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Generalized fractional differentiation of multivariable I-function involving general class of polynomials

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Abstract

In this research work, we study and obtain new results on the generalized fractional derivative operators. Initially, we establish two theorems of generalized fractional derivative of multivariable I-function involving general class of polynomial, that give the images of multivariable I-function in saigö operators[1]. On account of general nature of saigö operators and multivariable I-function and several special functions.

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Introduction

The generalized fractional differential derivative operators introduced by Saigö[1] are defined as

$$\left(D_{0+}^{\alpha,\beta,\eta}f\right)(x)=\left(\frac{d}{dx}\right)^{n}\left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n}f\right)(x),\mathrm{Re}(\alpha)\geq 0,n=[\mathrm{Re}(\alpha)+1] \quad (1)$$

$$\left(D_{-}^{\alpha,\beta,\eta}f\right)(x)=\left(-\frac{d}{dx}\right)^{n}\left(I_{-}^{-\alpha+n,-\beta-n,\alpha+n}f\right)(x),\mathrm{Re}(\alpha)\geq 0,n=[\mathrm{Re}(\alpha)]+1. \quad (2)$$

Where $\alpha,\beta,\eta\in C,\mathrm{Re}(\alpha)\geq 0$ and $I_{0+}^{\alpha,\beta,\eta},I_{-}^{\alpha,\beta,\eta}$ known as generalized fractional operators introduced by Saigö[1]. When $\beta=-\alpha$, in view of eq'n above, we have

$$\left(D_{0+}^{\alpha,-\alpha,\eta}f\right)(x)=\left(D_{0+}^{\alpha}f\right)(x)=\left(\frac{d}{dx}\right)^{n}\cdot\frac{1}{\Gamma(n-\alpha)}\cdot\int_{0}^{x}\frac{f(t)\,dt}{(x-t)^{\alpha-n+1}}\ ,\ x>0,n=[\mathrm{Re}(\alpha)+1] \quad (3)$$

$$\left(D_{-}^{\alpha,-\alpha,\eta}f\right)(x)=\left(D_{-}^{\alpha}f\right)(x)=\left(-\frac{d}{dx}\right)^{n}\cdot\frac{1}{\Gamma(n-\alpha)}\cdot\int_{x}^{\infty}\frac{f(t)\,dt}{(x-t)^{\alpha-n+1}},\ \ x>0,n=[\mathrm{Re}(\alpha)+1]. \quad (4)$$

Again if B=0, the equation (1) and (2) reduces the fractional differential operator defined as

$$\left(D_{0+}^{\alpha,0,\eta}f\right)(x)=\left(D_{n,\alpha}^{+}f\right)(x)=x^{-n}\left(\frac{d}{dx}\right)^{n}\cdot\frac{1}{\Gamma(n-\alpha)}\cdot\int_{0}^{x}\frac{t^{\alpha+n}f(t)\,dt}{(x-t)^{\alpha-n++1}}\quad x>0,n=[\mathrm{Re}(\alpha)+1] \quad (5)$$

$$\left(D_{-}^{\alpha,0,\eta}f\right)(x)=\left(D_{n\alpha}^{-}f\right)(x)=x^{\alpha+n}\left(-\frac{d}{dx}\right)^{n}\cdot\frac{1}{\Gamma(n-\alpha)}\cdot\int_{x}^{\infty}\frac{t^{-n}f(t)\,dt}{(t-x)^{\alpha-n+1}}\ ,\ x>0,n=[\mathrm{Re}(\alpha)+1]. \quad (6)$$

