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Analysis of an SIVR epidemic model with different optimal control strategies

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Abstract

This paper presents the optimal control applied to a non-linear mathematical SIVR epidemic model. To investigate optimal control strategy of the SIVR model to minimize the infection in minimum cost is discussed with help of three controls and are derived and analyzed by considering different objective functions with the same control variables in all strategies. It is demonstrated by the analytical findings, the effect of choosing different objective function on the state variables with the help of numerical results. This study show that different strategies using different objective functions for an epidemic results in a significant effect to slow down the epidemic.

Keywords: Optimal control theory, Pontryagin's maximum principle, Epidemics.

Introduction

The communicable diseases models have a very long history of support in public health planning and policy creation. The largest part of epidemics control is, if sufficient and timely steps (e.g. vaccination, treatment and awareness about disease etc.) are taken within the duration of the epidemic. Many of the communicable diseases are tending to endemic form due to shortage of appropriate interventions to control the diseases spread. Consequently, there is a necessity of suitable steps towards control of the disease outbreak, essentially those for which treatment is available. It is good to prevent the disease occurrence than treating infective individual. However, diseases like Influenza, Cholera, Tuberculosis and Measles have an approved vaccine and medical treatment Gaff1. Even though the treatment and vaccine are exist still the diseases are reaching the state of endemic. The significant tools to analyze and control the outbreak of the epidemics are mathematical modeling. The study on the optimal control of infectious diseases by constructing mathematical model for the analysis is reviewed from literatures [2-5] etc. Granish et al[2], has driven mathematical model through simulations for HIV model, and obtained the results that wide spread HIV testing followed by an immediate start of antiretroviral therapy for the infected individuals is a strategy towards controlling HIV epidemic in the population. Wang et al.[6] has designed an improved model for Hepatitis B virus (HBV) for control of the epidemic and investigate the model solution with different control strategy to reduce the disease induced death cases.

Pontryagin et al.[7] has introduced the optimal control theory by using Pontryagins maximum principle and it is developed by Fleming and Rishel[8]. Optimal control theory is effectively applied in a many different fields and explored mathematical models. In case of epidemics including HIV is studied by Adams et al.[9], Makinde and Okosun[10], Okosun et al.[11], Joshi[12] and pandemic influenza and vector–borne diseases studied by Blayneh[13]. Okosun et al.[11] investigated the elementary role of optimal control theory to discover the effect of treatment and observation of unaware infective on HIV/AIDS epidemic.

The model considered in the study is a continuation of SIR model of an epidemic by the inclusion of vaccination, time dependent control parameters and assuming that the infective individuals may spread the epidemic wildly. Sunmi et al.[14], in one of his article has discussed the comparison of objective functions with different control strategies, but in this study, optimal control analysis is carried out and discussed optimal control analysis qualitatively for the resulting model with different strategies of objective functions with all controls and observed the variation in the infection with respect to objective functions.

The results are presented and compared through numerical simulations. Rest of the paper continues with four sections; Section 2 deals with formulation of mathematical model, in Section 3 the model with optimal control is analyzed, in Section 4 has the numerical summary and concluding remarks are presented in Section 5.

Mathematical Model

The population at time '*t*' is sub-divided into the following sub-populations, Suppose '*S*' denotes the susceptible number at time '*t*', '*I*' denotes the infective number at time '*t*', '*V*' denotes the number of vaccinated individuals at time '*t*', '*R*' represents the number recovered individuals with permanent immunity at time '*t*' and '*D*' represents the number of disease induced death individuals at time '*t*'. The dynamics of population considered above are mathematically shown and expressed by following system of nonlinear differential equations:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \beta S(t) I(t) - \alpha S(t) - \mu s(t) + \omega V(t) \\
\frac{dI(t)}{dt} &= \beta S(t) I(t) - \gamma I(t) - \mu I(t) - \delta I(t) \\
\frac{dV(t)}{dt} &= \alpha S(t) - \omega V(t) - \mu V(t) \\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t) \\
\frac{dD(t)}{dt} &= \delta I(t)
\end{aligned} \quad (1)$$

Where,
$S(0) > 0, I(0) > 0, \ V(0) \geq 0, V(0) \geq 0, \ D(0) \geq 0$ and $N(t) = S(t) + R(t) + V(t) + R(t) + D(t)$. The parameters used in the model are described in Table-1.

Table 1: Parameters description of the model

Parameter	Description
Λ	The recruitment rate.
β	The infection rate.
γ	The recovery rate.
α	The vaccination rate.
δ	The disease induced death rate.
ω	The rate of vaccinated immunity wanes.
μ	Natural death rate from each class.

$$\begin{aligned}
\frac{dN(t)}{dt} &= \Lambda - \mu N(t) + \mu D(t) \\
&\leq \Lambda - \mu N(t)
\end{aligned} \quad (2)$$

For the solution of equation (2), we have

$$0 \leq N(t) \leq N(0) e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}) \quad (3)$$

where: $N(0)$ is a sum of the states initial values $S(0), I(0), V(0)$ *and* $R(0)$. As $t \to \infty, N(t) \to \frac{\Lambda}{\mu}$, then $\frac{\Lambda}{\mu}$ is the upper bound of $N(t)$. However if $N(0) > \frac{\Lambda}{\mu}$, then the solution approaches asymptotically to the feasible region Ω defined by

$$\Omega = \left\{ (I, V, R, N, D) \in \Re_+^5 : 0 \leq I + V + R \leq N \leq \frac{\Lambda}{\mu} \right\} \quad (4)$$

Hence, it sufficient to prove the positivity of states and understand the dynamics of the epidemic model.

