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Short Review Paper

Orthogonal contrasts and planned contrasts

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Abstract

In analysis of variance, the F-test fails to identify the cause behind the differential effect among the levels of the treatment under consideration. That cause can be easily pointed out through contrasts. Either the investigator plans the contrasts of interest prior to the study or the experimenter searches for something unexpected after the experiment is over, allowing the data itself to suggest additional interesting contrasts. Orthogonality of the contrasts is an important feature which facilitates the testing procedures.

Keywords: Orthogonal contrasts, partition of total sum of squares, planned contrasts.

Introduction

Suppose there are 'p' number of levels of an attribute, referred to as 'treatments', and we wish to know whether these p treatments affect the response variable differently or not. The one-way ANOVA proves its utility in such problems. In comparing p treatments, the null hypothesis is given by H_0: $\mu_1 = \mu_2 = ... = \mu_p$, where μ_i is the mean of the i-th group and the alternative hypothesis states that at least one of the means is not equal.

The above comparison is carried out through an F-test. However this test fails to specify which treatments lead to the differences in means. Hence, the ANOVA F-test provides limited information on the data available. It is here that we first feel the necessity of contrast analysis. There can be two kinds of contrasts[1] as stated below:

Planned Contrasts[2]: Following a significant one-way ANOVA, one may be interested in carrying out some comparisons specific to our problem. Planned or priori contrasts are used when we have certain questions or hypotheses in mind and hence want to reframe them as contrasts for the sake of hypotheses testing. It is just a reconstruction of the questions to which we want answers in a proper mathematical relation between the means.

Post hoc Contrasts[3]: These, on the other hand, are provided by the data itself. We frame the post hoc or posteriori contrast after completion of the experiment and they are framed as per the results of the experiment conducted. The objective of such contrasts is to check though different hypotheses (framed as contrasts) whether the results are reliable or not.

Before performing tests relating to planned contrasts, we are to check whether contrasts planned by the investigator are mutually orthogonal or not. Treatment means are selected so that they can be properly utilized to frame these questions into proper orthogonal contrasts which in turn help in the testing procedures.

These treatment means are so chosen that the sum of squares corresponding to treatments (abbreviated as SST) can be partitioned in such a way that the investigator is able to test for most of his questions through independently (orthogonally) framed contrasts.

Here, we can frame as many orthogonal contrasts as there are degrees of freedom for treatments in the ANOVA.

Orthogonal Contrasts[4,5]: Orthogonality of contrasts ensures independence of the questions to which the investigator seeks answers to. Earlier, we had considered p treatments. Let us define-

$C = \sum_{i=1}^{p} c_i T_i$, where $\sum_{i=1}^{p} c_i = 0$ and $D = \sum_{i=1}^{p} d_i T_i$, with $\sum_{i=1}^{p} d_i = 0.$

C and D will be said to be orthogonal contrasts if we have $\sum_{i=1}^{p} c_i d_i = 0.$

Property-1: For p treatments, we can have at most (p-1) orthogonal treatment contrasts.

Proof: Let $C_1, C_2,...,C_k$ be k treatment contrasts $C_j = \sum_{i=1}^{p} c_{ji} T_i$, where $\sum_{i=1}^{p} c_{ji} = 0 \quad \forall$ j=1(1)k.

The contrasts can be written in matrix form as-

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kp} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_p \end{bmatrix} = \mathrm{C}\underline{T}\,.$$

The matrix C must be orthogonal for the k contrasts to be mutually orthogonal. Thus k must equal p to satisfy the above condition. We know, an orthogonal matrix satisfies the following conditions: i. Sum of square of all row elements is 1. ii. Sum of product of elements of any two rows is 0.

As observed, condition (ii) is nothing but the main definition of two mutually orthogonal contrasts. However, for each of them to be contrasts, the matrix C must also satisfy the condition that sum of all row elements is 0. Due to imposition of this 3rd condition, the number of orthogonal contrasts diminishes by 1. Thus we get (p-1) orthogonal treatment contrasts for p treatments.