The I-function of r variables introduced by Prasad [2], is defined as

$$I[z_1\cdots z_r]=I_{p_2,q_2:p_3,q_3;\ldots;p_r,q_r:(p',q');\ldots;(p^{(r)},q^{(r)})}^{0,n_2:0,n_3;\ldots;0,n_r:(m',n');\ldots;(m^{(r)},n^{(r)})} \quad (7)$$

$$\left[\begin{matrix}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{matrix}\left|\begin{matrix}\left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2}:\left(a_{3j};\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j}\right)_{1,p_3}:\ldots:\left(a_{rj};\alpha'_{rj},\alpha^{(r)}_{rj}\right)_{1,p_r}:\ (a'_j,\alpha'_j)_{1,p'}\ldots(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}}\\ \\ \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2}:\left(b_{3j};\beta'_{3j},\beta''_{3j},\beta'''_{3j}\right)_{1,q_3}:\ldots:\left(b_{rj};\beta'_{rj},\ldots,\beta^{(r)}_{rj}\right)_{1,q_r}:\ (b'_j,\beta'_j)_{1,q'}\ldots(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}}\end{matrix}\right.\right]$$

$$=\frac{1}{(2\pi\omega)^{r}}\int_{L_1}\ldots\int_{L_r}\psi(\xi_1\ldots\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)\,z_1^{\xi_1}\ldots z_r^{\xi_r}\,d\xi_1\ldots d\xi_r$$

where $\omega=\sqrt{-1}\ldots$,

$$\psi(\xi_{1\ldots}\xi_r)=\frac{\prod_{k=2}^{r}\left[\prod_{j=1}^{n_k}\Gamma(1-a_{kj}+\sum_{i=1}^{k}\alpha_{kj}^{(i)}\xi_i)\right]}{\prod_{k=2}^{r}\left[\prod_{j=n_k+1}^{p_k}\Gamma(a_{kj}-\sum_{i=1}^{k}\alpha_{kj}^{(i)}\xi_i)\right]}\times\frac{1}{\prod_{k=2}^{r}\left[\prod_{j=1}^{q_k}\Gamma(1-b_{kj}+\sum_{i=1}^{k}\alpha_{kj}^{(i)}\xi_i)\right]} \quad (8)$$

and

$$\phi_i(\xi_i)=\frac{\left[\prod_{k=1}^{m^{(i)}}\Gamma(b_k^{(i)}-\beta_k^{(i)}\xi_i)\right]\left[\prod_{j=1}^{n^{(i)}}\Gamma(1-a_j^{(i)}+\alpha_j^{(i)}\xi_i)\right]}{\left[\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma(a_j^{(i)}-\alpha_j^{(i)}\xi_i)\right]\left[\prod_{k=m^{(i)}+1}^{q^{(i)}}\Gamma(1-b_k^{(i)}-\beta_k^{(i)}\xi_i)\right]} \quad \forall i\in\{1,...,r\} \tag{9}$$

Where: $n_i, p_i, q_i, m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)}$ are non–negative integers and $a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}, \alpha_j^{(i)}, \beta_j^{(i)}, a_j^{(i)}, b_j^{(i)}$ are complex numbers and the empty product denotes unity.

For convergence conditions and other details, in this present work it is considered that the above function always satisfied the existence and convergence in the range of Integration.

Preliminary results

The following lemmas are required to establish main results.

Lemma 1: Let $\alpha, \beta, \eta \in c$ such that $\mathrm{Re}(\alpha)\ge 0, \mathrm{Re}(\sigma) > -\min[0, \mathrm{Re}(\alpha+\beta+\eta)]$. Then, we have

$$\left(D_{0+}^{\alpha,\beta,\eta}\, t^{\sigma-1}\right)(x)=\frac{\Gamma(\sigma).\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta).\Gamma(\sigma+\eta)}x^{\sigma+\beta-1}, x>0 \tag{10}$$

In particular, for $x>0, and\ \beta=-\alpha$

$$\left(D_{0+}^{\alpha}\, t^{\sigma-1}\right)(x)=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}x^{\sigma-\alpha-1}, (\mathrm{Re}(\alpha)\ge 0, \mathrm{Re}(\sigma)>0) \tag{11}$$

and

$$\left(D_{\eta\alpha}^{+} t^{\sigma-1}\right)(x)=\frac{\Gamma(\sigma+\alpha+n)}{\Gamma(\sigma+\eta)}x^{\sigma-1}, for \beta=0, \mathrm{Re}(\alpha)\ge 0, \mathrm{Re}(\sigma)>-\mathrm{Re}(\alpha+\eta). \tag{12}$$