Optimal Control Analysis

In this paper, dynamics of SIVR model solutions of epidemic are examined through various objective functions. Here it is necessary to consider time dependent controls and then continued by applying Prontrygins maximum principle to investigate effective control in a finite time. Controls which are introduced in the model system (1) are as follows: i. Control $u_1 \in [0,1]$ is the successful practice of non-pharmaceutical interventions which are susceptible to protect themselves from attack of the disease on a time interval $[0, t_f]$. ii. Control $u_2 \in [0,1]$ is the hospitalization of infective individuals in the time interval $[0, t_f]$. iii. Control $u_3 \in [0,1]$ refers to successful practice of prevention of transmission in the time interval $[0, t_f]$

The Mathematical SIVR Model with controls is given as follows (5)

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \beta(1 - u_1(t)) S(t) I(t) - \alpha(1 + u_2(t)) S(t) - \mu s(t) + \omega V(t) \\
\frac{dI(t)}{dt} &= \beta(1 - u_1(t)) S(t) I(t) - \gamma(1 + u_3(t)) I(t) - \mu I(t) - \delta I(t) \\
\frac{dV(t)}{dt} &= (1 + u_2(t)) \alpha S(t) - \omega V(t) - \mu V(t) \\
\frac{dR(t)}{dt} &= (1 + u_3(t)) \gamma I(t) - \mu R(t) \\
\frac{dD(t)}{dt} &= \delta I(t)
\end{aligned} \quad (5)$$

where:
$S(0) > 0, I(0) > 0, V(0) \geq 0, R(0) \geq 0$ *and* $D(0) \geq 0$.

Controlling the serious threat of any epidemic requires (our goal is) to minimize the infection and cost of controls during the course of an epidemic. In the literature different objective functions were considered to analyses the effect of control variables on various epidemic models. Here our aim is to suggest best control course of action to control the serious threat of an epidemic. For that purpose explored constructed the objective functions of different behavior with three control variables. This approach helps to investigate best optimal strategy of the objective functions to control the epidemic.

Optimal control strategy of minimization of infective: It is introduced into the model (4), controls u_1, u_2 and u_3 as a time dependent controls to control the outbreak of an epidemic. To determine optimal strategy for controlling the epidemic, it is considered the objective functional J_1, which is of minimization of the number of infective and cost of controls.

$$J_1(u_1(t), u_2(t), u_3(t)) = \int_0^{tf} \left\{ A_1 I(t) + \frac{B_1}{2} u_1^2(t) + \frac{B_2}{2} u_2^2(t) + \frac{B_3}{2} u_3^2(t) \right\} dt \quad (6)$$

where A_1, $\frac{B_1}{2}$, $\frac{B_2}{2}$ and $\frac{B_3}{2}$ are non-negative weights and the terms $\frac{B_1}{2} u_1^2(t)$, $\frac{B_2}{2} u_2^2(t)$, and $\frac{B_3}{2} u_3^2(t)$ are the costs

associated with the controls u_1, u_2 and u_3 respectively. Here it is chosen the quadratic costs on the controls, by review of the literature of epidemic. The optimal controls u_1^*, u_2^* and u_3^* are such that

$$J_1(u_1^*, u_2^*, u_3^*) = \min_{(u_1,u_2,u_3)\in\Gamma} J_1(u_1, u_2, u_3) \quad (7)$$

where:

$$\Gamma = \left\{ \begin{matrix} J(u_1,u_2,u_3) \mid u_i(t) \text{ is piecewise continuous function} \\ \text{on } [0,t_f] \mid 0 \le u_i(t) \le 1, i = 1,2,3 \end{matrix} \right\}$$

is the control set. The Pontryagins maximum principle[7] must satisfied by optimal control problem is the necessary conditions to check. This principle converts equation (5) and (6) into a problem of minimization of Hamiltonian *H*, with respect to $u_1(t),\ u_2(t)$ and $u_3(t)$

$$\begin{aligned} H &= A_1 I(t) + \frac{B_1}{2}u_1^2(t) + \frac{B_2}{2}u_2^2(t) + \frac{B_3}{2}u_3^2(t) \\ &+ \lambda_S(\Lambda - (1-u_1(t))\beta S(t)I(t) - (1+u_2(t))\alpha S(t) - \mu s(t) + \omega V(t)) \\ &+ \lambda_I((1-u_1(t))\beta S(t)I(t) - (1+u_3(t))\gamma I(t) - \mu I(t) - \delta I(t)) \\ &+ \lambda_V((1+u_2(t))\alpha S(t) - \omega V(t) - \mu V(t)) \\ &+ \lambda_R((1+u_3(t))\gamma I(t) - \mu R(t)) \\ &+ \lambda_D(\delta I(t)) \end{aligned} \quad (8)$$

Where: $\lambda_S, \lambda_I, \lambda_V, \lambda_R$ *and* λ_D are the adjoint variables associated with the states variables S, I, V, R and D respectively. The system of adjoint variables is found by differentiating (8) with respect to the associate state variable.

Theorem 1: Optimal controls $u_1^*(t),\ u_2^*(t)$ *and* $u_3^*(t)$ solution $S^*(t),\ I^*(t)$ *and* $V^*(t)$ of the consequent state system (5) that minimizes $J_1(u_1(t), u_2(t), u_3(t))$ over the control set Γ, $\exists$ adjoint variables $\lambda_S, \lambda_I, \lambda_V, \lambda_R, \lambda_D$ satisfying.

$$-\frac{d\lambda_i}{dt} = \frac{\partial H}{\partial i} \quad (9)$$

with the condition of transversality

$$\lambda_S(t_f) = \lambda_I(t_f) = \lambda_V(t_f) = \lambda_R(t_f) = \lambda_D(t\ ) = 0, \quad \text{where} \quad i = S, I, V, R, D \quad (10)$$

Further,

$$\begin{aligned} u_1^* &= \min\left\{1,\ \max\left[0,\ \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1}\right]\right\} \\ u_2^* &= \min\left\{1,\ \max\left[0,\ \frac{\alpha S(t)(\lambda_S - \lambda_V)}{B_2}\right]\right\} \\ u_3^* &= \min\left\{1,\ \max\left[0,\ \frac{\gamma I(t)(\lambda_I - \lambda_R)}{B_3}\right]\right\} \end{aligned} \quad (11)$$

