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Solving Bessel differential equation of order zero using exponentially fitted collocation approximation method

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Abstract

This paper presents analytic-numeric solution Bessel differential equations of order zero using Exponentially Fitted Collocation Approximation Method (EFCAM). This technique was employed to obtain the analytic-numerical solutions of Bessel equations. The method introduces a significant improvement in solving differential equations on mathematical physics. The numerical results obtained by EFCAM are in good agreement with exact solution and available results in literature with little error and showed effectiveness of the proposed method.

Keywords: Bessel equation, order zero, exponentially fitted collocation approximate method (EFCAM), exact solution, analytic-numeric solution.

Introduction

Linear ordinary differential equations of the second order are frequently occur in both chemical and physical sciences. Among the most frequently encountered of such equations is Bessel equation of the form:

$$x^2y^{//} + xy^{/} + (x^2 - n^2)y = 0 \quad\quad 0 \le x \le 1 \quad\quad (1)$$

with initial conditions:

$$y(x_0) = \alpha\,, \quad y^{/}(x_0) = \beta \quad\quad (2)$$

When: $n = 0$ equation (1) is said to be order zero and α, β are constant coefficients.

Bessel differential equation is considered among the most important ordinary differential equations due to its wide applications in heat transfer, vibrations, stress analysis, optics, signal processing and fluid mechanics[1-3].

This equation is named after a German mathematician and astronomer Friedrich Wilhelm Bessel (1784-1846), however, Daniel Bernoulli was the first to introduce the concept of Bessel functions in 1732. Its comes up in many engineering applications such as heat transfer involving the analysis of circular fins, vibration analysis, stress analysis and fluid mechanics. Equations are modeled in all these fields to form differential equations which are in most times modified to form Bessel differential equation[4,5]. There are different methods to solve Bessel differential equation, Gray A. et al.[6] proposed the Laplace transform method that named after mathematician and astronomer Pierre-Simon Laplace which is a powerful integral transform method to solve linear differential equations with given initial conditions, the power series method which is very common method in solving differential equations was employed by McLachlan N.W.[7].

It may be important to employ numerical technique to obtain numerical solution of this equation quickly, though its require within a neighborhood of a singular point which can be defined as a series, rather than in terms of elementary functions. This article is to employ an easy, fast and accurate numerical technique to obtain numerical solution of Bessel equation of order zero.

Definition of Chebyshev polynomials: The Chebyshev polyomials of first kind can be defined by the recurrence relation given by

$$T_0(x) = 1$$

$$T_1(x) = 2x - 1$$

Thus, we have

$$T_{N+1}(x) = 2(2x-1)T_N(x) - T_{N-1}(x) \quad\quad N \ge 1 \quad\quad (3)$$

Formulation of exponentially fitted collocation approximation method (EFCAM)

This section describes the formulation of exponentially fitted collocation approximate method for the numerical solution of equations (1). The whole idea of the method is to use power series as a basis function and its derivative substituted into a slightly perturbed equation which eventually collocated.

Table-1: The first ten (10) Chebyshev polynomial is given as follows:

$T_N(x)$	Chebyshev Polynomials
$T_0(x)$	1
$T_1(x)$	$2x-1$
$T_2(x)$	$8x^2-8x+1$
$T_3(x)$	$32x^3-48x^2+18x-1$
$T_4(x)$	$128x^4-256x^3+160x^2-32x+1$
$T_5(x)$	$512x^5-1280x^4+1120x^3-400x^2+50x-1$
$T_6(x)$	$2048x^6-6144x^5+6912x^4-3584x^3+640x^2-72x+1$
$T_7(x)$	$8172x^7-28672x^6+39424x^5-26880x^4+9408x^3-1568x^2+98x-1$
$T_8(x)$	$32768x^8-131072x^7+212992x^6-180224x^5+84480x^4-21504x^3+2688x^2-128x+1$
$T_9(x)$	$131072x^9-589824x^8+1105920x^7-1118208x^6+658944x^5-228096x^4+44352x^3-4320x^2+162x-1$
$T_{10}(x)$	$52488x^{10}-2621440x^9+5570560x^8-6553600x^7+4659200x^6-2050048x^5+549120x^4-84480x^3+6600x^2-200x+1$

In this this work, we employed approximate solution used in Falade K.I.[8].