Lemma 2: Let $\alpha,\beta,\eta\in C$
$\mathrm{Re}(\alpha)\ge 0, \mathrm{Re}(\sigma)<1+\min[\mathrm{Re}(-\beta-\eta)-\mathrm{Re}(\alpha+\eta)], n=[\mathrm{Re}(\alpha)]+1$
then, we have

$$\left(D_{-}^{\alpha,\beta,\eta}\, t^{\sigma-1}\right)(x)=\frac{\Gamma(1-\sigma-\beta).\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma).\Gamma(1-\sigma+\eta-\beta)}x^{\sigma+\beta-1}, x>0 \tag{13}$$

In particular, for $x>0.$

$$\left(D_{-}^{\alpha}\, t^{\sigma-1}\right)(x)=\frac{\Gamma(1-\sigma+\alpha)}{\Gamma(1-\sigma)}x^{\sigma-\alpha-1}, \mathrm{Re}(\alpha)\ge 0, \mathrm{Re}(\sigma)<1+\mathrm{Re}(\alpha)-n \tag{14}$$

and

$$\left(D_{\eta,\alpha}^{-}\, t^{\sigma-1}\right)(x)=\frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma+\eta)}, \mathrm{Re}(\alpha)\ge 0, \mathrm{Re}(\sigma)<1+\mathrm{Re}(\alpha+\eta)-n. \tag{15}$$

The Binomial expansion used in our investigation is given by

$$(b+at)^{-\alpha}=b^{-\alpha}.\left(1+\frac{at}{b}\right)^{-\alpha}, \left|\frac{at}{b}\right|<1 \tag{16}$$

$$=b^{-\alpha}\left(\frac{1}{2\pi i}\right)\int_c \frac{\Gamma(-\xi).\Gamma(\xi+\alpha)}{\Gamma(\alpha)}.\left(\frac{at}{b}\right)^{\xi}d\xi$$

c runs between $-i_\infty$ to $+i_\infty$.

Main result

Theorem (I): $\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(at+b)^{-v}.s_N^M\left[t^{\lambda}.(at+b)^{-\delta}\right]\right.\right.$

$$.I\left[z_1 t^{\sigma_1}(at+b)^{-w_1}.....z_r t^{\sigma_r}(at+b)^{-w_r}\right]\Big\}(x)$$

$$=x^{\mu+\beta-1}.b^{-v}\sum_{k=0}^{[N/M]}\frac{(-N)_{Mk}}{k!}A_{N,k}\,b^{-\delta k}.x^{\lambda k}$$

$$.I^{0,n_2.....0,n_r+3\ :m^1,n^1.............m^{(r)},n^{(r)}\ :1\ 0}_{p_2,q_2.....p_r+3,q_r+3\ :p^1,q^1.....p^{(r)},q^{(r)}\ :0\ 1}\left[\begin{matrix} z_1 x^{\sigma_1}.b^{-w_1}\\ .\\ .\\ .\\ z_r x^{\sigma_r}.b^{-w_r}\\ \frac{ax}{b}\end{matrix}\right.$$

$$\begin{matrix}[1-v-\delta k;w_1..w_r,1]:[1-\mu-\lambda k;\sigma_1..\sigma_r,1]:[1-\mu-\lambda k-\alpha-\beta;\sigma_1..\sigma_r,1]:\\ (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}..(a_{rj};\alpha'_{rj},\alpha''_{rj})_{1,p_r}:\\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2}..(b_{rj};\beta'_{rj},\beta''_{rj})_{1,q_r}[1-v-\delta k;w_1..w_r,0]:\\ [1-\mu-\lambda k-\beta;\sigma_1..\sigma_r,1]:[1-\mu-\lambda k-\eta;\sigma_1..\sigma_r,1]\end{matrix}$$

$$\left.\begin{matrix}(a_j^1,\alpha_j^1)_{1,p^1}....(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}},....\\ .\\ .\\ .\\ (b_j^1,\beta_j^1)_{1,q^1}....(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}},....(0,1)\end{matrix}\right]$$