Proof: An optimal control existence is followed from the corollary 4.1 of Fleming and Rishel[8], since the integrand J_1 is a convex function of $u_1(t), u_2(t)$ *and* $u_3(t)$, Lipschitz property is satisfied by a priori boundedness of the state system and state solutions, with respect to the state variables. The adjoint system is

$$\begin{aligned} \frac{d\lambda_S}{dt} &= \lambda_S(1-u_1(t))\beta I(t) - \lambda_S(1+u_2(t))\alpha - \lambda_S\mu - \lambda_I(1-u_1(t))\beta I(t) - \lambda_V(1+u_2(t))\alpha\} \\ \frac{d\lambda_I}{dt} &= -A_1 + (1-u_1(t))\beta S(t)(\lambda_s - \lambda_I) + (1+u_3(t))\gamma(\lambda_I - \lambda_R) - \lambda_D\delta \\ \frac{d\lambda_V}{dt} &= \lambda_V\mu \\ \frac{d\lambda_R}{dt} &= \lambda_R\mu \\ \frac{d\lambda_D}{dt} &= 0 \end{aligned}$$

with the condition of transversality

$$\lambda_S(t_f) = \lambda_I(t_f) = \lambda_V(t_f) = \lambda_R(t_f) = \lambda_D(t_f) = 0, \quad \text{where } i = S, I, V, R, D \quad (13)$$

Due to the priori boundedness of the state system and the Lipschitz structure results of the ordinary differential equations, it is obtained the uniqueness of the optimal control for small t_f. The uniqueness of the optimality system is used for obtaining the optimal control uniqueness, which consist of equation (9) and (10) with characterization (11). To certify the uniqueness of the optimality system there is a limitation on the length of time interval. The certification of the uniqueness of optimality system has a limitation on the duration of time interval. This smallness restriction of the duration on time is due to the contradictory time operations of equation (9) and (10). The state system has initial values where as the adjoint system has terminal values. To minimize Hamiltonian with respect to the controls at the optimal point, it is derived with respect to $u_1(t), u_2(t)$ and $u_3(t)$ on the control set Γ, and equating to zero.

$$\begin{aligned} \frac{\partial H}{\partial u_1} = 0 \quad \text{gives} \quad u_1^*(t) &= \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1} \\ \frac{\partial H}{\partial u_1} = 0 \quad \text{gives} \quad u_2^*(t) &= \frac{\alpha S(t)(\lambda_S - \lambda_V)}{B_2} \\ \frac{\partial H}{\partial u_1} = 0 \quad \text{gives} \quad u_3^*(t) &= \frac{\gamma I(t)(\lambda_I - \lambda_R)}{B_3} \end{aligned} \quad (14)$$

By standard control arguments involved in the bonds on the controls, it is found that $u_1(t)$:

$$u_1^* = \begin{cases} 0 & \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1} \le 0 \\ \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1} & 0 < \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1} < 1 \\ 1 & \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1} \ge 1 \end{cases} \quad (15)$$

In compact form

$$u_1^* = \min\left\{1,\ \max\left[0,\ \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1}\right]\right\} \quad (16)$$

Similarly for u_1 and u_2

$$u_2^* = \min\left\{1, \max\left[0, \frac{\alpha S(t)(\lambda_S - \lambda_V)}{B_2}\right]\right\}$$
$$u_3^* = \min\left\{1, \max\left[0, \frac{\gamma I(t)(\lambda_I - \lambda_R)}{B_3}\right]\right\} \quad (17)$$

Hence the theorem.

Optimal strategy of minimization of infective and susceptible and maximization of vaccinated and recovered individuals: Here it is considered the objective functional $J_2(u_1, u_2, u_3)$ with the controls $u_1(t), u_2(t)$ and $u_3(t)$ such that

$$J_2(u_1(t),u_2(t),u_3(t)) = \int_0^{tf}\left\{\begin{matrix} A_1 I(t) + A_2 S(t) - A_3 V(t) - \\ A_4 R(t) + \frac{B_1}{2}u_1^2(t) + \frac{B_2}{2}u_2^2(t) + \frac{B_3}{2}u_3^2(t)\end{matrix}\right\} dt \quad (18)$$

where: A_1, A_2, A_3, A_4, $\frac{B_1}{2}$, $\frac{B_2}{2}$ and $\frac{B_3}{2}$ are positive weights. The terms $\frac{B_1}{2}u_1^2(t)$, $\frac{B_2}{2}u_2^2(t)$, and $\frac{B_3}{2}u_3^2(t)$ are the costs associated with the controls u_1, u_2 and u_3 respectively. Here it is chosen the quadratic costs on the controls, by referring the literature of epidemic. With the considered objective function $J_2(u_1, u_2, u_3)$ in (18) is implied to minimize the infectives, susceptible and cost of controls $u_1(t), u_2(t)$ and $u_3(t)$ and maximize the vaccinated individuals. The optimal controls u_1^*, u_2^* and u_3^* are such that

$$J_2(u_1^*, u_2^*, u_3^*) = \min_{(u_1,u_2,u_3)\in\Gamma} J_2(u_1, u_2, u_3) \quad (19)$$

where:

$$\Gamma = \left\{J(u_1,u_2,u_3) \mid u_i(t) \text{ is piecewise continuous function on } [0,t_f] \mid 0 \le u_i(t) \le 1, i = 1,2,3\right\}$$

is the control set. The Pontryagins maximum principle[7] must satisfied by optimal control problem is the necessary conditions to check. This principle converts equation (5) and (18) into a problem of minimizing point wise a Hamiltonian *H*, with respect to $u_1(t), u_2(t)$ and $u_3(t)$.

$$\begin{aligned} H = {} & A_1 I(t) + A_2 S(t) - A_3 V(t) - A_4 R(t) + \frac{B_1}{2}u_1^2(t) + \frac{B_2}{2}u_2^2(t) + \frac{B_3}{2}u_3^2(t) \\ & + \lambda_S(\Lambda - (1-u_1(t))\beta S(t)I(t) - (1+u_2(t))\alpha S(t) - \mu s(t) + \omega V(t)) \\ & + \lambda_I((1-u_1(t))\beta S(t)I(t) - (1+u_3(t))\gamma I(t) - \mu I(t) - \delta I(t)) \\ & + \lambda_V(1+u_2(t))\alpha S(t) - \omega V(t) - \mu V(t)) \\ & + \lambda_R((1+u_3(t))\gamma I(t) - \mu R(t)) \\ & + \lambda_D(\delta I(t)) \end{aligned} \quad (20)$$

Where $\lambda_S, \lambda_I, \lambda_V, \lambda_R$ *and* λ_D are the adjoint variables associated with the states variables S, I, V, R and D respectively. The system of adjoint variables is found by differentiating (20) with respect to the associate state variable.

