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Sublime estimators of population variance using parameter median of study variable

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Abstract

In the present article, we propose some enhanced ratio type and Searle type estimators for estimating parameter variance utilizing known parameter median of study variable under investigation. The study of the sampling properties of the suggested estimators are derived up to the approximation of order one. The proposed estimators are compared with the competing estimators of variance parameter of primary variable, utilizing the known auxiliary information. Numerical example depicts that the suggested estimators performs better than the competing estimators of population variance.

Keywords: Population, parameter, bias, mean squared error (MSE), efficiency.

Introduction

In general any parameter is estimated through its statistics but with some desirable properties. Thus variance parameter of the main variable Y is estimated by its sample variance. Being unbiased, but it does not possess one of the important properties of minimum variance.

So it requires the enhancement in the sample variance in terms of its sampling variance. This aim is achieved through the proper utilization of information on auxiliary variable or the information on the known parameter of the study variable which is easily available in practice.

Singh *et al.* as well as Searls and Intarapanich independently suggested the modified estimator of population variance by multiplying sample variance with a constant[1,2]. Das and Tripathi along with Srivastava and Jhajj suggested the improved estimators by using auxiliary variable parameters[3,4].

Isaki made the use of auxiliary variable having strong positive correlation with primary variable in the form of population variance and suggested the usual ratio as well as the traditional regression estimator for the population variance of main variable[5].

Singh *et al.* and Prasad and Singh proposed the modified ratio type along with the regression type estimators for estimating variance parameter using known auxiliary information[6,7].

Upadhyaya and Singh suggested a modified ratio type estimator for variance parameter utilizing known coefficient of kurtosis of auxiliary variable and Dubey and Kant suggested an improved regression type estimator for variance parameter utilizing auxiliary information[8,9]. Kadilar and Cingi presented three modified ratio estimator of population variance by utilizing known coefficient of variation, population variance and coefficient of kurtosis of auxiliary variable while Subramani and Kumarpandiyan suggested modified ratio estimator using known quartiles as well as the functions of quartiles of auxiliary variable[10,11].

Khan and Shabbir used known population third quartile and other parameters of auxiliary variable and suggested modified ratio estimator[12].

Other authors including Yadav and Kadilar, Yadav *et al.*, Singh and Pal and Bhatt *et al.* utilized known auxiliary parameters and suggested improved estimators for variance parameter of primary variable under investigation[13-19].

Let the finite population for the study has N different units and $(x_i, y_i),\ i = 1, 2, ..., n$ is the sample drawn from bivariate set (*X,* Y) using the technique of simple random sampling without replacement (SRSWOR). *Y* and *X* are the primary (study) and the secondary (auxiliary) variable.

Let $\bar{X}$ and $\bar{Y}$ respectively be the population means, and $\bar{x}$ and $\bar{y}$ be the respective sample means.

Review of Existing Estimators

The following Table-1 represents various estimators of variance parameter of primary variable utilizing and without using auxiliary information in terms of its various known parameters along with their bias and MSE.

Table-1: Various estimators of population variance, their biases and MSEs.

Estimator	Bias and MSE	References
$t_0 = s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$	$V(t_0) = \gamma S_y^4(\lambda_{40} - 1)$	Sample Variance
$t_R = s_y^2\left[\frac{S_x^2}{s_x^2}\right]$	$B(t_R) = \gamma S_y^2[(\lambda_{04} - 1) - (\lambda_{22} - 1)]$ $MSE(t_R) = \gamma S_y^4[(\lambda_{40} - 1) + (\lambda_{04} - 1) - 2(\lambda_{22} - 1)]$	5
$t_{\mathrm{Re}\,g} = s_y^2 + b_{(s_y^2, s_x^2)}(S_x^2 - s_x^2)$	$V(t_{\mathrm{Re}\,g}) = \gamma S_y^4[(\lambda_{40} - 1)(1 - \rho^2_{(s_y^2, s_x^2)})]$	5
$t_1 = \kappa_1 s_y^2 + \kappa_2(S_x^2 - s_x^2)$	$B(t_1) = (\kappa_1 - 1)S_y^2$ $MSE_{\min}(t_1) = \frac{V(t_{\mathrm{Re}\,g})}{1 + S_y^{-4}V(t_{\mathrm{Re}\,g})}$ For, $\kappa_{1(opt)} = \frac{V_{20}}{(V_{02} + V_{02}V_{20} - V_{11}^2)}$ and $\kappa_{2(opt)} = \frac{S_x^2}{S_y^2}\frac{V_{11}}{(V_{02} + V_{02}V_{20} - V_{11}^2)}$ Where, $V_{20} = \gamma(\lambda_{40} - 1)$, $V_{02} = \gamma(\lambda_{04} - 1)$, $V_{11} = \gamma(\lambda_{22} - 1)$	6
$t_2 = s_y^2\left[\frac{S_x^2 + \beta_{2(x)}}{s_x^2 + \beta_{2(x)}}\right]$	$B(t_2) = \gamma S_y^2 R_2[R_2(\lambda_{04} - 1) - (\lambda_{22} - 1)]$ $MSE(t_2) = \gamma S_y^4[(\lambda_{40} - 1) + R_2^2(\lambda_{04} - 1) - 2R_2(\lambda_{22} - 1)]$	8
$t_3 = s_y^2\left[\frac{\bar{X}}{\bar{x}}\right]$	$B(t_3) = \gamma S_y^2[C_x^2 - \lambda_{21}C_x]$ $MSE(t_3) = \gamma S_y^4[(\lambda_{40} - 1) + C_x^2 - 2\lambda_{21}C_x]$	20
$t_4 = s_y^2\left[\frac{S_x^2 + C_x}{s_x^2 + C_x}\right]$ $t_5 = s_y^2\left[\frac{S_x^2\beta_{2(x)} + C_x}{s_x^2\beta_{2(x)} + C_x}\right]$ $t_6 = s_y^2\left[\frac{S_x^2 C_x + \beta_{2(x)}}{s_x^2 C_x + \beta_{2(x)}}\right]$	$B(t_i) = \gamma S_y^2 R_i[R_i(\lambda_{04} - 1) - (\lambda_{22} - 1)]$, $(i = 4, 5, 6)$ $MSE(t_i) = \gamma S_y^4[(\lambda_{40} - 1) + R_i^2(\lambda_{04} - 1) - 2R_i(\lambda_{22} - 1)]$	10
$t_7 = [d_1 s_y^2 + d_2(S_x^2 - s_x^2)]\exp\left(\frac{(S_x^2 - s_x^2)}{(S_x^2 + s_x^2)}\right)$	$B_{opt}(t_7) = \frac{\gamma S_y^2\left[\frac{1}{8}(\lambda_{04} - 1) - (\lambda_{40} - 1)\{1 - \rho^2_{s_y^2, s_x^2}\}\right]}{1 + \gamma(\lambda_{40} - 1)\{1 - \rho^2_{s_y^2, s_x^2}\}}$ $MSE_{\min}(t_7) = \left[\frac{V(t_{\mathrm{Re}\,g})}{1 + S_y^{-4}V(t_{\mathrm{Re}\,g})}\right]$	21