The sufficient conditions of theorem (I) are

$$\alpha,\beta,\eta,\mu,z_i \in C \ \ and \ \ \sigma_i > 0 \ \ \forall i = 1....r$$

$$|\arg\ z_i| < \frac{1}{2} T_i \pi \ \ and \ \ T_i > 0$$, where

$$T_i = \sum_{j=1}^{n^{(i)}} \alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} \alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}} \beta_j^{(i)} + \left(\sum_{j=1}^{n_2} \alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2} \alpha_{2j}^{(i)} \right)$$
$$+ \left(\sum_{j=1}^{n_3} \alpha_{3j}^{(i)} - \sum_{j=n_3+1}^{p_3} \alpha_{3j}^{(i)} \right) + ... \left(\sum_{j=1}^{n_r} \alpha_{rj}^{(i)} - \sum_{j=n_r+1}^{p_r} \alpha_{rj}^{(i)} \right) - \left(\sum_{j=1}^{q_2} \beta_{2j}^{(i)} + ... + \sum_{j=n_2+1}^{q_r} \beta_{rj}^{(i)} \right)$$

$$\mathrm{Re}(\alpha) \geq 0$$

$$\left| \frac{at}{b} \right| < 1$$

Proof

Proof of Theorem (I): Initially ,we obtain the general class of polynomial in series form given by (10) and multivariate I-function in type contour integral given by (7).Interchanging the orders of summation and integration and taking the generalized fractional derivative operator . inside (which is permissible Next we express binomial expansion for $(b+at)^{-(v+\delta k+w_1\xi_1+...+w_r\xi_r)}$ in terms type contour integral given by equ'n (7), which reduces to

$$\Delta = \sum_{k=0}^{\left[\frac{N}{M}\right]} \frac{(-N)_{Mk}}{k!} A_{N,k} b^{-v-\delta k} . \frac{1}{(2\pi i)^{r+1}} . \int_{L_1} ... \int_{L_1} \psi(\xi_1..\xi_r) . \prod_{i=1}^{r} \left\{ \phi_i(\xi_i)(z_i b^{-w_i})^{\xi_i} \right\} d\xi_1..d\xi_r \cdot$$

$$\int_L \frac{\Gamma(v+\delta k+w_1\xi_1+...+w_r\xi_r+\xi)}{\Gamma(v+\delta k+w_1\xi_1+...+w_r\xi_r)} \Gamma(-\xi) \left(\frac{a}{b}\right)^{\xi} d\xi . \left\{ D_{0+}^{\alpha,\beta,\eta} \left(t^{\mu+\lambda k+\sigma_1\xi_1+..+\sigma_r\xi_r+\xi-1} \right) \right\}(x).$$

Finally in view of eq'n (11) and interpreting the result in the form of multivariable I-function of (r+1) variables, we get the result.

Corollary 1: If we put $\beta = -\alpha$, then under the condition stated in theorem (I), we have

$$\left\{ D_{0+}^{\alpha} \left(t^{\mu-1}(at+b)^{-v} . s_N^M \left[t^{\lambda} . (at+b)^{-\delta} \right] . I \left[z_1 t^{\sigma_1} (at+b)^{-w_1} .....z_r t^{\sigma_r} (at+b)^{-w_r} \right] \right) \right\}(x)$$

Conditions of validity are same as in theorem (I).

Corollary 2: If we put and $\beta = 0$ in theorem (I), we have

$$\left\{ D_{n,\alpha}^{+} \left(t^{\mu-1} (at+b)^{-v} . s_N^M \left[t^{\lambda} . (at+b)^{-\delta} \right] . I \left[z_1 t^{\sigma_1} (at+b)^{-w_1} ....z_r t^{\sigma_r} (at+b)^{-w_r} \right] \right) \right\}(x)$$

