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Short Communication

On the characteristic and moment generating functions of type-2 (*Fréchet*) and type-3 (*reversed Weibull*) distributions

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Abstract

We are able to derive for the first time the simplest forms of characteristic functions (CHFs) and moment generating functions (MGF)of type-2(Fréchet)and type-3 (reversed Weibull)in explicit closed forms by direct and unique methodology. CHF and MGF have wide applications in statistical theories such as in inversion and convolution.

Keywords: Characteristic Function, Moment generating function, *GEV* distribution, *Fréchet* distribution, *reversed Weibull* distribution.

Introduction

The generalised extreme value distribution was introduced by Jenkinson[1,2] and recommended by Natural Environment Research Council of Great Britain[3]. The *GEV* distribution is the most widely accepted distribution for flood frequency data[4-8].

Let X be a generalised extreme value (*GEV*) random variable represented by the probability density function as

$$f_X(x)dx = \frac{1}{\alpha}\left[1-\kappa\left(\frac{x-\xi}{\alpha}\right)\right]^{\frac{1}{\kappa}-1} exp\left\{-\left[1-\frac{\kappa(x-\xi)}{\alpha}\right]^{\frac{1}{\kappa}}\right\}dx \qquad (1)$$

for range of x: $-\infty < x \leq \xi + \frac{\alpha}{\kappa}\ if\ \kappa > 0;\ -\infty < x < \infty\ if\ \kappa = 0;\ \xi + \frac{\alpha}{\kappa} \leq x \leq \infty\ \ if\ \kappa < 0$

In this paper we derived a closed form for characteristic function (*CHF*) and moment generating function (*MGF*) of generalised extreme value distribution (*GEV*) for $\kappa>0$ (also known as type 3 *EVD* or reversed *Weibull* distribution) and type-2 *EVD*($\kappa< 0$; also known as *Fréchet* distribution).

When $\kappa = 0$, then

$$f_X(x)dx = \frac{1}{\alpha} exp\left[-\left(\frac{x-\xi}{\alpha}\right)\right] exp\left\{-\left[exp-\left(\frac{x-\xi}{\alpha}\right)\right]\right\} \qquad (2)$$

is the *Gumbel* probability density function.

The probability density function of *Fréchet* distribution is given as:

$$f_X(x)dx = \frac{\kappa}{\alpha}\left(\frac{x-\xi}{\alpha}\right)^{-\kappa-1} exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{-\kappa}\right]dx; x > \xi \qquad (3)$$

ξ, α and κ are respectively location, scale and shape parameters.

Derivations of type-2 and type-3 extreme value distributions

CHF of type-3 EVD: The probability density function of generalised extreme value distribution (*GEV*) can be given as:

$$f_Z(z)dz = (1-\kappa z)^{\frac{1}{\kappa}-1} exp\left[-(1-\kappa z)^{\frac{1}{\kappa}}\right]dz, z = \frac{x-\xi}{\alpha};\ -\infty < z \leq \frac{1}{\kappa}\ ;\ \kappa > 0 \qquad (4)$$

When $\kappa > 0$, then (1) is the probability density function of type-3 extreme value distribution and also known as the *reversed Weibull* distribution.

The *CHF* of (1) can be computed as

$$\phi_Z(t;\xi,\alpha,\kappa) = \int_R [cos(t\alpha z)f(z) + isin(t\alpha z)f(z)]dz \qquad (5)$$

t-any arbitrary real constant

$i = \sqrt{-1}$

Integrating Ist term by parts ⇒

$$\int_R cos(t\alpha z)f(z)dz = exp(it\xi)\left\{cos\left(\frac{t\alpha}{\kappa}\right) + t\alpha\int_R sin(t\alpha z) exp\left[-(1-\kappa z)^{\frac{1}{\kappa}}\right]dz\right\} \textbf{A}$$

Now integrating IInd term by parts ⇒

$$\int_R isin(t\alpha z)f(z)dz = exp(it\xi)\left\{\left[isin\left(\frac{t\alpha}{\kappa}\right) - it\alpha\int_R cos(t\alpha z)exp\left[-(1-\kappa z)^{\frac{1}{\kappa}}\right]dz\right]\right\}$$

B

A+B ⇒

$$\phi_Z(t;\xi,\alpha,\kappa) = exp(it\xi)\left\{exp\left(\frac{it\alpha}{\kappa}\right) - it\alpha\int_R exp\left[it\alpha z - (1-\kappa z)^{\frac{1}{\kappa}}\right]dz\right\} \quad (6)$$

Adding the integral of each product obtained by multiplying each term of Taylor´s series expansion of *exp* (*itαz*) with $exp\left[-(1-\kappa z)^{\frac{1}{\kappa}}\right]$ and substituting $(1-\kappa z)^{\frac{1}{\kappa}} = y$ in the integrals of the products leads to the *CHF* of type-3 *EVD* as:

$$\phi_X(t;\xi,\alpha,\kappa) = exp\left[it\left(\xi+\frac{\alpha}{\kappa}\right)\right]\sum_{r=0}^{\infty}\frac{(-it\alpha)^r}{r!\kappa^r}\Gamma(1+r\kappa)\,; r = 0,1,2, \quad (7)$$

CHF of type-2 EVD: In this work, the *CHF* of the *Fréchet* distribution is derived by the direct and lucid methodology discussed in the previous section. The *CHF* can be derived as:

$$\phi_X(t;\xi,\alpha,\kappa) = \int_R[cos(tx)f(x) + isin(tx)f(x)]dx \quad (8)$$

Integrating Ist term by parts⇒

$$\int_R cos(tx)f(x)dx = cos(t\xi) + t\int_R sin(tx)exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{-\kappa}\right]dx\mathbf{C}$$