The Power series of the form

$$y_N(x) \approx \sum_{k=0}^{N} Y(k)x^k \quad (4)$$

The exponentially fitted approximate solution of the form

$$y_N(x) \approx \sum_{k=0}^{N} Y(k)x^k + \tau_2 e^x \quad (5)$$

Where: x represents the dependent variables, $y(x), Y(k), (k \geq 0), \tau_2$ are the unknown constants to be determined and N is the length of computation (Cbebysev polynomials, Table-1).

The construction process of the EFCAM starts by obtain second derivation of equation (4) and substitutes into equation (1), we have

$$\sum_{k=2}^{N} k(k-1)Y(k)x^{k-2} + \frac{1}{x}\sum_{k=1}^{N} kY(k)x^{k-1} + \sum_{k=0}^{N} Y(k)x^k = 0 \quad (6)$$

Expand equation (6) leads

$$\begin{cases} (2Y(2)+6xY(3)+12x^2Y(4)+\cdots+N(N-1)Y(N)x^{N-2})+ \\ \frac{1}{x}(Y(1)+2xY(2)+3x^2Y(3)+4x^3Y(4)+\cdots+NY(N)x^{N-1})+ \\ (Y(0)+xY(1)+x^2Y(2)+x^3Y(3)+x^4Y(4)+\cdots+Y(N)x^N)=0 \end{cases} \quad (7)$$

Collect the likes terms of equation (7), we obtained

$$\begin{cases} Y(0)+\left(\frac{1}{x}+x\right)Y(1)+ \\ (4+x^2)Y(2)+(9x+x^3)Y(3)+ \\ (9x+x^3)Y(3)+(16x^2+x^4)Y(4)+ \\ \vdots \\ + \\ \left(N(N-1)x^{N-2}+N\frac{1}{x}x^{N-1}+x^N\right)Y(N)=0 \end{cases} \quad (8)$$

Slightly perturbed equation (8) with perturbation the term added to the right hand side of the equation. The addition of the perturbation term is to minimize the error of the problem in consideration[9].

$$\begin{cases} Y(0)+\left(\frac{1}{x}+x\right)Y(1)+ \\ (4+x^2)Y(2)+(9x+x^3)Y(3)+ \\ (9x+x^3)Y(3)+(16x^2+x^4)Y(4)+ \\ \vdots \\ + \\ \left(N(N-1)x^{N-2}+N\frac{1}{x}x^{N-1}+x^N\right)Y(N)=H(x) \end{cases} \quad (9)$$

Where $H(x)=\tau_1 T_N(x)+\tau_2 T_{N-1}(x)$

Where: τ_2 and τ_1 are free tau parameters to be determined and $T_N(x)$ and $T_{N-1}(x)$ are the Chebyshev Polynomials defined by (Table-1) and N is the computational length.

Collocate the equation (9) at point $x=x_q$, we have,

$$\begin{cases} Y(0)+\left(\frac{1}{x_q}+x_q\right)Y(1)+ \\ \left(4+{x_q}^2\right)Y(2)+\left(9x_q+{x_q}^3\right)Y(3)+ \\ \left(9x_q+{x_q}^3\right)Y(3)+\left(16{x_q}^2+{x_q}^4\right)Y(4)+ \\ \vdots \\ + \\ \left(N(N-1){x_q}^{N-2}+N\frac{1}{x_q}{x_q}^{N-1}+{x_q}^N\right)Y(N) \\ -\tau_1 T_N(x_q)-\tau_2 T_{N-1}(x_q)=0 \end{cases} \quad (10)$$

Where $x_q = a+\frac{(b-a)q}{N+2};\ q=1,2,3,\ldots N+1$

Hence, equation (10) gives rise to (N+1) algebraic linear system of equations in (N+1) unknown constants. Two extra equations are obtained by fitting one tau-parameter to the initial condition as suggested by Taiwo O.A. et al.[10] to the initial conditions given:

$$y(x_0) \approx \sum_{k=0}^{N} Y(k)x^k + \tau_2 e^{x_0} = \alpha \tag{11}$$

$$y^{/}(x_0) \approx \sum_{k=0}^{N} kY(k)x^{k-1} + \tau_2 e^{x_0} = \beta \tag{12}$$