Property-2: Treatment sum of squares can be expressed as the sum of squares of estimates of (p-1) mutually orthogonal treatment contrasts.

Proof: Let $C_1, C_2,...,C_{p-1}$ be p-1 mutually orthogonal treatment contrasts $C_j = \sum_{i=1}^{p} c_{ji} T_i$, with $\sum_{i=1}^{p} c_{ji} = 0 \quad \forall$ j=1(1)p-1 and $\sum_{i=1}^{p} c_{ji} c_{j'i} = 0 \ \forall\ j \neq j'$. Then, estimate of C_j is given by –

$\hat{C}_j = \sum_{i=1}^{p} c_{ji} \bar{y}_{i0}$, where $\bar{y}_{i0}$ is the group mean of the group receiving treatment T_i.

Let $\begin{bmatrix} \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} & \cdots & \frac{1}{\sqrt{p}} \\ c_{11} & c_{12} & \cdots & c_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ c_{p-1,1} & c_{p-1,2} & \cdots & c_{p-1,p} \end{bmatrix} \begin{bmatrix} \bar{y}_{10} \\ \bar{y}_{20} \\ \vdots \\ \bar{y}_{p0} \end{bmatrix} = \underset{\sim}{z}$, say. $\Rightarrow \mathrm{C}\underset{\sim}{y} = \underset{\sim}{z}$.

Now, $\underset{\sim}{z}^{\mathrm{T}}\underset{\sim}{z} = \square\, \underset{\sim}{y}^{\mathrm{T}}\underset{\sim}{y} \Rightarrow \sum_{i=1}^{p} z_i^2 = \sum_{i=1}^{p} \bar{y}_{i0}^2$

$\Rightarrow (\sqrt{p}\,.\bar{y}_{00})^2 + \hat{C}_1^{\,2} + \hat{C}_2^{\,2} + \cdots + \hat{C}_{p-1}^{\,2} = \sum_{i=1}^{p} \bar{y}_{i0}^2$

$\Rightarrow \mathrm{SST} = \sum_{i=1}^{p} (\bar{y}_{i0} - \bar{y}_{00})^2 = \hat{C}_1^{\,2} + \hat{C}_2^{\,2} + \cdots + \hat{C}_{p-1}^{\,2}$.

SST has df (p-1). Thus the contrast squares will have df 1.

Planned orthogonal contrasts[6-8] are equivalent to independent enquiries pertaining to our data. Neither can all questions of the investigator be framed through orthogonal contrasts, nor do all possible orthogonal contrasts lead to meaningful questions about the experimental results. Here, we provide some ways to set the contrasts depending on the research questions.

Let us consider the following data which gives the seed yield of a plant under implementation of type I fertilizing procedure, type II fertilizing procedure, type III fertilizing procedure and a control.

In Table-1 we have 4 treatments. Hence, at most 3 research questions can be planned – i. Do fertilizing procedures affect the yield?, ii. Is there a difference between type I, type II fertilizers and the type III fertilizer? iii. Do the fertilizers I and II affect the yield differently?

To form the planned contrasts, we must reframe these research questions of interest as linear combination of the treatments. The coefficients of the planned contrasts are show in Table-2.

The 3 research questions can now be easily framed as statistical hypotheses. For instance, the first contrast is same as asking whether the population mean of control is same as the means of population for type I, II and III. If μ_0 , μ_1 , μ_2, μ_3 denote the population means of yields of the control group, type I, II, III fertilizers receiving groups respectively, then the hypothesis can be given by-

H_o : $3\mu_0 - \mu_1 - \mu_2 - \mu_3 = 0$ or $H_o: C_1 = 0$

The hypothesis $3\mu_0 - \mu_1 - \mu_2 - \mu_3 = 0 \Rightarrow \mu_0 = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3)$

$\Rightarrow$Control group mean = Mean of all fertilizer-receiving groups

Similarly, questions 2 and 3 can be expressed as the following statistical hypotheses –

H_{o2}: $\mu_1 + \mu_2 - 2\mu_3 = 0$ and H_{o3}: $\mu_1 - \mu_2 = 0$ or equivalently H_{o2}: $C_2 = 0$ and H_{o3}: $C_3 = 0$

Table-1: Seed yield under different fertilizers and a control.

