{r} f(x) d x$
Where $F(x)$ and $f(x)$ are cdf and pdf respectively of the given distribution. For TLCR distribution we consider
$\int_{0}^{\infty} x\{F(x)\}^{r} f(x) d x$
$=\frac{4 \alpha \theta}{\beta} \sum_{j=0}^{\theta(r+1)-1}(-1)^{j}\binom{\theta(r+1)-1}{j} \int_{0}^{\infty} x^{2}(1$
$\left.+\frac{x^{2}}{\beta}\right)^{-[2 \alpha(1+j)+1]} d x$
Suppose that $y=\left(1+x^{2} / \beta\right)^{-1}$ then above equation can take the form as

$$
\begin{aligned}
& =\beta^{\frac{1}{2}} \sum_{j=0}^{\theta(r+1)-1}(-1)^{j}\binom{\theta(r+1)-1}{j} \int_{0}^{1} y^{2 \alpha(1+j)+\frac{1}{2}-1}(1 \\
& -y)^{\frac{r}{2}} d y
\end{aligned} \quad \begin{gathered}
\beta_{r}=\beta^{\frac{1}{2}} \sum_{j=0}^{\theta(r+1)-1}(-1)^{j}\binom{\theta(r+1)-1}{j} B\left(2 \alpha(1+j)+\frac{1}{2}, \frac{r}{2}\right. \\
+1)
\end{gathered}
$$

## Maximum Likelihood Estimates

Considering $X \sim \operatorname{TLCR}(\alpha, \beta, \theta)$ distribution and let $x_{1}, x_{2}, \ldots, x_{n}$ be a random sample of size $n$ drawn from the distribution then $\log$ likelihood function $l$ is given by $l=n \ln 4+n \ln \alpha+$ $n \ln \theta+$
$n \ln \beta+2 n \alpha \ln \beta+\sum_{i=0}^{n} \log x_{i}-(2 \alpha+1) \sum_{i=0}^{n} \ln \left(\beta+x_{i}^{2}\right)$
$+(\theta-1) \sum_{i=0}^{n} \ln \left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]$
Now differentiating the above equation with respect to $\alpha, \beta$ and $\theta$, we get the following equations.
$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}+2 n \ln \beta-2 \sum_{i=0}^{n} \ln \left(\beta+x_{i}^{2}\right)$
$-(\theta-1) \sum_{i=0}^{n}\left[\frac{2 \beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\left\{\ln \beta\left(\beta+x^{2}\right)\right\}}{\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]}\right]$
$\frac{\partial l}{\partial \beta}=\frac{n}{\beta}+\frac{2 n \alpha}{\beta}-(2 \alpha+1) \sum_{i=0}^{n}\left(\beta+x_{i}^{2}\right)^{-1}$
$-(\theta-1) \sum_{i=0}^{n} \frac{2 \alpha \beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\left\{\beta^{-1}+\left(\beta+x^{2}\right)^{-1}\right\}}{\left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]}$
$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}+\sum_{i=0}^{n} \ln \left[1-\beta^{2 \alpha}\left(\beta+x^{2}\right)^{-2 \alpha}\right]$
Due to complications, these equations are difficult to solve analytically. To overcome this problem, there are no of softwares are available with different iterative methods in which Newton-Raphson method is commonly used.

## Applications

Now we compare TLCR model with its baseline distribution. All these results are obtained by using the R Language with package adequacy model with "N" (Nekdor-Mead) algorithm. The package provides goodness of fit statistics such as W (Cramer-von Misses), A (Anderson Darling) and K-S (Kolmogorov Smirnov test) and second one most popular comparison criteria are $l$ (log likelihood) AIC, BIC, CAIC and

HQIC. As for selection of best model among other models, generally values of these measures should be smaller as compare to others. Here we use three data sets; first data set consists of 20 values which show the ordered failure of components. This data also been used by Nigm et al ${ }^{18}$ and data are $0.0009,0.004,0.0142,0.0221,0.0261,0.0418,0.0473$, $0.0834,0.1091,0.1252,0.1404,0.1498,0.15,0.2031,0.2099$, $0.2168,0.2918,0.3465,0.4035,0.6143$. The second data consists of 10 observations of fatigue life in hours of bearings that initially by $\mathrm{McCool}^{19}$ analyzed by several researchers, the data are $152.7,172.0,172.5,173.3,193.0,204.7,216.5,234.9$, 262.6, 422.6. The third data that we use consists of 214 values of successive failure of the air conditioning system for more detail see Proschan ${ }^{20}$ and $\mathrm{Kus}^{21}$ the data are :50, 130, 487, 57, $102,15,14,10,57,320,261,51,44,9,254,493,33,18,209$, $41,58,60,48,56,87,11,102,12,5,14,14,29,37,186,29$, $104,7,4,72,270,283,7,61,100,61,502,220,120,141,22$, $603,35,98,54,100,11,181,65,49,12,239,14,18,39,3,12$, $5,32,9,438,43,134,184,20,386,182,71,80,188,230,152$, $5,36,79,59,33,246,1,79,3,27,201,84,27,156,21,16,88$, $130,14,118,44,15,42,106,46,230,26,59,153,104,20,206$, $5,66,34,29,26,35,5,82,31,118,326,12,54,36,34,18,25$, $120,31,22,18,216,139,67,310,3,46,210,57,76,14,111$, $97,62,39,30,7,44,11,63,23,22,23,14,18,13,34,16,18$, $130,90,163,208,1,24,70,16,101,52,208,95,62,11,191$, 14, 71. Detail of these statistics is presented with the help of tables; from these results one can easily see that TLCR is best fitted model than its baseline model.