$$= x^{\mu-\alpha-1} b^{-v} \sum_{k=0}^{\left[\frac{N}{M}\right]} \frac{(-N)_{Mk}}{k!} A_{N,k} b^{-\delta k} x^{\lambda k} . I^{0,n_2,...,0,n_r+2 \,:\, m^1,n^1,......,m^{(r)},n^{(r)} \,:\, 1\,0}_{p_2,q_2,...,p_r+2,q_r+2 \,:\, p^1,q^1,...,p^{(r)},q^{(r)} \,:\, 0\,1} \left[\begin{matrix} z_1 x^{\sigma_1} b^{-w_1} \\ . \\ . \\ . \\ z_r x^{\sigma_r} b^{-w_r} \\ \frac{ax}{b} \end{matrix} \right.$$

$$[1-v-\delta k; w_1...w_r,1] : [1-\mu-\lambda k; \sigma_1...\sigma_r,1] :$$
$$(a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2} ... (a_{rj}; \alpha'_{rj}, \alpha''_{rj})_{1,p_r} :$$
$$(b_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2} ... (b_{rj}; \beta'_{rj}, \beta''_{rj})_{1,q_r} [1-v-\delta k; w_1...w_r,0] :$$
$$[1-\mu-\lambda k+\alpha; \sigma_1...\sigma_r,1] :$$

$$\left. \begin{matrix} (a_j^1, \alpha_j^1)_{1,p^1} .... (a_j^{(r)}, \alpha_j^{(r)})_{1,p^{(r)}} \text{\_\_\_\_\_\_} \\ . \\ . \\ . \\ (b_j^1, \beta_j^1)_{1,q^1} .... (b_j^{(r)}, \beta_j^{(r)})_{1,q^{(r)}} ,....(0,1) \end{matrix} \right]$$

$$= x^{\mu-1} . b^{-v} \sum_{k=0}^{\left[\frac{N}{M}\right]} \frac{(-N)_{Mk}}{k!} A_{N,k} b^{-\delta k} . x^{\lambda k}$$

$$. I^{0,n_2.....0,n_r+2 \,:\, m^1,n^1 ....m^{(r)},n^{(r)} \,:\, 1\,0}_{p_2,q_2.....p_r+2,q_r+2 \,:\, p^1,q^1.....p^{(r)},q^{(r)} \,:\, 0\,1} \left[\begin{matrix} z_1 x^{\sigma_1} . b^{-w_1} \\ . \\ . \\ . \\ z_r x^{\sigma_r} . b^{-w_r} \\ \frac{ax}{b} \end{matrix} \right]$$

$$[1-v-\delta k; w_1..w_r,1] : [1-\mu-\lambda k-\alpha-\eta, \sigma_1..\sigma_r,1] :$$
$$(a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2} .. (a_{rj}; \alpha'_{rj}, \alpha''_{rj})_{1,p_r} :$$
$$(b_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2} .. (b_{rj}; \beta'_{rj}, \beta''_{rj})_{1,q_r} [1-v-\delta k; w_1..w_r,0] :$$
$$[1-\mu-\lambda k-\eta, \sigma_1..\sigma_r,1] :$$

$$\left.\begin{matrix}(a_j^1,\alpha_j^1)_{1,p^1}\ldots.(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\_\_\_\_\_\_\\ \cdot\\ \cdot\\ \cdot\\ (b_j^1,\beta_j^1)_{1,q^1}\ldots.(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}},\ldots.(0,1)\end{matrix}\right]$$

Conditions of validity are same as in theorem (I).

Theorem (II)

$$\left\{D_{-}^{\alpha,\beta,\eta}\left(\begin{matrix}t^{\mu-1}(at+b)^{-v}.s_N^M\left[t^{\lambda}.(at+b)^{-\delta}\right].I\\ \left[z_1t^{-\sigma_1}(at+b)^{-w_1}\ldots.z_rt^{-\sigma_r}(at+b)^{-w_r}\right]\end{matrix}\right)\right\}(x)$$