Theorem 2: Optimal controls $u_1(t), u_2(t)$ *and* $u_3(t)$ solution $S^*(t), I^*(t)$ *and* $V^*(t)$ of the corresponding state (5) that minimizes $J_2(u_1(t), u_2(t), u_3(t))$ over control set Γ, $\exists$ adjoint variables $\lambda_S, \lambda_I, \lambda_V, \lambda_R, \lambda_D$ satisfying.

$$-\frac{d\lambda_i}{dt} = \frac{\partial H}{\partial i} \quad (21)$$

with the condition of transversality

$$\lambda_S(t_f) = \lambda_I(t_f) = \lambda_V(t_f) = \lambda_R(t_f) = \lambda_D(t_f) = 0, \quad (22)$$

where $i = S, I, V, R, D$

$$\begin{aligned} \frac{d\lambda_S}{dt} &= -A_2 + \lambda_S(1-u_1(t))\beta I(t) - \lambda_S(1+u_2(t))\alpha - \lambda_S\mu - \lambda_I(1-u_1(t))\beta I(t) - \lambda_V(1+u_2(t))\alpha \\ \frac{d\lambda_I}{dt} &= -A_1 + (1-u_1(t))\beta S(t)(\lambda_S - \lambda_I) + (1+u_3(t))\gamma(\lambda_I - \lambda_R) - \lambda_D\delta \\ \frac{d\lambda_V}{dt} &= A_3 + \lambda_V\mu \\ \frac{d\lambda_R}{dt} &= A_4 + \lambda_R\mu \\ \frac{d\lambda_D}{dt} &= 0 \end{aligned} \quad (23)$$

Further,

$$u_1^* = \min\left\{1, \max\left[0, \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1}\right]\right\}$$
$$u_2^* = \min\left\{1, \max\left[0, \frac{\alpha S(t)(\lambda_S - \lambda_V)}{B_2}\right]\right\}$$
$$u_3^* = \min\left\{1, \max\left[0, \frac{\gamma I(t)(\lambda_I - \lambda_R)}{B_3}\right]\right\} \quad (24)$$

Proof : On the same lines as Theorem 1.

Optimal strategy of minimization of infective and susceptible: Here it is considered the objective functional of minimizing infective and susceptible with all controls

$$J_3(u_1(t),u_2(t),u_3(t)) = \int_0^{tf}\left\{A_1 I(t) + A_2 S(t) + \frac{B_1}{2}u_1^2(t) + \frac{B_2}{2}u_2^2(t) + \frac{B_3}{2}u_3^2(t)\right\} dt \quad (25)$$

where: A_1, A_2, $\frac{B_1}{2}$, $\frac{B_2}{2}$ and $\frac{B_3}{2}$ are positive weights. The terms $\frac{B_1}{2}u_1^2(t)$, $\frac{B_2}{2}u_2^2(t)$, and $\frac{B_3}{2}u_3^2(t)$ are the costs associated with the controls u_1, u_2 and u_3 respectively. Here it is chosen quadratic costs on the controls, by referring the literature of epidemic. With the above given objective functional $J_3(u_1, u_2, u_3)$. The optimal controls u_1^*, u_2^* and u_3^* are such that

$$J_3(u_1^*, u_2^*, u_3^*) = \min_{(u_1,u_2,u_3)\in\Gamma} J_3(u_1, u_2, u_3) \quad (26)$$

where

$$\Gamma = \left\{\begin{matrix} J_3(u_1,u_2,u_3) \mid u_i(t) \text{is piecewise continuous} \\ \text{functionon}\,[0,t_f] \mid 0 \le u_i(t) \le 1, i = 1,2,3 \end{matrix}\right\}$$

is the control set. The Pontryagins maximum principle[7] must satisfied by optimal control problem is the necessary conditions to check. This principle converts equation (5) and (25) into a problem of minimizing point wise a Hamiltonian *H*, with respect to $u_1(t), u_2(t)$ and $u_3(t)$

Then the Hamiltonian function is constructed as

$$\begin{aligned} H &= A_1 I(t) + A_2 S(t) + \frac{B_1}{2} u_1^2(t) + \frac{B_2}{2} u_2^2(t) + \frac{B_3}{2} u_3^2(t) \\ &+ \lambda_S(\Lambda - (1-u_1(t))\beta S(t) I(t) - (1+u_2(t))\alpha S(t) - \mu s(t) + \omega V(t)) \\ &+ \lambda_I((1-u_1(t))\beta S(t) I(t) - (1+u_3(t))\gamma I(t) - \mu I(t) - \delta I(t)) \\ &+ \lambda_V(1+u_2(t))\alpha S(t) - \omega V(t) - \mu V(t)) \\ &+ \lambda_R((1+u_3(t))\gamma I(t) - \mu R(t)) \\ &+ \lambda_D(\delta I(t)) \end{aligned} \tag{27}$$

Where $\lambda_S, \lambda_I, \lambda_V, \lambda_R$ *and* λ_D are the adjoint variables associated with the states variables S, I, V, R and D respectively. The system of adjoint variables is found by differentiating (27) with respect to the associate state variable.