Altogether, we obtain (N+3) algebraic linear equations in (N+3) unknown constants. Thus, we put the (N+3) algebraic equations in Matrix form as:

$$MX = G \tag{13}$$

Where: M = the system of equations of N + 3
$X = (Y(0), Y(1), Y(2), \dots\dots, Y(N)\ \tau_1, \tau_2)^T$
$G = (0,0,0 \dots\dots\dots..0, \alpha, \beta\ )^T$

MAPLE 18 software is used to obtain the unknown constants $Y(0), Y(1), Y(2) \dots\dots\dots..Y(N), \tau_1, \tau_2$.

Which then substitute into the exponentially fitted approximate solution (5).

Numerical Implementation

Example-1: Consider the Bessel differential equation[11].

$$y^{//} + \frac{1}{x}y^{/} + y = 0 \tag{14}$$

with initial conditions

$$y(0) = 1,\ y^{/}(0) = 0 \tag{15}$$

Exact solution is given as $y(x) = BesselJ(0, x)$ (16)

EFCAM Technique: Comparing equation (14) with equation (10) and taking computational length = 8 , we obtained.

$$\begin{cases} Y(0) + \left(x_q + \frac{1}{x_q}\right)Y(1) + (4 + {x_q}^2)Y(2) + \\ (9x_q + {x_q}^3)Y(3) + (16{x_q}^2 + {x_q}^4)Y(4) + \\ (25{x_q}^3 + {x_q}^5)Y(5) + (36{x_q}^4 + {x_q}^6)Y(6) + \\ (49{x_q}^5 + {x_q}^7)Y(7) + (64{x_q}^6 + {x_q}^8)Y(8) + \\ -\begin{pmatrix} 32768{x_q}^8 - 131072{x_q}^7 + 212992{x_q}^6 - 180224{x_q}^5 + \\ 84480{x_q}^4 - 21504{x_q}^3 + 2688{x_q}^2 - 128x_q + 1 \end{pmatrix}\tau_1 \\ -\begin{pmatrix} 8192{x_q}^7 - 28672{x_q}^6 + 39424{x_q}^5 - 26884{x_q}^4 + \\ 9410{x_q}^3 - 1568{x_q}^2 + 98x_q - 1 \end{pmatrix}\tau_2 = 0 \end{cases} \tag{17}$$

Collocate equation (17) as follows:
$$x_q = a + \frac{(b-a)q}{N+2};\ q = 1,2,3 \dots\dots..N+1$$

where $a = 0$, $b = 1$, $N = 8$

$$\begin{cases} x_1 = \frac{1}{10}, x_2 = \frac{2}{10}, x_3 = \frac{3}{10} \\ x_4 = \frac{4}{10}, x_5 = \frac{5}{10}, x_6 = \frac{6}{10} \\ x_7 = \frac{7}{10}, x_8 = \frac{8}{10}, x_9 = \frac{9}{10} \end{cases}$$

Consider initial conditions (15) and Matrix equation (13). Thus using MAPLE 18 software to obtain eleveen (11) unkown constants of equations (17), we obtained the following constants

$$\begin{cases} Y(0) = 0.999999996200 \\ Y(1) = -0.0000000037912 \\ Y(2) = -0.249999997300 \\ Y(3) = -0.0000000925470 \\ Y(4) = 0.01562507695000 \\ Y(5) = 0.000000419934581 \\ Y(6) = -0.00043553379104 \\ Y(7) = 0.000001517416246 \\ Y(8) = 0.000006080127317 \\ \tau_1 = -0.00000000013333416 \\ \tau_2 = 0.0000000037918135080 \end{cases}$$

Substitute the above values into approximation solution (5)

Thurs, the approximate solution of the Bessel differential equation (14) is given as

$$y_8(x) \approx Y(0) + Y(1)x + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + Y(5)x^5 + Y(6)x^6 + Y(7)x^7 + Y(8)x^8 + \tau_2 e^x \tag{18}$$

$$y_8(x) \approx 0.999999996200 - 0.0000000037912\, x - 0.249999997300x^2 - 0.0000000925470x^3 + 0.01562507695000x^4 + 0.000000419934581x^5 - 0.00043553379104x^6 + 0.000001517416246x^7 + 0.000006080127317x^8 + 0.0000000037918135080e^x \tag{19}$$

