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Characterization of pareto distribution through expectation

Bhatt Milind B.
Department of Statistics, Sardar Patel University, V.V. Nagar-388120, Gujarat, India
bhattmilind_b@yahoo.com

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Abstract

In statistical inference we often encounter the situation, for instant 60 observations of random phenomena observed and one group of students fit normal distribution whereas other group fits log-normal distribution with almost same p-value. This is one of cases where characterization results provide navigation tools for correct direction for further study (research). One of the research materials which receives room in both pure (mathematics) and applied (probability) is characterization rules because they are boundary points of both. That is why mathematical community as well as probabilists and statisticians contribute research are of characterizations. Characterization is not concern to academician or researcher only who deals with foundation and application of probabilistic model building but also concern to operation research, behavior science, natural science, decision making process in industrial and engineering problem. For characterization of Pareto distribution one needs any arbitrary non constant function only by approach of identity of distribution and equality of expectation of function of random variable in place of approaches such as relation (linear) in (economic variation) reported and true income, independency of suitable function of order statistics, mean and the extreme observation of the sample etc. Examples are given for illustrative purpose.

Keywords: Characterization; Pareto distribution.

Introduction

Economic variation in reported income and true income is studied by error-in-variable model and certain invariance properties of Pareto law. Income reported and true are important subject for ruler (government) for revenue generation. The relation between reported and true (regression) turns out be linear, repotted by Krishnaji and also he asserted that truncated reported income (suitably) agrees to true income in distribution[1]. Nagesh asserted that average under-reporting error for given reported income is linear function of reported income if and only if income follows Pareto law[2].

Henrick[3], Ahsanullah[4,5], Shah[6] and Dimaki[7], used independence of suitable function of order statistics whereas Srivastava[8] used mean and the extreme observation of the sample and characterized Pareto distribution.

Other attempts were made for characterization of exponential and related distributions assuming linear relation of conditional expectation by Beg[9] and Dallas[10], characterization of some types of distributions using recurrence relations between expectations of function of order statistics by Alli[11] and characterization results on exponential and related distributions by Tavangar[12] included characterization of Pareto distribution.

This research note characterizes Pareto distribution by using expectation. Main results of characterization of Pareto distribution is provided in section 2 with proof and section 3 provides examples for illustrative purpose.

The characterization given in section 2 proves the main results and section 3 is gives examples for illustrative purpose.

This research let X have Pareto distribution with probability density function (pdf) as

$$f(x;\theta)=\begin{cases}\dfrac{c\theta^{c}}{x^{c+1}};0<\theta<x<b,c>0,\\ \\ 0;otherwise\end{cases} \tag{1}$$

Where: $-\infty\le a<b\le\infty$ are known constants, x^{-c-1} is positive absolutely continuous function and $c\theta^{c}$ is everywhere differentiable function. Since the range of derivative $\left(1/c\theta^{c}\right)$ being negative and since the range is truncated by θ from left $\left(1/cb^{c}\right)=0$.

Characterization

In this more general result of characterization through expectation is stated by following theorem.

Theorem 2.1 Let X be a continuous random variable (r.v.) with distribution function $F(X)$, having pdf $f(x;\theta)$. Assume that $F(X)$ is continuous on the interval (a,b) where $-\infty \le a < b \le \infty$. Let $g(X)$ be differentiable functions of X on the interval (a,b) where $-\infty \le a < b \le \infty$ and more over $g(X)$ be non constant. Then $f(x;\theta)$ is the p.d.f. of Pareto distribution defined in (1) if and only if

$$E\left[g(X)-\frac{X}{c}\frac{d}{dX}g(X)\right]=g(\theta) \quad (2)$$

Proof Given $f(x;\theta)$ defined in (1), if $\phi(X)$ is such that $g(\theta)=E\left[\phi(X)\right]$ where $g(\theta)$ is differentiable function then

$$g(\theta)=\int_{\theta}^{\infty}\phi(x)f(x;\theta)\,dx \quad (3)$$

Differentiating with respect to θ on both sides of (3) and replacing X for θ after simplification one gets

$$\phi(X)=g(X)-\frac{X}{c}\frac{d}{dX}g(X) \quad (4)$$

which establishes necessity of (2). Conversely given (2), let $k(x;\theta)$ be the p.d.f. of r.v. X such that

$$g(\theta)=\int_{\theta}^{\infty}\left[g(x)-\frac{x}{c}\frac{d}{dx}g(x)\right]k(x;\theta)\,dx \quad (5)$$