Theorem 3: Optimal controls $u_1(t), u_2(t)$ *and* $u_3(t)$ solution $S^*(t), I^*(t)$ *and* $V^*(t)$ of the corresponding state (5) that minimizes $J_3(u_1(t), u_2(t), u_3(t))$ over control set Γ, $\exists$ adjoint variables $\lambda_S, \lambda_I, \lambda_V, \lambda_R, \lambda_D$ satisfying.

$$-\frac{d\lambda_i}{dt} = \frac{\partial H}{\partial i} \tag{28}$$

with condition of transversality

$$\lambda_S(t_f) = \lambda_I(t_f) = \lambda_V(t_f) = \lambda_R(t_f) = \lambda_D(t_f) = 0, \quad where \ i = S, I, V, R, D$$

$$\begin{aligned} \frac{d\lambda_S}{dt} &= -A_2 + \lambda_S(1-u_1(t))\beta I(t) - \lambda_S(1+u_2(t))\alpha - \lambda_S\mu - \lambda_I(1-u_1(t)\beta I(t) - \lambda_V(1+u_2(t))\alpha \\ \frac{d\lambda_I}{dt} &= -A_1 + (1-u_1(t))\beta S(t)(\lambda_s - \lambda_I) + (1+u_3(t))\gamma(\lambda_I - \lambda_R) - \lambda_D\delta \\ \frac{d\lambda_V}{dt} &= \lambda_V\mu \\ \frac{d\lambda_R}{dt} &= \lambda_R\mu \\ \frac{d\lambda_D}{dt} &= 0 \end{aligned} \tag{29}$$

Further,

$$\begin{aligned} u_1^* &= \min\left\{1, \max\left[0, \frac{\beta S(t) I(t)(\lambda_I - \lambda_S)}{B_1}\right]\right\} \\ u_2^* &= \min\left\{1, \max\left[0, \frac{\alpha S(t)(\lambda_S - \lambda_V)}{B_2}\right]\right\} \\ u_3^* &= \min\left\{1, \max\left[0, \frac{\gamma I(t)(\lambda_I - \lambda_R)}{B_3}\right]\right\} \end{aligned} \tag{30}$$

Proof : On same lines as Theorem 1.

Numerical Analysis

In this section, it is discussed the numerical simulations of the optimal system and parallel results of varying the objective functional J_1, J_2, J_3. Parameter values and description are considered by refering various articles on epidemics given in Table-2. Numerical results to the state system (5) and the adjoint systems (12,22,29) are carried out using parameters from the Table-2 with the weight factors A = 20, B = 10, C = 10 and initial conditions S(0) = 1,000 ,I(0) = 10, V (0) = 0, R(0)=0.

Table-2: Parameter description and values

Baseline parameter	Description	Value	Reference
Λ	The recruitment rate of susceptible.	10	Assumed
β	The infection rate	0.009	Assumed
γ	The recovery rate	0.01	Assumed
α	The vaccination rate	0.15	Assumed
δ	The death rate of disease induced individuals.	1	Assumed
ω	The rate of vaccinated immunity wanes.	0.5	Assumed
μ	Natural death rate	0.02	Assumed

The algorithm which is considered for simulation is the forward-backward scheme starting with an initial supposition for the set of optimal controls. The Runge-Kutta fourth order method is used to solve forward in time for the state variables. Then, to solve adjoint system with given final conditions (7) state variables and initial supposition of the controls backward in time are used, again employing a Runge-Kutta method of fourth order. The controls are updated and used to solve the state system (5) and adjoint system (12, 24) and (29). When the current state, adjoint variable and control values converge sufficiently the iterative process terminates. The results of three different strategies are computed numerically by considering the results of the associated control system (5), and adjoint systems (12, 24) and (29).

Implications of optimal control strategy with objective of minimization of infective: With this strategy, the successful practice of non-pharmaceutical interventions control u_1, the successful practice of prevention of transmission by immunization control u_2 and the treatment of infected individuals control u_3 are involved for the optimization of objective functional J_1. In Figure-1(a), results the decrease in the infective cases with help of control strategy. The control profile is shown in Figure-1(b), control u_1 remains at the upper bound till the final time; control u_2 stays at the lower bound up to the end time; control u_3 is at the upper bound up to 700 units of time before gradually dropping down to the lower bound at the terminal time.