Example-2 Consider the Bessel differential equation[12].

$$x\, y^{//} + y^{/} + 4y = 0 \tag{20}$$

With initial conditions:
$$y(0) = 3,\ \ y^{/}(0) = 0 \tag{21}$$

Exact solution is given as $y(x) = 3BesselJ(0,2x)$ (22)

Comparing equation (20) with equation (10) and taking computational length $N = 10$, we obtained

$$\begin{cases} 4Y(0) + \left(4x_q + \frac{1}{x_q}\right)Y(1) + (4 + 4{x_q}^2)Y(2) + \\ (9x_q + 4{x_q}^3)Y(3) + (16{x_q}^2 + 4{x_q}^4)Y(4) + \\ (25{x_q}^3 + 4{x_q}^5)Y(5) + (36{x_q}^4 + 4{x_q}^6)Y(6) + \\ (49{x_q}^5 + 4{x_q}^7)Y(7) + (64{x_q}^6 + 4{x_q}^8)Y(8) + \\ (81{x_q}^7 + 4{x_q}^9)Y(9) + (100{x_q}^8 + 4{x_q}^{10})Y(10) + \\ -\begin{pmatrix} 52488{x_q}^{10} - 2621440{x_q}^9 + 5570560{x_q}^8 - 6553600{x_q}^7 + 4659200{x_q}^6 - \\ 2050048{x_q}^5 + 549120{x_q}^4 - 84480{x_q}^3 + 6600{x_q}^2 - 200{x_q}^{10} + 1 \end{pmatrix}\tau_1 \\ -\begin{pmatrix} 131072{x_q}^9 - 589824{x_q}^8 + 1105920{x_q}^7 - 1118208{x_q}^6 + \\ 658944{x_q}^5 - 228096{x_q}^4 + 44352{x_q}^3 - 4320{x_q}^2 + 162x_q - 1 \end{pmatrix}\tau_2 = 0 \end{cases} \tag{23}$$

Collocate equation (23) as follows:

$x_q = a + \frac{(b-a)q}{N+2};\ q = 1,2,3 \dots\dots N+1$ where $a = 0,\ b = 1,\ N = 10$

$$\begin{cases} x_1 = \frac{1}{12}, x_2 = \frac{2}{12}, x_3 = \frac{3}{12}, x_4 = \frac{4}{12} \\ x_5 = \frac{5}{12}, x_6 = \frac{6}{12}, x_7 = \frac{7}{12}, x_8 = \frac{8}{12} \\ x_9 = \frac{9}{12}, x_{10} = \frac{10}{12}, x_{11} = \frac{11}{12} \end{cases}$$

Consider initial conditions (21) and Matrix equation (10). Thus using MAPLE 18 software to obtain thirteen (13) unkown constants of equations (23), we obtained the following constants

$$\begin{cases} Y(0) = 3.000000024 \\ Y(1) = 0.000000023914509 \\ Y(2) = -3.0000002990000 \\ Y(3) = 0.000001913327839 \\ Y(4) = -0.74999386380000 \\ Y(5) = 0.0000009714370262 \\ Y(6) = -0.0832794071300 \\ Y(7) = -0.0001752334755 \\ Y(8) = 0.005475650818000 \\ Y(9) = -0.00021514553890 \\ Y(10) = -0.0001300186039 \\ \tau_1 = -0.00000000741764409 \\ \tau_2 = -0.00000002391450891 \end{cases}$$

Substitute the above values into approximation solution (5)

Hence, the approximate solution of the Bessel differential equation (20) is given as

$$y_{10}(x) \approx Y(0) + Y(1)x + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + Y(5)x^5 +$$

$$Y(6)x^6 + Y(7)x^7 + Y(8)x^8 + Y(9)x^9 + Y(10)x^{10} + \tau_2 e^x \quad (24)$$

$$y_{10}(x) \approx 3.000000024 + 0.00000002391509x - -3.000000299000x^2 + 0.000001913327839x^3 - 0.7499938638000x^4 + 0.0000009714370262x^5 - 0.0832794071300x^6 - 0.0001752334755x^7 + 0.005475650818000x^8 - 0.00021514553890x^9 - 0.0001300186037x^{10} - 0.00000002391450891e^x \quad (25)$$