$t_8 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$	$B(t_8) = \gamma \frac{S_y^2}{2}\left[\frac{3}{4}(\lambda_{04}-1)-(\lambda_{22}-1)\right]$ $MSE(t_8) = \gamma S_y^4[(\lambda_{40}-1)+\frac{(\lambda_{04}-1)}{4}-(\lambda_{22}-1)]$	22
$t_9 = s_y^2\left[\frac{S_x^2+Q_1}{s_x^2+Q_1}\right]$ $t_{10} = s_y^2\left[\frac{S_x^2+Q_3}{s_x^2+Q_3}\right]$ $t_{11} = s_y^2\left[\frac{S_x^2+Q_r}{s_x^2+Q_r}\right]$ $t_{12} = s_y^2\left[\frac{S_x^2+Q_d}{s_x^2+Q_d}\right]$ $t_{13} = s_y^2\left[\frac{S_x^2+Q_a}{s_x^2+Q_a}\right]$	$B(t_i) = \gamma S_y^2 R_i[R_i(\lambda_{04}-1)-(\lambda_{22}-1)]$ $MSE(t_i) = \gamma S_y^4[(\lambda_{40}-1)+R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)]$ $(i = 9,10,11,12,13)$	11
$t_{14} = s_y^2\left[\frac{S_x^2\rho+Q_3}{s_x^2\rho+Q_3}\right]$	$B(t_{14}) = \gamma S_y^2 R_{14}[R_{14}(\lambda_{04}-1)-(\lambda_{22}-1)]$ $MSE(t_{14}) = \gamma S_y^4[(\lambda_{40}-1)+R_{14}^2(\lambda_{04}-1)-2R_{14}(\lambda_{22}-1)]$	12
$t_{15} = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2+(\alpha-1)s_x^2}\right]$	$B(t_{15}) = \gamma \frac{S_y^2}{2}\frac{(\lambda_{04}-1)}{2\alpha^2}\left[2\alpha\left\{1-\frac{(\lambda_{22}-1)}{(\lambda_{04}-1)}\right\}-1\right]$ $MSE_{\min}(t_{15}) = \gamma S_y^4\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]$ For $\alpha_{opt} = \frac{(\lambda_{04}-1)}{(\lambda_{22}-1)}$	13
$t_{16} = \lambda s_y^2 \exp\left[\alpha\left(\frac{\bar{X}-\bar{x}}{\bar{X}+(a-1)\bar{x}}\right)\right]$	$B(t_{16}) = \gamma \frac{S_y^2}{2}\left[\lambda\left\{1+\frac{\alpha^2}{2a^2}C_x^2-\frac{\alpha}{a}\lambda_{21}C_x\right\}\right]-S_y^2$ $MSE_{\min}(t_{16}) = \gamma S_y^4\left[1-\frac{1}{(\lambda_{04}-\lambda_{21}^2)}\right]$ For $\lambda = (\lambda_{40}-\lambda_{21}^2)^{-1}$, $a = (C_x)^{-1}$ and $\alpha = \lambda_{21}(C_x)^{-1}$	23
$t_{17} = s_y^2\left[\varphi\left\{2-\left(\frac{\bar{X}+\beta_{2(x)}}{\bar{x}+\beta_{2(x)}}\right)\right\} + (1-\varphi)\left\{2-\left(\frac{\bar{x}+\beta_{2(x)}}{\bar{X}+\beta_{2(x)}}\right)\right\}\right]$	$B(t_{17}) = \gamma S_y^2[R_{17}(2\varphi-1)\lambda_{21}-\varphi R_{17}^2 C_x^2]$ $MSE_{\min}(t_{17}) = \gamma S_y^4[(\lambda_{40}-1)-\lambda_{21}^2]$ For $\varphi = \frac{1}{2}\left[1-\frac{\lambda_{21}}{R_{15}C_x}\right]$	15

$t_{18}=s_y^2\left[\bar{y}^2\dfrac{C_y^2}{s_y^2}\right]^{\alpha}$	$B(t_{18})=\gamma S_y^2\begin{bmatrix}2\alpha(\alpha-1)C_y^2-4\alpha(\alpha-1)\gamma_{1(y)}C_y\\+\alpha(\alpha-1)(\gamma_{2(y)}+2)\end{bmatrix}$ $MSE_{\min}(t_{18})=\gamma S_y^4\left[(\gamma_{2(y)}+2)-\dfrac{(2\gamma_{1(y)}C_y-\gamma_{2(y)}-2)^2}{(4C_y^2-4\gamma_{1(y)}C_y+\gamma_{2(y)}+2)}\right]$ For $\alpha=\dfrac{-(2\gamma_{1(y)}C_y-\gamma_{2(y)}-2)}{(4C_y^2-4\gamma_{1(y)}C_y+\gamma_{2(y)}+2)}$	24
$t_{19}=s_y^2\left[\dfrac{S_x^2\beta_{2(x)}+M_x^2}{s_x^2\beta_{2(x)}+M_x^2}\right]$	$B(t_{19})=\gamma S_y^2R_{19}[R_{19}(\lambda_{04}-1)-(\lambda_{22}-1)]$ $MSE(t_{19})=\gamma S_y^4[(\lambda_{40}-1)+R_{19}^2(\lambda_{04}-1)-2R_{19}(\lambda_{22}-1)]$	25
$t_{20}=s_y^2\left[\dfrac{S_x^2+C_xS_x}{s_x^2+C_xS_x}\right]$ $t_{21}=s_y^2\left[\dfrac{S_x^2+C_x\bar{X}}{s_x^2+C_x\bar{X}}\right]$ $t_{22}=s_y^2\left[\dfrac{S_x^2+C_xM_x}{s_x^2+C_xM_x}\right]$	$B(t_i)=\gamma S_y^2R_i[R_i(\lambda_{04}-1)-(\lambda_{22}-1)]$ $MSE(t_i)=\gamma S_y^4[(\lambda_{40}-1)+R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)]$ $(i=20,21,22)$	26