Since $b^{-c}=0$, the following identity holds

$$g(\theta)=-c\theta^{c}\int_{\theta}^{\infty}\frac{d}{dx}\left(\frac{x^{-c}}{c}g(x)\right)dx \quad (6)$$

which can be rewritten as

$$g(\theta)=-c\theta^{c}\int_{\theta}^{\infty}g(x)\frac{d}{dx}\left(\frac{x^{-c}}{c}\right)+\left(\frac{x^{-c}}{c}\right)\frac{d}{dx}g(x)dx \quad (7)$$

which reduces to

$$g(\theta)=\int_{\theta}^{\infty}\left[g(x)-\frac{x}{c}\frac{d}{dx}g(x)\right]\left\{-c\theta^{c}\frac{d}{dx}\left(\frac{x^{-c}}{c}\right)\right\}dx \quad (8)$$

Using (5) and (8) by uniqueness theorem it follows that p.d.f. of r.v. X

$$k(x;\theta)=-c\theta^{c}\frac{d}{dx}\left(\frac{x^{-c}}{c}\right) \quad (9)$$

Since b^{c} is increasing function for $-\infty \le a < b \le \infty$ and $b^{-c}=0$ is satisfied only when range of X is truncated by θ from left and integrating (9) on the interval (θ,b) on both sides, one gets (9) as

$$1=\int_{\theta}^{\infty}k(x;\theta)\,dx$$

and

$$k(x;\theta)=-c\theta^{c}\frac{d}{dx}\left(\frac{x^{-c}}{c}\right);0<\theta<x<b,c>0 \quad (10)$$

Substituting $\frac{d}{dx}\left(\frac{x^{-c}}{c}\right)$ in (9), $k(x;\theta)$ reduces to $f(x,\theta)$ defined in (1) which establishes sufficiency of (2).

Remark 2.1.Using $\phi(X)$ derived in (4), the $f(x;\theta)$ given in (1) can be determined by

$$M(X)=\frac{\frac{d}{dx}g(x)}{\phi(X)-g(x)} \quad (11)$$

and pdf is given by

$$f(x;\theta)=-\frac{\frac{d}{dx}T(x)}{T(\theta)} \quad (12)$$

where $T(x)$ is decreasing function in the interval (a,b) for $-\infty \le a < b \le \infty$ with $T(b)$ such that it satisfies

$$M(X)=\frac{d}{dx}\log\left[T(x)\right] \quad (13)$$

Examples

Example 3.1 Using method described in the remark characterization of Pareto distribution through survival function $g(\theta)=\overline{F}(t)=\left(\frac{\theta}{t}\right)^{c}$ is illustrated.

$$g(X)=\left(\frac{X}{t}\right)^{c}$$

$$\phi(X)=g(X)-\frac{X}{c}\frac{d}{dX}g(X)=0$$

$$M(X)=\frac{\frac{d}{dx}g(X)}{\phi(X)-g(X)}=-\frac{X}{c}$$

$$\frac{d}{dX}\log\left[T(X)\right]=-\frac{X}{c}=M(X)$$

$$T(x)=\frac{1}{cX^{c}}$$

$$f(x;\theta)=-\frac{\frac{d}{dx}T(x)}{T(\theta)}=-\frac{\frac{d}{dx}T\left(\frac{1}{cx^{c}}\right)}{T\left(\frac{1}{c\theta^{c}}\right)}=\frac{c\theta^{c}}{x^{c+1}} \quad \text{for} \quad x<\theta .$$

Note that as Hazard function for Pareto distribution being constant it cannot characterize pdf $f(x;\theta)$ given in (1).

Example 3.2 The pdf $f(x;\theta)$ defined in (1) can be characterized through expectation of non constant functions of θ such as

$$g(\theta)=\begin{cases} \frac{c}{c-1}\theta; mean, for-i=1 \\ \frac{c}{c-r}\theta^{r}; r^{th} rowmoment, for-i=2 \\ e^{\theta}, for-i=3 \\ e^{-\theta}, for-i=4 \\ \theta(1-p)^{-1/c}, p^{th} quantile, for-i=5 \\ 1-\left(\frac{X}{t}\right)^{c}; distribution-function, for-i=6 \\ \left(\frac{X}{t}\right)^{c}; reliability-function, for-i=7 \end{cases}$$

by using

$$\left[\phi_i(X)-g_i(X)\right]=\begin{cases} -\frac{1}{c-1}X; mean, for-i=1 \\ -\frac{r}{c-r}X^{r}; r^{th} rowmoment, for-i=2 \\ \frac{X}{t}e^{-X}, for-i=3 \\ -\frac{X}{t}e^{X}, for-i=4 \\ -\frac{X}{c}(1-p)^{-1/c}, p^{th} quantile, for-i=5 \\ \left(\frac{X}{t}\right)^{c}; distribution-function, for-i=6 \\ -\left(\frac{X}{t}\right)^{c}; reliability-function, for-i=7 \end{cases}$$

and defining $M(X)$ given in (11) and using $T(X)$ as appeared in (13), for (12).

Conclusion

To characterize pdf given in (1) one needs any arbitrary non constant function only.

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