Conclusion

In this study, we implement numerical technique: Exponentially Fitted Collocation Approximation Method (EFCAM) proposed by Falade K.I.[8] to solve Bessel differential equations of order zero. The results obtained in both examples was compared to exact solution and result obtained in Shiralashetti S.C.[11], Entisar A.S.[12]. The numerical solutions showed that EFCAM are more accurate, the computation of the components of the solution are easy and take less time in comparison with other available techniques.

Furthermore, the proposed technique showed a good agreement with exact solution with relative error.

Table-2: Numerical Results.

Example-1: Solution of Bessel Equation			
x	Exact Solution	EFCAM N=8	CWCM[11]
0.0	1.0000000000	1.000000000	1.00000000
0.1	0.9975015621	0.9975015621	0.99749000
0.2	0.9900249722	0.9900249727	0.99001000
0.3	0.9776262465	0.9776262475	0.97761000
0.4	0.9603982267	0.9603982285	0.96039000
0.5	0.9384698072	0.9384698096	0.93847000
0.6	0.9120048635	0.9120048667	0.91200000
0.7	0.8812008886	0.8812008886	0.88119000
0.8	0.8462873528	0.8462873576	0.84628000
0.9	0.8075237981	0.8075238040	0.80752000
1.0	0.7651976866	0.7651976934	0.76518000

Table-3: Numerical Results.

Example-2: Solution of Bessel Equation			
x	Exact Solution	EFCAM N=10	Entisar A.S. et al[12]
0.0	3.000000000	3.000000000	3.000000000
0.1	2.970074917	2.970074915	2.970074917
0.2	2.881194680	2.881194675	2.881194680
0.3	2.736014590	2.736014581	2.736014592
0.4	2.538862058	2.538862045	2.538862080
0.5	2.295593060	2.295593039	2.295593262
0.6	2.013398233	2.013398207	2.013399480
0.7	1.700565361	1.700565327	1.700571167
0.8	1.366206503	1.366206463	1.366228480
0.9	1.019959233	1.019959184	1.020030267
1.0	0.671672337	0.671672279	0.671875000