Where, $\lambda_{rs}=\dfrac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}}$,

$$\mu_{rs}=\frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})^r(X_i-\bar{X})^s,$$

$\gamma=\dfrac{1-f}{n}$ and $f=\dfrac{n}{N}$, $b_{(s_y^2,s_x^2)}=\dfrac{s_y^2(\lambda_{22}-1)}{s_x^2(\lambda_{04}-1)}$,

$\rho_{(s_y^2,s_x^2)}=\dfrac{(\lambda_{22}-1)}{\sqrt{(\lambda_{40}-1)(\lambda_{04}-1)}}$, $R_2=\dfrac{S_x^2}{S_x^2+\beta_{2(x)}}$,

$R_4=\dfrac{S_x^2}{S_x^2+C_x}$, $R_5=\dfrac{S_x^2\beta_{2(x)}}{S_x^2\beta_{2(x)}+C_x}$,

$R_6=\dfrac{S_x^2C_x}{S_x^2C_x+\beta_{2(x)}}$, $Q_r=Q_3-Q_1$ $Q_d=\dfrac{Q_3-Q_1}{2}$,

$Q_a=\dfrac{Q_3+Q_1}{2}$, $R_9=\dfrac{S_x^2}{S_x^2+Q_1}$, $R_{10}=\dfrac{S_x^2}{S_x^2+Q_3}$,

$R_{11}=\dfrac{S_x^2}{S_x^2+Q_r}$, $R_{12}=\dfrac{S_x^2}{S_x^2+Q_d}$, $R_{13}=\dfrac{S_x^2}{S_x^2+Q_a}$,

$R_{14}=\dfrac{S_x^2\rho}{S_x^2\rho+Q_3}$,

$$d_{1(opt)}=\left[1+\gamma\left\{(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{40}-1)}\right\}\right]^{-1},$$

$$d_{2(opt)}=d_{1(opt)}S_y^2\left(\frac{(\lambda_{22}-1)-\frac{1}{2}(\lambda_{40}-1)}{S_x^2(\lambda_{04}-1)}\right),$$

$R_{17}=\dfrac{\bar{X}}{\bar{X}+\beta_{2(x)}}$, $\gamma_{1(y)}=\dfrac{\mu_{30}}{\mu_{20}^{3/2}}$, $\gamma_{2(y)}=\dfrac{\mu_{40}}{\mu_{20}^2}-3$,

$R_{19}=\dfrac{S_x^2\beta_{2(x)}}{S_x^2\beta_{2(x)}+M_x^2}$, $R_{20}=\dfrac{S_x^2}{S_x^2+C_xS_x}$,

$R_{21}=\dfrac{S_x^2}{S_x^2+C_x\bar{X}}$, $R_{22}=\dfrac{S_x^2}{S_x^2+C_xM_x}$.

Proposed Estimators

Motivated by Subramani[27] and applying this technique, we propose the following ratio and regression type estimators for variance parameter as,

$$t_{p1}=s_y^2+b_{(s_y^2,m)}(M-m), \tag{1}$$

$$t_{p2} = \kappa_1 s_y^2 + \kappa_2 (M - m), \quad (2)$$

$$t_{p3} = s_y^2 \exp\left[\frac{M-m}{M+m}\right], \quad (3)$$

$$t_{p4} = s_y^2 \exp\left[\frac{M-m}{M+(\alpha-1)m}\right], \quad (4)$$

$$t_{p5} = \lambda s_y^2 \exp\left[\alpha\left(\frac{M-m}{M+(a-1)m}\right)\right], \quad (5)$$

$$t_{p6} = [d_1 s_y^2 + d_2 (M-m)]\exp\left(\frac{(M-m)}{(M+m)}\right), \quad (6)$$

The sampling properties of the suggested estimators are studied using the approximations given as,

$s_y^2 = S_y^2(1+e_0)$ and $m = M(1+e_1)$ with $E(e_0) = 0$, $E(e_1) = \frac{Bias(m)}{M} = \frac{\overline{M} - M}{M}$ and $E(e_0^2) = \gamma(\lambda_{40} - 1)$, $E(e_1^2) = \gamma C_m^2$, $E(e_0 e_1) = \gamma \lambda_{21(m)} C_m$

Where, $\gamma = \frac{1-f}{n}$, $\lambda_{rs(m)} = \frac{\mu_{rs(m)}}{\mu_{20}^{r/2}\mu_{02(m)}^{s/2}}$,

$\mu_{rs(m)} = \frac{1}{{}^N C_n}\sum_{i=1}^{{}^N C_n}(\bar{y}_i - \bar{Y})^r (m_i - M)^s$, $\lambda_{21(m)} = \frac{\mu_{21(m)}}{\mu_{20}\sqrt{\mu_{02(m)}}}$

$M =$ Population median, $\overline{M} = \frac{1}{{}^N C_n}\sum_{i-1}^{{}^N C_n} m_i$ (average of sample medians), $m_i =$ sample median of i[th] sample ($i = 1, 2, ..., {}^N C_n$).

The mathematical forms for the biases and the MSEs of the suggested estimators are given in Table-2 below.

Table-2: Various estimators of population variance, their biases and MSEs

Estimator	Bias and MSE
$t_{p1} = s_y^2 + b_{(s_y^2, m)}(M - m)$	$B(t_{p1}) \cong -B(m) \times B$ $MSE(t_{p1}) = \gamma S_y^4[(\lambda_{40} - 1)(1 - \rho^2_{(s_y^2, m)})]$ Where, $B = \frac{Cov(s_y^2, m)}{V(m)}$
$t_{p2} = \kappa_1 s_y^2 + \kappa_2 (M - m)$	$B(t_{p2}) = (\kappa_1 - 1)S_y^2 - \kappa_2 B(m)$ $MSE_{\min}(t_{p2}) = \frac{V(t_{p1})}{1 + S_y^{-4} V(t_{p1})}$ For, $\kappa_{1(opt)} = \frac{V_{20}}{(V'_{02} + V'_{02} V_{20} - V'^2_{11})}$ and $\kappa_{2(opt)} = \frac{S_m^2}{S_y^2}\frac{V'_{11}}{(V'_{02} + V'_{02} V_{20} - V'^2_{11})}$ Where, $V_{20} = \gamma(\lambda_{40} - 1)$, $V'_{02} = \gamma C_m^2$, $V'_{11} = \gamma \lambda_{21(m)} C_m$
$t_{p3} = s_y^2 \exp\left[\frac{M-m}{M+m}\right]$	$B(t_{p3}) = S_y^2\left[\gamma(\frac{3}{8}C_m^2 - \lambda_{21(m)} C_m) - \frac{B(m)}{2M}\right]$ $MSE(t_{p3}) = \gamma S_y^4\left[(\lambda_{40} - 1) + \frac{C_m^2}{4} - \lambda_{21(m)} C_m\right]$