Now integrating IInd term by parts ⇒

$$\int_R isin(tx)f(x)dx = isin(t\xi) - it\int_R cos(tx)exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{-\kappa}\right]dx$$

D

C+D ⇒

$$\phi_X(t;\xi,\alpha,\kappa) = exp(it\xi) - it\int_R exp\left[itx - \left(\frac{x-\xi}{\alpha}\right)^{-\kappa}\right]dx \quad (9)$$

Adding the integral of each product obtained by multiplying each term of *Taylor´s* series expansion of *exp* (*itx*) with $exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{-\kappa}\right]$ and substituting $\left(\frac{x-\xi}{\alpha}\right)^{-\kappa} = y$ in the integrals of the products leads to the *CHF* of *Fréchet* distribution as:

$$\phi_X(t;\xi,\alpha,\kappa) = exp(it\xi)\sum_{r=0}^{\infty}\frac{(it\alpha)^r}{r!}\Gamma\left(1-\frac{r}{\kappa}\right); r = 0,1,2, \quad (10)$$

The recent paper by Nadarajah and Pogány[9] obtained the *CHF* of type-2 *EVD* that uses the integral referred to as the complex parameter *Kratzel* function. The methodology applied by us is direct, unique and lucid and the expression for characteristic function of type-2 *EVD* derived here are in closed form and are simpler than that obtained by Nadarajah and Pogány[9] and by Muraleedharan et.al[10] and hence suitable for further statistical applications.

MGF of type-3 EVD: The *MGF*, $M_X(\theta;\,\xi,\alpha,\kappa)$ of reversed *Weibull* distribution (*κ*> 0) can be computed as

$$M_Z(\theta;\xi,\alpha,\kappa) = exp(\theta\xi)\int_R(1-\kappa z)^{\frac{1}{\kappa}-1}exp\left[\theta\alpha z - (1-\kappa z)^{\frac{1}{\kappa}}\right]dz \quad (11)$$

θ – arbitrary real constant

Adding the integral of each product obtained by multiplying each term of *Taylor´s* series expansion of $exp(\theta\alpha z)$with$(1-\kappa z)^{\frac{1}{\kappa}-1}exp\left[-(1-\kappa z)^{\frac{1}{\kappa}}\right]$ and substituting $(1-\kappa z)^{\frac{1}{\kappa}} = y$ in the integrals of the products leads to the *MGF* of type-3 EVD as

$$M_X(\theta;\xi,\alpha,\kappa) = exp(\theta\xi)\sum_{r=0}^{\infty}\left[\frac{(-\theta\alpha)^r}{r!\kappa^r}\Gamma(1+r\kappa)\right]\;; r = 0,1,2,\ldots \quad (12)$$

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The raw moments of the reversed *Weibull* distribution can be obtained from $M_X(\theta;\xi,\alpha,\kappa)$. Ie.

$$M_X^{(n)}(0;\xi,\alpha,\kappa) = \acute{\mu_n} = \left[\left(\frac{d^n M_X(\theta;\xi,\alpha,\kappa)}{d\theta^n}\right)\right]_{\theta=0} \quad (13)$$

$\acute{\mu_n}$ $-n^{th}$ raw moment

MGF of type-2 EVD: The *MGF* of *Fréchet* distribution can also be computed by the same methodology as:

$$M_X(\theta;\xi,\alpha,\kappa) = \int_R\frac{\kappa}{\alpha}\left(\frac{x-\xi}{\alpha}\right)^{-\kappa-1}exp\left[\theta x - \left(\frac{x-\xi}{\alpha}\right)^{-\kappa}\right]dx \quad (14)$$

Adding the integral of each product obtained by multiply each term of *Taylor´s* series expansion of $exp(\theta x)$ with $\frac{\kappa}{\alpha}\left(\frac{x-\xi}{\alpha}\right)^{-\kappa-1}exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{\kappa}\right]$ and substituting $\left(\frac{x-\xi}{\alpha}\right)^{\kappa} = y$ in the integrals of the products leads to the *MGF* of *Fréchet* distribution as

$$M_X(\theta;\xi,\alpha,\kappa) = exp(\theta\xi)\sum_{r=0}^{\infty}\frac{(\theta\alpha)^r}{r!}\Gamma\left(1-\frac{r}{\kappa}\right);\; r = 0,1,2, \quad (15)$$

Discussion

The *CHFs* and *MGFs* of *reversed Weibull* (*κ*>0;) and *Fréchet* (*κ*< 0) distributions derived here are new and are in simplest closed forms. The methodology is also unique, lucid and direct. The *CHFs* satisfy the tests for a function to be a characteristic function[11,12] such as: i. That $\phi_X(t)$ must be continuous in *t*, ii. That$\phi_X(t)$ is defined in every finite *t* interval, iii. 3) That $\phi_X(0) = 1$ and 4) That $\phi_X(t)$ and $\phi_X(-t)$ shall be conjugate quantities.

CHFs are very useful some of which to mention are to, develop sampling distributions based on the method of characteristic functions, generate cumulants, apply in the theory of decomposability of random variables etc. The *MGFs* can also find applications such as in inversion, convolution of individual probability density functions of independent random variables, generation of moments of random variables, in continuity theorem etc.

Conclusion

The characteristic and moment generating functions of generalised extreme value distribution (*GEV*) for its shape parameter $\kappa > 0$ (*reversed Weibull*) and $\kappa < 0$ (*Fréchet* distribution) are derived for the first time by a direct, lucid and unique methodology. The expressions are new and simple. The *CHFs* satisfied the tests for a function to be a characteristic function and the *MGFs* are able to generate all the raw moments of their respective probability distributions.

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