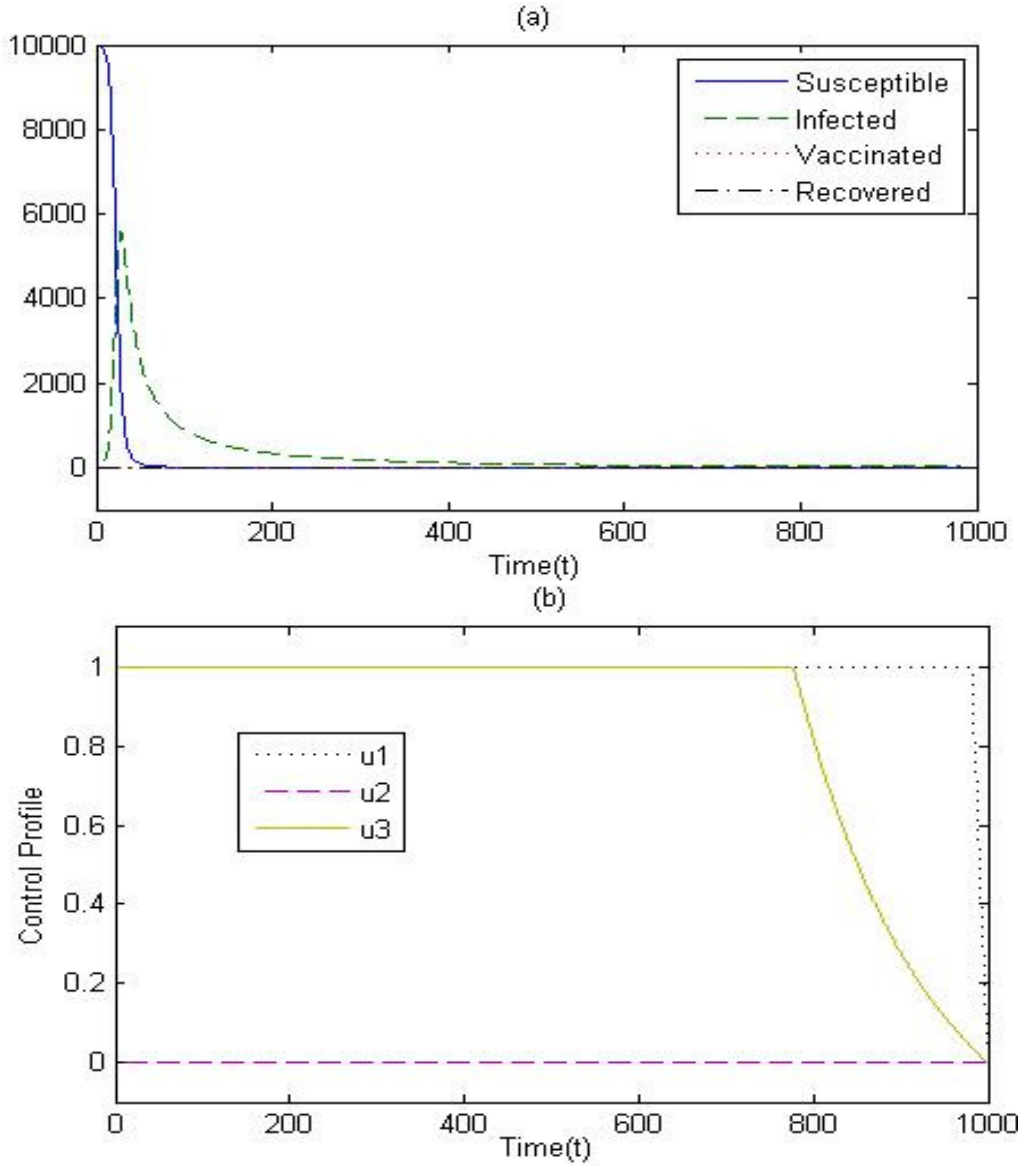

Figure-1: Simulation result of the model solution (a) and control profile (b) of objective function J_1.

Implications of optimal control strategy with objective of minimization of infective and susceptible, and maximization of vaccinated and recovered individuals: With this strategy, the successful practice of non-pharmaceutical interventions control u_1, the successful practice of prevention of transmission by immunization control u_2 and the treatment of infected individuals control u_3 are involved for the optimization of objective functional J_1. In Figure-2(a), it is clear that control strategy results in a significant decrease in the infections number. Figure-2(b) shows the control profile of controls, control u_1 and u_3 remains at the upper bound up to 925 units of time before gradually dropping to the lower bound at the final time; control u_2 remains at the upper bound up to the final time.

Implications of optimal control strategy with objective of minimization of infective and susceptible: With this strategy, the successful practice of non-pharmaceutical interventions control u_1, the successful practice of prevention of transmission by immunization control u_2 and the treatment of infected individuals control u_3 are involved for the optimization of objective function J_3. In Figure-3(a), it is observed that control strategy results in increase in the number of infections as comparing to second strategy, but same number of infections as that of strategy first. The control profile given in the Figure 3(b), has control u_1 remains at the upper bound up to 850 units of time then suddenly reached lower bound at 900; control u_2 remains at the lower bound up to 200 units of time and then gradually increasing up to 0.7 at 600 units of time and then again gradually decreasing to the lower bound till final time; control u_3 is at the upper bound up to 750 units of time before gradually dropping down to the lower bound at the final time.

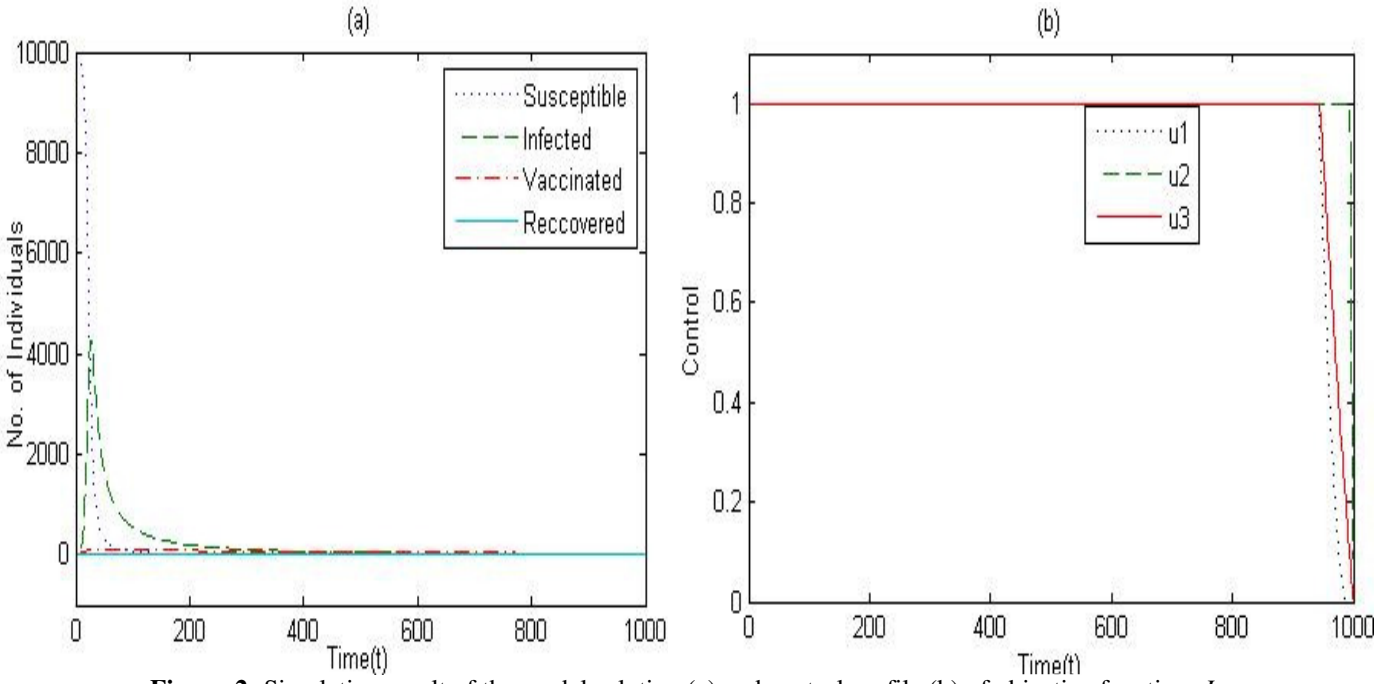

Figure-2: Simulation result of the model solution (a) and control profile (b) of objective function J_2.