$t_{p4} = s_y^2 \exp\left[\frac{M-m}{M+(\alpha-1)m}\right]$	$B(t_{p4}) = S_y^2\left[\left(\alpha_1 - \frac{\alpha_1^2}{2}\right)\gamma C_m^2 - \alpha_1\gamma\lambda_{21(m)}C_m - \alpha_1\frac{B(m)}{M}\right]$ $MSE_{\min}(t_{p4}) = \gamma S_y^4[(\lambda_{40}-1)-\lambda_{21(m)}^2]$ For $\alpha_{1(opt)} = \lambda_{21(m)}/C_m$
$t_{p5} = \lambda s_y^2 \exp\left[\alpha\left(\frac{M-m}{M+(a-1)m}\right)\right]$	$B(t_{p5}) = S_y^2\left[\lambda\left(1 - w\frac{B(m)}{M} + \frac{w^2}{2}C_m^2 - w\lambda_{21(m)}C_m\right) - 1\right]$ $MSE(t_{p5}) = S_y^4\left[\lambda^2\left\{\begin{matrix}1+(\lambda_{40}-1)+w^2C_m^2\\ -w\lambda_{21(m)}C_m - 2w\frac{B(m)}{M}\end{matrix}\right\} + 1 - 2\lambda\left\{1 - w\frac{B(m)}{M}\right\}\right]$ with $w_{opt} = \frac{\left\{\frac{B(m)}{M}\right\}^2 + \lambda_{21(m)}C_m\frac{B(m)}{M} - C_m^2}{\gamma C_y^2\frac{B(m)}{M} - \lambda_{21(m)}C_m}$, $\lambda_{opt} = \frac{\frac{B(m)}{M}}{\left[\lambda_{21(m)}C_m + \frac{B(m)}{M} - wC_m^2\right]}$, where, $w = \frac{\alpha}{a}$
$t_{p6} = [d_1 s_y^2 + d_2(M-m)]\exp\left(\frac{(M-m)}{(M+m)}\right)$	$B(t_{p6}) = \left[\begin{matrix}d_1S_y^2\left\{1 - \frac{B(m)}{M} + \frac{3}{8}\gamma C_m^2 - \frac{\lambda_{21(m)}C_m}{2}\right\}\\ -d_2M\left\{\frac{B(m)}{M} + \frac{1}{2}\gamma C_m^2\right\} - S_y^2\end{matrix}\right]$ $MSE(t_{p6}) = \left[\begin{matrix}(d_1-1)^2S_y^4 + d_1^2V_{20}S_y^4 + (d_1S_y^2 - d_2M)^2\gamma C_m^2\\ 2d_1S_y^2(d_1S_y^2 - d_2M)\gamma\lambda_{21(m)}C_m\\ +2(d_1-1)S_y^2(d_1S_y^2 - d_2M)\frac{B(m)}{M}\end{matrix}\right]$ With $d_{1(opt)} = \frac{M^2\gamma C_m^2A_3 - A_2B(m)}{M^2\gamma C_m^2A_1 - A_2^2}$, $d_{2(opt)} = \frac{A_2A_3 - A_1B(m)}{M^2\gamma C_m^2A_1 - A_2^2}$ Where $A_1 = \left[S_y^4 + V_{20}S_y^4 + S_y^4\gamma C_m^2 + 2S_y^4\gamma\lambda_{21(m)}C_m + 2S_y^4\frac{B(m)}{M}\right]$ $A_2 = [MS_y^2\gamma C_m^2 + MS_y^2\gamma\lambda_{21(m)}C_m + B(m)]$, $A_3 = S_y^4[1+B(m)]$

Efficiency Comparison

Comparison of suggested estimator t_{p1} with the competing estimators in Table-1: The proposed estimator t_{p1} is more efficient compared to the competing estimators in Table-1 under the following conditions respectively as,

$V(t_0) - MSE(t_{p1}) > 0$, if $\rho^2_{(s_y^2,m)} > 0$

$MSE(t_R) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + (\lambda_{04} - 1) - 2(\lambda_{22} - 1)] > 0$

$MSE(t_{\mathrm{Re}\,g}) - MSE(t_{p1}) > 0$, if $\rho^2_{(s_y^2,m)} - \rho^2_{(s_y^2,s_x^2)} > 0$

$MSE(t_1) - MSE(t_{p1}) > 0$, if $\dfrac{V(t_{\mathrm{Re}\,g})}{1 + S_y^{-4} V(t_{\mathrm{Re}\,g})} - \gamma S_y^4 (\lambda_{40} - 1)\rho^2_{(s_y^2,m)} > 0$

$MSE(t_2) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + R_2^2(\lambda_{04} - 1) - 2R_2(\lambda_{22} - 1)] > 0$

$MSE(t_3) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + C_x^2 - 2\lambda_{21} C_x] > 0$

$MSE(t_i) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + R_i^2(\lambda_{04} - 1) - 2R_i(\lambda_{22} - 1)] > 0$, $(i = 4,5,6)$

$MSE(t_7) - MSE(t_{p1}) > 0$, if $\left[\dfrac{\gamma^2(\lambda_{04} - 1)^2 + 16\{\gamma(\lambda_{04} - 1) - 4\} S_y^{-4} V(t_{\mathrm{Re}\,g})}{-1 - S_y^{-4} V(t_{\mathrm{Re}\,g})}\right] - 64\gamma(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} > 0$

$MSE(t_8) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + \dfrac{(\lambda_{04} - 1)}{4} - (\lambda_{22} - 1)] > 0$

$MSE(t_i) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + R_i^2(\lambda_{04} - 1) - 2R_i(\lambda_{22} - 1)] > 0$, $(i = 9,10,11,12,13)$

$MSE(t_{14}) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + R_{14}^2(\lambda_{04} - 1) - 2R_{14}(\lambda_{22} - 1)] > 0$

$MSE(t_{15}) - MSE(t_{p1}) > 0$, if $\left[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} - \dfrac{(\lambda_{22} - 1)^2}{(\lambda_{04} - 1)}\right] > 0$

$MSE(t_{16}) - MSE(t_{p1}) > 0$, if $\left[1 - \dfrac{1}{(\lambda_{04} - \lambda_{21}^2)}\right] - [(\lambda_{40} - 1)(1 - \rho^2_{(s_y^2,m)})] > 0$

$MSE(t_{17}) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} - \lambda_{21}^2] > 0$