Control	7.6	8.3	9.6	8.8	10.1	8.5	7.9	8.5	9.4	9.8
Type I	12.2	12.4	11.9	11.3	11.8	12.1	13.1	12.7	12.4	11.4
Type II	16.6	15.8	16.5	15	15.4	15.6	15.8	15.8	16	15.8
Type III	9.5	9.5	9.6	8.8	9.5	9.8	9.1	10.3	9.5	8.5

Table-2: Coefficients of Planned Contrasts.

Contrasts	Control	Type I	Type II	Type III
C_1	+3	-1	-1	-1
C_2	0	+1	+1	-2
C_3	0	+1	-1	0

It can be easily verified that the three contrasts are orthogonal and hence the SST can be split up as $\hat{C}_1^2 + \hat{C}_2^2 + \hat{C}_3^2$, each containing df 1. The F-tests for testing each of the contrast hypotheses are carried out through F-statistics, whose numerators are nothing but the contrast sum of squares. All calculations are given in the following ANOVA Table-3.

Table-3: ANOVA

Source	df	SS	MS	F
Treatment	3	305.243	101.748	272.21
Control vs. fertilizers	1	97.56033	97.56033	261.011
Types I & II vs. type III	1	139.2327	139.2327	372.501
Type I & type II	1	68.45	68.45	183.130
Error	36	13.456	0.374	

The contrast sum of squares are computed by $SS(C) = MS(C) = \frac{(\sum c_i \bar{y}_{i0.})^2}{\sum (c_i^2 / r)}$ where r is the number of units receiving the i^{th} treatment.

We observe that in the above analysis, fertilizers significantly enhance plant yield, that the types I and II cause significantly more growth than the type III fertilizer, and that the difference between type I and type II fertilizers is also significant.

The problem that arises when the planned contrasts are not orthogonal (i.e., the comparisons are not independent) is that the sum of square corresponding to one contrast is either a subset or a superset of the sum of squares due to any other contrast. As a result, the different F-tests carried out for the different contrasts will get confounded or contaminated or mixed up with one another and hence the sum of squares of the individual contrasts will never add up to the treatment sum of squares SST.

General rule for formulation of planned contrasts[9,10]

Here, we first decide the groups of treatments in the contrast under consideration, i.e., a particular contrast divides the treatments into groups which confront each other in the hypothesis. Once we decide upon the groups, we assign to the means of the first group, the coefficients equaling the number of treatments in the second group. Similarly, we assign to the means of the second group, the coefficients equaling the number of treatments in the first group, but this time with an opposite sign. For example, we have 5 treatments and we are interested in comparing the first two to the last three. Then we assign the coefficients respectively as +3, +3, -2, -2, -2 or -3, -3, +2, +2, +2. Both these will give the same contrast sum of squares in the corresponding F-test.

Also, the coefficients of the means in any contrast must be reduced to the smallest possible integers, i.e., instead of +4, +4, -2, -2, -2, we must have +2, +2, -1, -1, -1.

Conclusion

Contrast analysis comes into play when information from ANOVA F-test is limited. There can be two kinds of contrasts-planned and post hoc. This paper mainly deals with the planned contrasts. The only thing that is to be kept in mind before performing a planned contrast analysis is that the planned questions must correspond to orthogonal contrasts. This actually establishes the independence of the enquiries of the investigator. When comparisons are not orthogonal, the contrast sum of squares may be jumbled up. The method of determining coefficients of planned contrasts has also been discussed.

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