Table-1: Presents statistics for data set 1.

| Distribution | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | A | W | K-S <br> P-value |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| TLCR | 10.1732867 <br> $(113.07049014)$ | 2.2446899 <br> $(26.31105589)$ | 0.3209663 <br> $(0.09336739)$ | 0.132755 | 0.02098258 | 0.09 <br> $\mathbf{0 . 9 9 2 1}$ |
| CR | 0.508883380 <br> $(0.153371281)$ | 0.002488151 <br> $(0.001391307)$ | - | 0.9838921 | 0.1704624 | 0.1906 <br> $\mathbf{0 . 4 1 0 1}$ |

Table-2: Presents comparison for data set 1.

| Distribution | $l$ | AIC | BIC | CAIC | HQIC |
| :---: | :---: | :---: | :---: | :---: | :---: |
| TLCR | -17.27782 | -28.55565 | -25.56845 | -27.05565 | -27.97252 |
| CR | -9.742232 | -15.48446 | -13.493 | -14.77858 | -15.09571 |

Table-3: Presents statistics for data set 2.

| Distribution | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | A | W | K-S <br> P-value |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| TLCR | 1.386029 <br> $(0.3753598)$ | 2547.827579 <br> $(2862.4807598)$ | 1766.859635 <br> $(4995.5623227)$ |  | 0.0354948 | 0.1793 <br> $\mathbf{0 . 8 5 0 3}$ |
| CR | $4.741579 \mathrm{e}+01$ <br> $(14.9939)$ | $2.533440 \mathrm{e}+06$ <br> $(1712.3686)$ | - | 0.884331 | 0.1314875 | 0.3524 <br> $\mathbf{0 . 1 2 9 2}$ $\mathbf{~}$ |

Table-4: Presents comparison for data set 2.

| Distribution | $l$ | AIC | BIC | CAIC | HQIC |
| :---: | :---: | :---: | :---: | :---: | :---: |
| TLCR | 53.24555 | 112.4911 | 113.3989 | 116.4911 | 111.4953 |
| CR | 58.58196 | 121.1639 | 121.7691 | 122.8782 | 120.5001 |

Table-5: Presents statistics for data set 3 .

| Distribution | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | A | W | K-S <br> P-value |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| TLCR | $\begin{gathered} 0.4467717 \\ (0.05653137) \end{gathered}$ | $\begin{gathered} 7557.4159384 \\ (768.72498724) \end{gathered}$ | $\begin{gathered} 0.4947277 \\ (0.04442210) \end{gathered}$ | 0.8191072 | 0.1221583 | $\begin{gathered} \hline 0.0562 \\ \mathbf{0 . 6 2 5} \end{gathered}$ |
| CR | $\begin{gathered} 0.5278747 \\ (0.06815637) \end{gathered}$ | $\begin{gathered} 687.2208302 \\ (209.46294715) \end{gathered}$ | - | 1.661247 | 0.2588549 | $\begin{aligned} & \hline 0.0769 \\ & \mathbf{0 . 2 4 0 8} \end{aligned}$ |

Table-6: Presents comparison for data set 3.

| Distribution | $l$ | AIC | BIC | CAIC | HQIC |
| :---: | :---: | :---: | :---: | :---: | :---: |
| TLCR | 982.3427 | 1970.685 | 1980.248 | 1970.823 | 1974.563 |
| CR | 990.2483 | 1984.497 | 1990.871 | 1984.565 | 1987.082 |



Figure-3: Fitted density to data 1.


Figure-4: fitted density to data 3.

## Conclusion

In this research an extension form of compound Rayleigh distribution presented so called Topp-Leone compound Rayleigh distribution. Its basic properties in which skewness, kurtosis means, variance, an expression of coefficient of variation, rth moments, incomplete moments, mean deviation are also examined. From reliability point of view, we obtained reliability function, hazard rate function, cumulative hazard rate function, reverse hazard rate function, mean residual life and mean waiting time. Income inequality measures were also the part of this research. Alternate methods of parameter estimation are also obtained called probability weighted moments. MLE of TLCR distribution are also obtained. In the end three applications with real data sets are performed and noticed that Topp-Leone compound Rayleigh distribution is better than its baseline model. It is hoped that this work will be helpful for further research.

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