$MSE(t_{18}) - MSE(t_{p1}) > 0$, if $\left[(\gamma_{2(y)} + 2) - \dfrac{(2\gamma_{1(y)} C_y - \gamma_{2(y)} - 2)^2}{(4C_y^2 - 4\gamma_{1(y)} C_y + \gamma_{2(y)} + 2)}\right] - [(\lambda_{40} - 1)(1 - \rho^2_{(s_y^2,m)})] > 0$

$MSE(t_{19}) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + R_{19}^2(\lambda_{04} - 1) - 2R_{19}(\lambda_{22} - 1)] > 0$

$MSE(t_i) - MSE(t_{p1}) > 0$, if $[(\lambda_{40} - 1)\rho^2_{(s_y^2,m)} + R_i^2(\lambda_{04} - 1) - 2R_i(\lambda_{22} - 1)] > 0$, $(i = 20, 21, 22)$

Comparison of suggested estimator t_{p2} with the competing estimators in Table-1: The proposed estimator t_{p2} performs better than the competing estimators in Table-1 under the following conditions if,

$$V(t_0) - MSE_{\min}(t_{p2}) = \gamma S_y^4(\lambda_{40} - 1) - \frac{V(t_{p1})}{1 + S_y^{-4} V(t_{p1})} > 0$$

$$MSE(t_R) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+(\lambda_{04}-1)-2(\lambda_{22}-1)] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$V(t_{\mathrm{Re}\,g}) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)(1-\rho^2_{(s_y^2,s_x^2)})] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE_{\min}(t_1) - MSE_{\min}(t_{p2}) = \frac{V(t_{\mathrm{Re}\,g})}{1+S_y^{-4}V(t_{\mathrm{Re}\,g})} - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE(t_2) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+R_2^2(\lambda_{04}-1)-2R_2(\lambda_{22}-1)] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE(t_3) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+C_x^2-2\lambda_{21}C_x] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE(t_i) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0, \quad (i=4,5,6)$$

$$MSE_{\min}(t_7) - MSE_{\min}(t_{p2}) = \frac{S_y^4}{64}\left[\frac{\gamma^2(\lambda_{04}-1)^2+16\{\gamma(\lambda_{04}-1)-4\}S_y^{-4}V(t_{\mathrm{Re}\,g})}{-1-S_y^{-4}V(t_{\mathrm{Re}\,g})}\right] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE(t_8) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+\frac{(\lambda_{04}-1)}{4}-(\lambda_{22}-1)] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE(t_i) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0, \quad (i=9,10,11,12,13)$$

$$MSE(t_{14}) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+R_{14}^2(\lambda_{04}-1)-2R_{14}(\lambda_{22}-1)] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE_{\min}(t_{15}) - MSE_{\min}(t_{p2}) = \gamma S_y^4\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE_{\min}(t_{16}) - MSE_{\min}(t_{p2}) = \gamma S_y^4\left[1-\frac{1}{(\lambda_{04}-\lambda_{21}^2)}\right] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE_{\min}(t_{17}) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)-\lambda_{21}^2] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE_{\min}(t_{18}) - MSE_{\min}(t_{p2}) = \gamma S_y^4\left[(\gamma_{2(y)}+2)-\frac{(2\gamma_{1(y)}C_y-\gamma_{2(y)}-2)^2}{(4C_y^2-4\gamma_{1(y)}C_y+\gamma_{2(y)}+2)}\right] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE(t_{19}) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+R_{19}^2(\lambda_{04}-1)-2R_{19}(\lambda_{22}-1)] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0$$

$$MSE(t_i) - MSE_{\min}(t_{p2}) = \gamma S_y^4[(\lambda_{40}-1)+R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)] - \frac{V(t_{p1})}{1+S_y^{-4}V(t_{p1})} > 0, \quad (i=20,21,22)$$

Comparison of suggested estimator t_{p3} with the competing estimators in Table-1: The suggested estimator t_{p3} performs more efficiently than estimators in Table-1 under the following conditions if,

$$MSE(t_{p3}) = \gamma S_y^4\left[(\lambda_{40}-1)+\frac{C_m^2}{4}-\lambda_{21(m)}C_m\right]$$

$$V(t_0)-MSE(t_{p3}) = \gamma S_y^4\left[\lambda_{21(m)}C_m-\frac{C_m^2}{4}\right]>0$$

$$MSE(t_R)-MSE(t_{p3}) = \gamma S_y^4[(\lambda_{04}-1)-2(\lambda_{22}-1)+\lambda_{21(m)}C_m-\frac{C_m^2}{4}]>0$$

$$V(t_{\mathrm{Re}g})-MSE(t_{p3}) = \gamma S_y^4[\lambda_{21(m)}C_m-\frac{C_m^2}{4}-(\lambda_{04}-1)\rho^2_{(s_y^2,s_x^2)}]>0$$

$$MSE_{\min}(t_1)-MSE(t_{p3}) = \frac{V(t_{\mathrm{Re}g})}{1+S_y^{-4}V(t_{\mathrm{Re}g})}-\gamma S_y^4\left[(\lambda_{40}-1)+\frac{C_m^2}{4}-\lambda_{21(m)}C_m\right]>0$$

$$MSE(t_2)-MSE(t_{p3}) = \gamma S_y^4[R_2^2(\lambda_{04}-1)-2R_2(\lambda_{22}-1)+\lambda_{21(m)}C_m-\frac{C_m^2}{4}]>0$$

$$MSE(t_3)-MSE(t_{p3}) = \gamma S_y^4[C_x^2-2\lambda_{21}C_x+\lambda_{21(m)}C_m-\frac{C_m^2}{4}]>0$$

$$MSE(t_i)-MSE(t_{p3}) = \gamma S_y^4[R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)+\lambda_{21(m)}C_m-\frac{C_m^2}{4}]>0,\ (i=4,5,6)$$

$$MSE_{\min}(t_7)-MSE(t_{p3}) = \frac{S_y^4}{64}\left[\frac{\gamma^2(\lambda_{04}-1)^2+16\{\gamma(\lambda_{04}-1)-4\}S_y^{-4}V(t_{\mathrm{Reg}})}{-1-S_y^{-4}V(t_{\mathrm{Reg}})}\right]-\gamma S_y^4\left[(\lambda_{40}-1)+\frac{C_m^2}{4}-\lambda_{21(m)}C_m\right]>0$$

$$MSE(t_8)-MSE(t_{p3}) = \gamma S_y^4[\frac{(\lambda_{04}-1)}{4}-(\lambda_{22}-1)+\lambda_{21(m)}C_m-\frac{C_m^2}{4}]>0$$