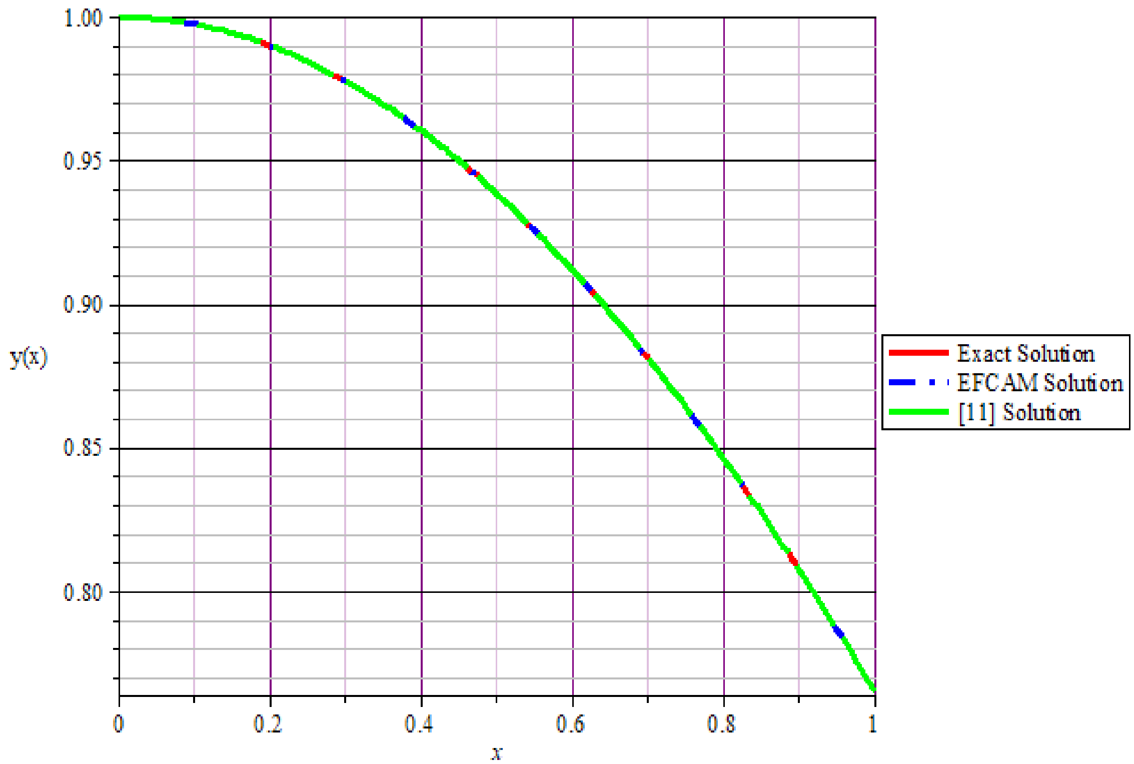

Figure-1: Exact Solution, EFCAM solution and [11] solution Example 1.

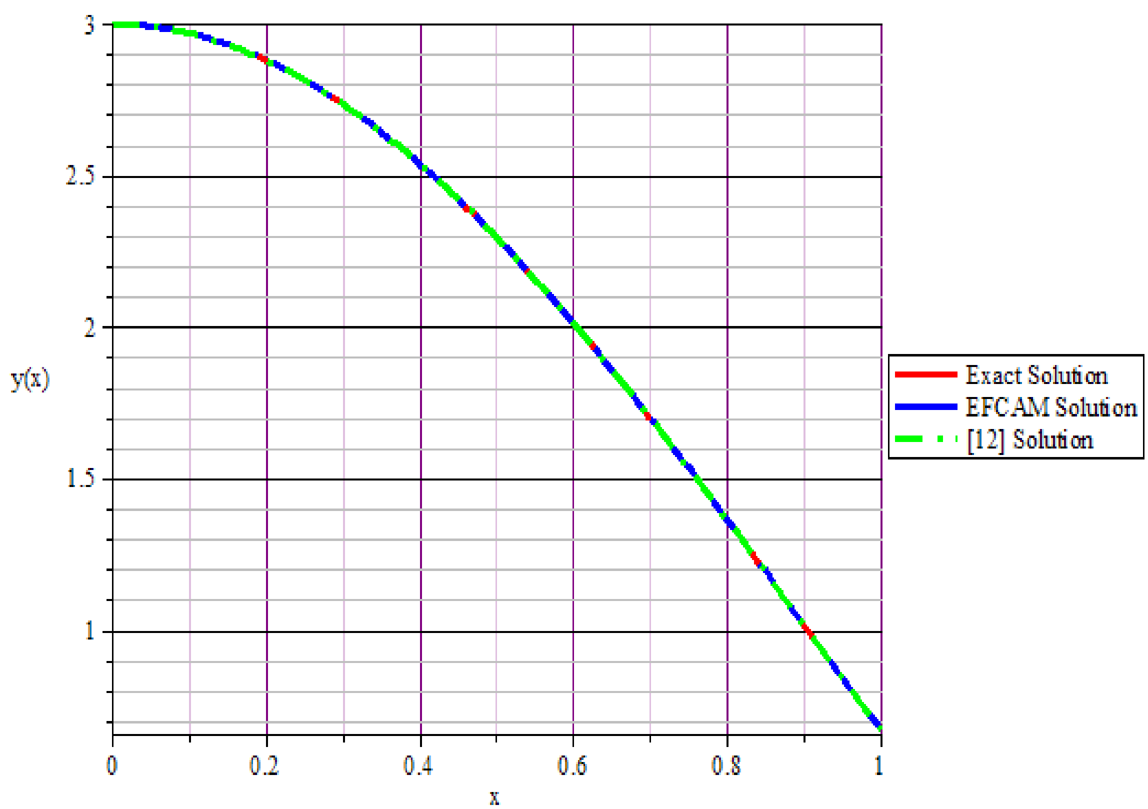

Figure-2: Exact Solution, EFCAM solution and [12] solution Example 2.

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