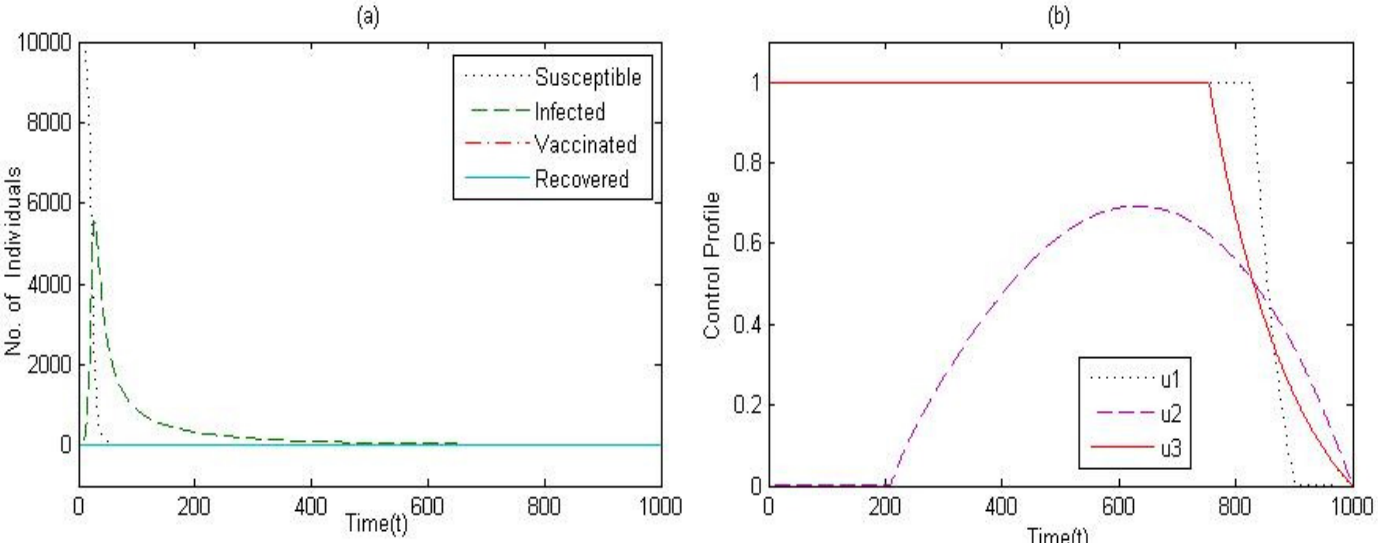

Figure-3: Simulations result of the model solution (a) and control profile (b) of objective function J_3.

Three different strategies are used to optimize objective functions J_1, J_2, J_3. In Figure 4(a), it is observed that the susceptible are gradually decreasing up to 150 in all the strategies; in Figure 4(b), the variation in infective individuals is observed, and the second strategy with objective function J_2 has minimum infection as compared to the first and third strategies with J_1, J_3 objective functions for same model with same controls. In Figure 4(c), it is observed that only strategy J_2 has shown variation vaccinated individuals but other two strategies are at the lower bound till the final time. This is because in the first and third strategies minimization of vaccinated individuals were not considered. In Figure 4(d), it is observed that recovered individuals are minimum for strategy J_2 because of less number of infection and for other two strategies recovered individuals are maximum and both seem to be same. In Figure 5, it is observed that cost function J_2 is at minimum level till the final time compared to other two cost functions J_1, J_2, J_3.