$$MSE(t_i)-MSE(t_{p3}) = \gamma S_y^4[R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)+\lambda_{21(m)}C_m-\frac{C_m^2}{4}]>0,\ (i=9,10,11,12,13)$$

$$MSE(t_{14})-MSE(t_{p3}) = \gamma S_y^4[R_{14}^2(\lambda_{04}-1)-2R_{14}(\lambda_{22}-1)+\lambda_{21(m)}C_m-\frac{C_m^2}{4}]>0$$

$$MSE_{\min}(t_{15})-MSE(t_{p3}) = \gamma S_y^4\left[\lambda_{21(m)}C_m-\frac{C_m^2}{4}-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]>0$$

$$MSE_{\min}(t_{16})-MSE(t_{p3}) = \gamma S_y^4\left[1-\frac{1}{(\lambda_{04}-\lambda_{21}^2)}-(\lambda_{40}-1)-\frac{C_m^2}{4}+\lambda_{21(m)}C_m\right]>0$$

$$MSE_{\min}(t_{17})-MSE(t_{p3}) = \gamma S_y^4[\lambda_{21(m)}C_m-\frac{C_m^2}{4}-\lambda_{21}^2]>0$$

$$MSE_{\min}(t_{18})-MSE(t_{p3}) = \gamma S_y^4\left[(\gamma_{2(y)}+2)-\frac{(2\gamma_{1(y)}C_y-\gamma_{2(y)}-2)^2}{(4C_y^2-4\gamma_{1(y)}C_y+\gamma_{2(y)}+2)}-(\lambda_{40}-1)-\frac{C_m^2}{4}+\lambda_{21(m)}C_m\right]>0$$

$$MSE(t_{19})-MSE(t_{p3}) = \gamma S_y^4[R_{19}^2(\lambda_{04}-1)-2R_{19}(\lambda_{22}-1)+\frac{C_m^2}{4}-\lambda_{21(m)}C_m]>0$$

$$MSE(t_i)-MSE(t_{p3})=\gamma S_y^4[R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)+\frac{C_m^2}{4}-\lambda_{21(m)}C_m]>0,\ (i=20,21,22)$$

Comparison of suggested estimator t_{p4} with the estimators in Table-1: The proposed estimator t_{p4} is more efficient than the estimators in Table-1 under the following conditions if,

$$V(t_0)-MSE_{\min}(t_{p4})=\gamma S_y^4\lambda_{21(m)}^2>0$$

$$MSE(t_R)-MSE_{\min}(t_{p4})=\gamma S_y^4[(\lambda_{04}-1)-2(\lambda_{22}-1)+\lambda_{21(m)}^2]>0$$

$$V(t_{\mathrm{Re}g})-MSE_{\min}(t_{p4})=\gamma S_y^4[\lambda_{21(m)}^2-(\lambda_{40}-1)\rho_{(s_y^2,s_x^2)}^2)]>0$$

$$MSE_{\min}(t_1)-MSE_{\min}(t_{p4})=\frac{V(t_{\mathrm{Re}g})}{1+S_y^{-4}V(t_{\mathrm{Re}g})}-\gamma S_y^4[(\lambda_{40}-1)-\lambda_{21(m)}^2]>0$$

$$MSE(t_2)-MSE_{\min}(t_{p4})=\gamma S_y^4[R_2^2(\lambda_{04}-1)-2R_2(\lambda_{22}-1)+\lambda_{21(m)}^2]>0$$

$$MSE(t_3)-MSE_{\min}(t_{p4})=\gamma S_y^4[C_x^2-2\lambda_{21}C_x+\lambda_{21(m)}^2]>0$$

$$MSE(t_i)-MSE_{\min}(t_{p4})=\gamma S_y^4[R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)+\lambda_{21(m)}^2]>0,\ (i=4,5,6)$$

$$MSE_{\min}(t_7)-MSE_{\min}(t_{p4})=\frac{S_y^4}{64}\left[\frac{\gamma^2(\lambda_{04}-1)^2+16\{\gamma(\lambda_{04}-1)-4\}S_y^{-4}V(t_{\mathrm{Re}g})}{-1-S_y^{-4}V(t_{\mathrm{Re}g})}\right]-\gamma S_y^4[(\lambda_{40}-1)-\lambda_{21(m)}^2]>0$$

$$MSE(t_8)-MSE_{\min}(t_{p4})=\gamma S_y^4[\frac{(\lambda_{04}-1)}{4}-(\lambda_{22}-1)+\lambda_{21(m)}^2]>0$$

$$MSE(t_i)-MSE_{\min}(t_{p4})=\gamma S_y^4[R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)+\lambda_{21(m)}^2]>0,\ (i=9,10,11,12,13)$$

$$MSE(t_{14})-MSE_{\min}(t_{p4})=\gamma S_y^4[R_{14}^2(\lambda_{04}-1)-2R_{14}(\lambda_{22}-1)+\lambda_{21(m)}^2]>0$$

$$MSE_{\min}(t_{15})-MSE_{\min}(t_{p4})=\gamma S_y^4\left[\lambda_{21(m)}^2-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]>0$$

$$MSE_{\min}(t_{16})-MSE_{\min}(t_{p4})=\gamma S_y^4\left[1-\frac{1}{(\lambda_{04}-\lambda_{21}^2)}-(\lambda_{40}-1)+\lambda_{21(m)}^2\right]>0$$

$$MSE_{\min}(t_{17})-MSE_{\min}(t_{p4})=\gamma S_y^4[\lambda_{21(m)}^2-\lambda_{21}^2]>0$$

$$MSE_{\min}(t_{18})-MSE_{\min}(t_{p4})=\gamma S_y^4\left[(\gamma_{2(y)}+2)-\frac{(2\gamma_{1(y)}C_y-\gamma_{2(y)}-2)^2}{(4C_y^2-4\gamma_{1(y)}C_y+\gamma_{2(y)}+2)}\right]-\gamma S_y^4[(\lambda_{40}-1)-\lambda_{21(m)}^2]>0$$

$$MSE(t_{19})-MSE_{\min}(t_{p4})=\gamma S_y^4[R_{19}^2(\lambda_{04}-1)-2R_{19}(\lambda_{22}-1)+\lambda_{21(m)}^2]>0$$

$$MSE(t_i)-MSE_{\min}(t_{p4})=\gamma S_y^4[R_i^2(\lambda_{04}-1)-2R_i(\lambda_{22}-1)+\lambda_{21(m)}^2]>0,\ (i=20,21,22)$$

Note: Similar conditions are for the estimators t_{p5} and t_{p6}.