$$\begin{matrix}[1-v-\delta k;w_1..w_r,1]:[\mu+\lambda k+\beta;\sigma_1..\sigma_r,-1]:[\mu+\lambda k-\alpha-\eta;\sigma_1..\sigma_r,-1]:(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}..(a_{rj};\alpha'_{rj},\alpha''_{rj})_{1,p_r}:\\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2}..(b_{rj};\beta'_{rj},\beta''_{rj})_{1,q_r}[1-v-\delta k;w_1..w_r,0]:[\mu+\lambda k-\eta+\beta;\sigma_1..\sigma_r,-1]:[\mu+\lambda k-\eta+\beta;\sigma_1..\sigma_r,-1]\end{matrix}$$

$$\left.\begin{matrix}(a_j^1,\alpha_j^1)_{1,p^1}\ldots(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}},\ldots.\\ \cdot\\ \cdot\\ \cdot\\ (b_j^1,\beta_j^1)_{1,q^1}\ldots(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}},\ldots.(0,1)\end{matrix}\right]$$

$$=x^{\mu+\beta-1}.b^{-v}\sum_{k=0}^{\left[\frac{N}{M}\right]}\frac{(-N)_{Mk}}{k!}A_{N,k}\,b^{-\delta k}.x^{\lambda k}.I^{0,n_2\ldots.0,n_r+3\ :m^1,n^1\ldots\ldots\ldots.m^{(r)},n^{(r)}\ :1\,0}_{p_2,q_2\ldots.p_r+3,q_r+3\ :p^1,q^1\ldots.p^{(r)},q^{(r)}\ :0\,1}\left[\begin{matrix}z_1x^{\sigma_1}.b^{-w_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{\sigma_r}.b^{-w_r}\\ \frac{ax}{b}\end{matrix}\right]$$

The conditions of validity are same as theorem (I).

Proof of Theorem (II): The proof of theorem (II) can be formed on the lines alike to those of theorem (I) we get the form as below

$$\Delta=\sum_{k=0}^{\left[\frac{N}{M}\right]}\frac{(-N)_{Mk}}{k!}A_{N,k}\,b^{-v-\delta k}.\frac{1}{(2\pi i)^{r+1}}.\int_{L_1}\ldots\int_{L_1}\psi(\xi_1\ldots\xi_r).\prod_{i=1}^{r}\left\{\phi_i(\xi_i)(z_ib^{-w_i})^{\xi_i}\right\}d\xi_1\ldots d\xi_r$$

$$\int_L\frac{\Gamma(v+\delta k+w_1\xi_1+\ldots+w_r\xi_r+\xi)}{\Gamma(v+\delta k+w_1\xi_1+\ldots+w_r\xi_r)}.\Gamma(-\xi).\left(\frac{a}{b}\right)^{\xi}d\xi.\left\{D_{-}^{\alpha,\beta,\eta}\left(t^{\mu+\lambda k-\sigma_1\xi_1-\ldots-\sigma_r\xi_r+\xi-1}\right)\right\}(x)$$

and then by applying equation (14), we get the required result.

Special Cases

The special cases are also obtained by Kilbas[3], Kilbas and Sebastian and Saxena[4], Ram and Suthar[5].

Special Cases of theorem (I)

If we put

$n_2=n_3=\ldots n_{r-1}=0,\ p_2=p_3=\ldots p_{r-1}=0$ and

$q_2=q_3=\ldots q_{r-1}=0$

In theorem (I), the multivariable I-function reduces to H-function of r-variables and we get,

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(at+b)^{-v}.s_N^M\left[t^{\lambda}.(at+b)^{-\delta}\right].H\left[z_1t^{\sigma_1}(at+b)^{-w_1}\ldots.z_rt^{\sigma_r}(at+b)^{-w_r}\right]\right)\right\}(x)$$

$$=x^{\mu+\beta-1}.b^{-v}\sum_{k=0}^{\left[\frac{N}{M}\right]}\frac{(-N)_{Mk}}{k!}A_{N,k}\,b^{-\delta k}.x^{\lambda k}$$

$$.H^{0,n_r+3\ \ :m^1,n^1\ldots\ldots\ldots.m^{(r)},n^{(r)}\ :1\,0}_{p_r+3,q_r+3:p^1,q^1\ldots.p^{(r)},q^{(r)}\ :0\,1}\left[\begin{matrix}z_1x^{\sigma_1}.b^{-w_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{\sigma_r}.b^{-w_r}\\ \frac{ax}{b}\end{matrix}\right]$$