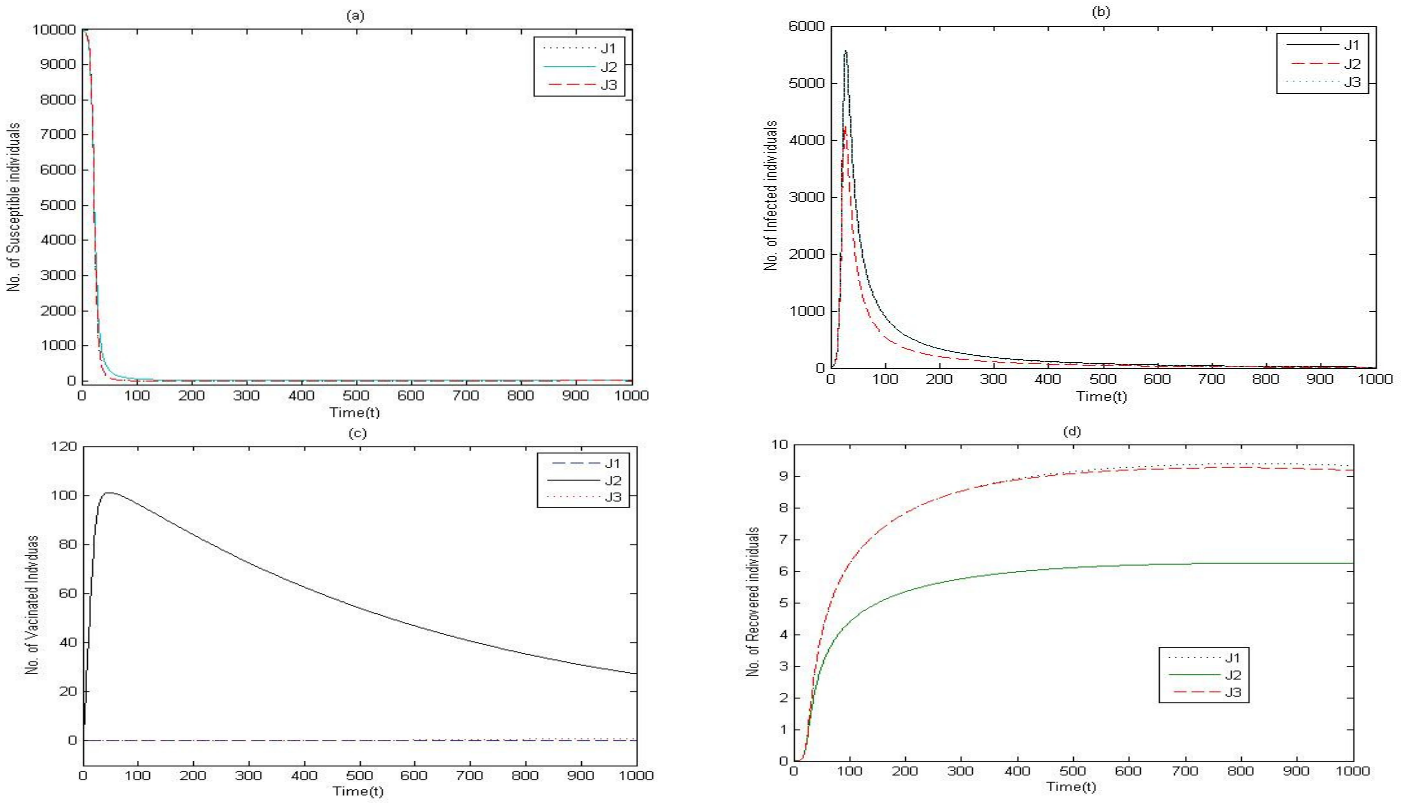

Figure-4: All states of the model for three strategies of objective functions.

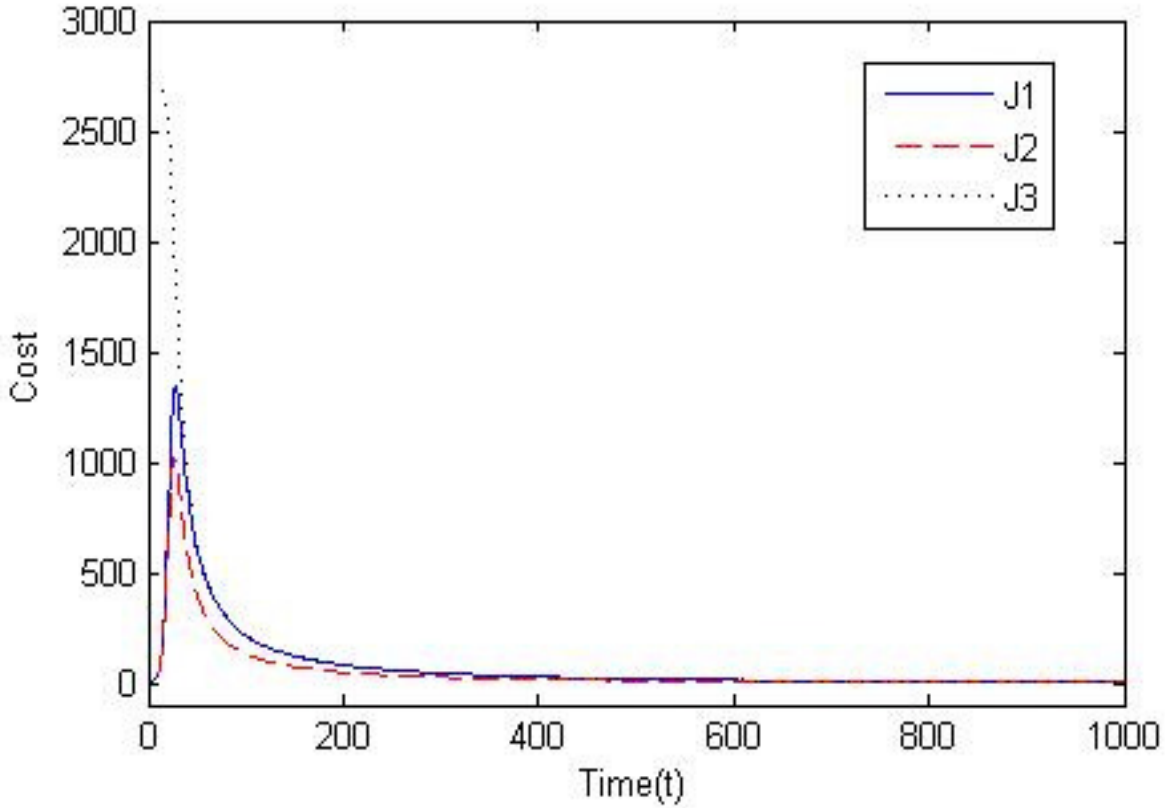

Figure-5: Cost functional for three different strategies of objective functions.

Conclusion

The optimal control analysis of any mathematical epidemic model mainly depends on the number of control variables chosen and objective functions selected. The aim of this study explore how analysis of optimal control shows a discrepancy with respect to state variables of the model by taking into consideration different objective functions. The conditions for optimal control of the SIVR model for different strategies are derived and analyzed by considering different objective functions with the same controls u_1, u_2, u_3 in all strategies. The study demonstrates the analytical findings regarding the effect of choosing different objective function on the state variables with the help of numerical results. This study can suggest a public health planner to choose a strategy from different strategies that needs to decrease the infected individuals as well as associated cost of controls for implementation of it. Also, from numerical analysis it is very clear that choosing different strategies for controlling the disease results in considerable change in the cost on implementation.

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