Numerical Verification

The theoretical conditions are verified through the numerical examples using two natural populations given below. The parameters, constants, MSE and the PRE of the proposed estimator over competing estimators are presented in different tables.

Population-: (Cochran, page 325)[28]: x= 60, 52, 58, 56, 62, 51, 72, 48, 71, 58, y = 115, 80, 82, 93, 105, 109,130, 93, 109, 95.

Table-3: Constants of Population-1.

Constant	Value	Constant	Value	Constant	Value
N	10	C_{yx}	0.3653	R_6	0.791867
n	3	λ_{40}	2.1171	R_9	0.543366
$\overline{X}$	58.8	λ_{04}	2.0149	R_{10}	0.506289
$\overline{Y}$	101.1	λ_{22}	1.3889	R_{11}	0.881230
S_x	7.9415	Q_1	0.4304	R_{12}	0.936866
S_y	15.4449	Q_2	53	R_{13}	0.524172
C_x	0.1351	Q_3	58	R_{14}	0.400519
C_y	0.1528	Q_r	61.5	R_{17}	0.96332
M_x	58	Q_d	8.5	R_{19}	0.040281
M_y	100	Q_a	4.25	R_{20}	0.983278
ρ_{yx}	0.6515	R_2	0.96572	R_{21}	0.888161
$\beta_1(x)$	0.2363	R_4	0.997863	R_{22}	0.889515
$\beta_2(x)$	2.2388	R_5	0.999044		

Table-4: MSE of suggested and other estimators for Population-1.

Estimator	MSE	Estimator	MSE	Estimator	MSE
t_0	14831.92	t_9	13198.58	t_{20}	17705.03
t_R	17979.24	t_{10}	13057.07	t_{21}	16288.73
$t_{\mathrm{Re}\,g}$	12852.98	t_{11}	16195.05	t_{22}	16307.17
t_1	10484.75	t_{12}	16983.49	t_{p1}	2105.84
t_2	17425.22	t_{13}	13120.70	t_{p2}	2030.69
t_3	13530.40	t_{14}	12857.01	t_{p3}	7799.09
t_4	17943.77	t_{15}	12852.98	t_{p4}	4158.24
t_5	17963.36	t_{16}	6020.57	t_{p5}	23507.42
t_6	15103.26	t_{17}	12372.04	t_{p6}	84972.56
t_7	10484.75	t_{18}	1239.48		
t_8	13036.75	t_{19}	14437.76		

Table-5: PRE of suggested over other estimators for Population-1.

Percentage Relative Efficiency (PRE)						
Estimator	t_{p1}	t_{p2}	t_{p3}	t_{p4}	t_{p5}	t_{p6}
t_0	704.32	730.39	190.17	356.69	63.09	17.45
t_R	853.78	885.38	230.53	432.38	76.48	21.16
$t_{\text{Re}g}$	610.35	632.94	164.80	309.10	54.68	15.13
t_1	497.89	516.32	134.44	252.14	44.60	12.34
t_2	827.47	858.10	223.43	419.05	74.13	20.51
t_3	642.52	666.30	173.49	325.39	57.56	15.92
t_4	852.10	883.63	230.08	431.52	76.33	21.12
t_5	853.03	884.60	230.33	431.99	76.42	21.14
t_6	717.21	743.75	193.65	363.21	64.25	17.77
t_7	497.89	516.32	134.44	252.14	44.60	12.34
t_8	619.08	641.99	167.16	313.52	55.46	15.34
t_9	626.76	649.96	169.23	317.41	56.15	15.53
t_{10}	620.04	642.99	167.42	314.00	55.54	15.37
t_{11}	769.06	797.52	207.65	389.47	68.89	19.06
t_{12}	806.50	836.34	217.76	408.43	72.25	19.99
t_{13}	623.06	646.12	168.23	315.53	55.82	15.44
t_{14}	610.54	633.14	164.85	309.19	54.69	15.13
t_{15}	610.35	632.94	164.80	309.10	54.68	15.13
t_{16}	285.90	296.48	77.20	144.79	25.61	7.09
t_{17}	587.51	609.25	158.63	297.53	52.63	14.56
t_{18}	58.86	61.04	15.89	29.81	5.27	1.46
t_{19}	685.61	710.98	185.12	347.21	61.42	16.99
t_{20}	840.76	871.87	227.01	425.78	75.32	20.84
t_{21}	773.50	802.13	208.85	391.72	69.29	19.17
t_{22}	774.38	803.04	209.09	392.17	69.37	19.19

Population 2: (Singh and Choudhary, page177)[29]: x=401, 630, 1194, 1170, 1065, 827, 1737, 1060, 360, 946, 4170, 1625, 827, 96, 1304, 377, 259, 186, 1767, 604, 700, 524, 571, 962, 407, 715, 845, 1016, 184, 282, 194, 439, 854, 820. y=50, 149, 284, 381, 278, 111, 634, 278, 112, 355, 99, 498, 111, 6, 339, 80, 105, 27, 515, 249, 85, 221, 133, 144, 103, 175, 335, 219, 62, 79, 60, 100, 141, 263.

Table-6: Constants of Population-2.

Constant	Value	Constant	Value	Constant	Value
N	34	C_{yx}	0.0337	R_6	0.99997
n	3	λ_{40}	3.6161	R_9	0.99925
$\overline{X}$	856.4118	λ_{04}	12.9735	R_{10}	0.99805
$\overline{Y}$	199.4412	λ_{22}	1.1884	R_{11}	0.99880
S_x	733.1407	Q_1	0.4820	R_{12}	0.99940
S_y	150.2150	Q_2	402.5	R_{13}	0.99865
C_x	0.8561	Q_3	767.5	R_{14}	0.99564
C_y	0.7532	Q_r	1049	R_{17}	0.98463
M_x	767.5	Q_d	646.5	R_{19}	0.92422
M_y	142.5	Q_a	323.25	R_{20}	0.99883
ρ_{yx}	0.4453	R_2	0.99997	R_{21}	0.99864
$\beta_1(x)$	7.9550	R_4	0.99999	R_{22}	0.99878
$\beta_2(x)$	13.3667	R_5	0.99999		

Table-7: MSE of suggested and other estimators for Population-2.

Estimator	MSE	Estimator	MSE	Estimator	MSE
t_0	404828860	t_9	2196631996	t_{20}	2195108815
t_R	2199360225	t_{10}	2192262749	t_{21}	2194395446
$t_{\mathrm{Re}g}$	404370139	t_{11}	2194981108	t_{22}	2194909936
t_1	225377307	t_{12}	2197168682	t_{p1}	339386026
t_2	2199269524	t_{13}	2194445394	t_{p2}	203644446
t_3	390518789	t_{14}	2183480367	t_{p3}	543558462
t_4	2199354416	t_{15}	404370139	t_{p4}	263576046
t_5	2199359791	t_{16}	142599242	t_{p5}	301767916
t_6	2199254274	t_{17}	368871635	t_{p6}	12404399
t_7	225377307	t_{18}	315648033		
t_8	838884872	t_{19}	1933613779		

Table-8: PRE of suggested over other estimators for Population-2.

Percentage Relative Efficiency(PRE)						
Estimator	t_{p1}	t_{p2}	t_{p3}	t_{p4}	t_{p5}	t_{p6}
t_0	119.28	198.79	74.48	153.59	134.15	3263.59
t_R	648.04	1080.00	404.62	834.43	728.83	17730.49
$t_{\text{Re}g}$	119.15	198.57	74.39	153.42	134.00	3259.89
t_1	66.41	110.67	41.46	85.51	74.69	1816.91
t_2	648.01	1079.96	404.61	834.40	728.80	17729.75
t_3	115.07	191.77	71.84	148.16	129.41	3148.23
t_4	648.04	1080.00	404.62	834.43	728.82	17730.44
t_5	648.04	1080.00	404.62	834.43	728.82	17730.48
t_6	648.01	1079.95	404.60	834.39	728.79	17729.63
t_7	66.41	110.67	41.46	85.51	74.69	1816.91
t_8	247.18	411.94	154.33	318.27	277.99	6762.80
t_9	647.24	1078.66	404.12	833.40	727.92	17708.49
t_{10}	645.95	1076.51	403.32	831.74	726.47	17673.27
t_{11}	646.75	1077.85	403.82	832.77	727.37	17695.18
t_{12}	647.40	1078.92	404.22	833.60	728.10	17712.82
t_{13}	646.59	1077.59	403.72	832.57	727.20	17690.86
t_{14}	643.36	1072.20	401.70	828.41	723.56	17602.47
t_{15}	119.15	198.57	74.39	153.42	134.00	3259.89
t_{16}	42.02	70.02	26.23	54.10	47.25	1149.59
t_{17}	108.69	181.14	67.86	139.95	122.24	2973.72
t_{18}	93.01	155.00	58.07	119.76	104.60	2544.65
t_{19}	569.74	949.50	355.73	733.61	640.76	15588.13
t_{20}	646.79	1077.91	403.84	832.82	727.42	17696.21
t_{21}	646.58	1077.56	403.71	832.55	727.18	17690.46
t_{22}	646.73	1077.81	403.80	832.74	727.35	17694.61

Results and discussion

It is evident from Table-4 and Table-7 that most of the competing estimators have their MSE in interval [10484.75, 17979.24] for population-1 and in the interval [142599242, 2199360225] for the population-2 respectively. The MSE of the members of the suggested class of estimators lie in the interval [2105.84 84972.56] for population-1 while in interval [12404399, 339386026] respectively. From Table-5 and Table-8, it can be observed that the PRE of the suggested class, over the competing estimators lie in interval [15.13, 884.60] for population-1 and that for population-2, in interval [26.23, 17730.49] respectively.

Conclusion

In the present paper, we suggested six estimators for variance parameter utilizing known median parameter of primary variable. The biases and the mean squared errors of these estimators have been acquire up to the approximation of order one. A comparison of the suggested estimators has been performed with the competing estimators of population variance using auxiliary variables. The efficiency conditions under which suggested estimators performs better than the competing estimators are derived. The theoretical findings are verified using two natural populations. The efficiencies of the suggested estimators over competing estimators are presented in Table-5 and Table-8 for Population-1 and Population-2 respectively. From these tables, it can be seen that the suggested estimators are more efficient than the competing estimators in most of the cases. Thus the suggested estimators may be used for enhanced estimation of variance parameter of the main variable.

Appendix

Expressing t_{p2} in terms of $e_i{}'s$, we get

$$t_{p2} = \kappa_1 S_y^2 (1+e_0) - \kappa_2 M e_1$$

Subtracting S_y^2 on both sides, we have

$$t_{p2} - S_y^2 = \kappa_1 S_y^2 (1+e_0) - \kappa_2 M e_1 - S_y^2$$

Taking expectations and putting values of different expectations, we get bias of S_y^2 as,

$$B(t_{p2}) = (\kappa_1 - 1) S_y^2 - \kappa_2 B(m)$$

Now, we have,

$$t_{p2} - S_y^2 = (\kappa_1 - 1) S_y^2 e_0 - \kappa_2 M e_1 + \kappa_1 S_y^2 e_0$$

Squaring on both sides and simplifying, we get

$$(t_{p2} - S_y^2)^2 = \begin{bmatrix} (\kappa_1 - 1)^2 S_y^4 + \kappa_2^2 M^2 e_1^2 + \kappa_1^2 S_y^4 e_0^2 \\ + 2 S_y^4 \kappa_1 (\kappa_1 - 1) e_0 \\ - 2 S_y^2 \kappa_2 (\kappa_1 - 1) M e_1 - 2\kappa_1 \kappa_2 M S_y^2 e_0 e_1 \end{bmatrix}$$

Taking expectation and putting values of various expectations, we get MSE of t_{p2} as,

$$MSE(t_{p2}) = \begin{bmatrix} (\kappa_1 - 1)^2 S_y^4 + \kappa_2^2 M^2 \gamma C_m^2 + \kappa_1^2 S_y^4 \gamma (\lambda_{40} - 1) \\ -2 S_y^2 \kappa_2 (\kappa_1 - 1) B(m) - 2\kappa_1 \kappa_2 M S_y^2 \gamma \lambda_{21(m)} C_m \end{bmatrix}$$

The $MSE(t_{p2})$ is minimum for,

For, $\kappa_{1(opt)} = \dfrac{V_{20}}{(V'_{02} + V'_{02} V_{20} - V'^2_{11})}$ and

$$\kappa_{2(opt)} = \frac{S_m^2}{S_y^2} \frac{V'_{11}}{(V'_{02} + V'_{02} V_{20} - V'^2_{11})}$$

Where, $V_{20} = \gamma(\lambda_{40} - 1)$, $V'_{02} = \gamma C_m^2$, $V'_{11} = \gamma \lambda_{21(m)} C_m$

And the minimum MSE is,

$$MSE_{\min}(t_{p2}) = \frac{V(t_{p1})}{1 + S_y^{-4} V(t_{p1})}$$

Note: By the same fashion the biases and the mean squared errors have been obtained for other proposed estimators.

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