$$\begin{matrix}[1-v-\delta k;w_1\ldots w_r,1],[1-\mu-\lambda k;\sigma_1\ldots\sigma_r,1],[1-\mu-\lambda k-\alpha-\beta-\eta;\sigma_1\ldots\sigma_r,1],(a_{rj};\alpha'_{rj},\alpha''_{rj})_{1,p_r}:\\ (b_{rj};\beta'_{rj},\beta''_{rj})_{1,q_r}[1-v-\delta k;w_1\ldots w_r,0]:[1-\mu-\lambda k-\beta;\sigma_1\ldots\sigma_r,1]:\end{matrix}$$

$$\left.\begin{matrix}(a_j^1,\alpha_j^1)_{1,p^1}\ldots.(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\_\_\_\_\_\_\\ \cdot\\ \cdot\\ \cdot\\ (b_j^1,\beta_j^1)_{1,q^1}\ldots.(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}},\ldots.(0,1)\end{matrix}\right].$$

Conditions of validity are same as in theorem (I).

If we put $n_2=n_3=..n_r=0,\ p_2=p_3=\ldots p_r=0$ and $q_2=q_3=\ldots q_r=0$

In theorem (I), the multivariable I-function reduces to Fox H-function and result becomes,

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(at+b)^{-v}.s_N^M\left[t^{\lambda}.(at+b)^{-\delta}\right].\prod_{i=1}^{r}H^{m^{(i)}n^{(i)}}_{p^{(i)}q^{(i)}}\left[z_it^{\sigma_i}(at+b)^{-w_i}\left|\begin{matrix}(a_j^{(i)}\alpha_j^{(i)})_{1,p^{(i)}}\\ (b_j^{(i)}\beta_j^{(i)})_{1,q^{(i)}}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=x^{\mu+\beta-1}b^{-v}\sum_{k=0}^{\left[\frac{N}{M}\right]}\frac{(-N)_{Mk}}{k!}A_{N,k}\,b^{-\delta k}\,x^{\lambda k}.H^{0,3:m^1,n^1\ldots m^{(r)},n^{(r)},10}_{3,3:p^1,q^1\ldots p^{(r)},q^{(r)},01}\left[\begin{matrix} z_1x^{\sigma_1}b^{-w_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{\sigma_r}b^{-w_r}\\ \frac{ax}{b}\end{matrix}\middle|\begin{matrix}(a_j^{(i)}\alpha_j^{(i)})_{1,p^{(i)}}\\ (b_j^{(i)}\beta_j^{(i)})_{1,q^{(i)}}\end{matrix}\right.$$

$$\begin{matrix}[1-v-\delta k;w_1\ldots w_r,1],[1-\mu-\lambda k;\sigma_1\ldots\sigma_r,1],[1-\mu-\lambda k-\alpha-\beta-\eta;\sigma_1\ldots\sigma_r,1]:\\ [1-v-\delta k;w_1\ldots w_r,0]:[1-\mu-\lambda k-\beta;\sigma_1\ldots\sigma_r,1]:[1-\mu-\lambda k-\eta;\sigma_1\ldots\sigma_r,1]:\end{matrix}$$

$$\left.\begin{matrix}(a_j^1,\alpha_j^1)_{1,p^1}\ldots\,(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\;\_\_\_\_\_\_\\ \cdot\\ \cdot\\ \cdot\\ (b_j^1,\beta_j^1)_{1,q^1}\ldots\,(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}},\ldots(0,1)\end{matrix}\right].$$

Conditions of validity are same as in theorem (I).

The general class of polynomial $S_N^M[x]$ reduces to several classical orthogonal polynomials by suitable choice of M and $A_{N,k}$.

For example $If\ M=2, A_{N,K}=(-1)^k, S_N^2[x]\to x^{\frac{n}{2}}H_N\left(\frac{1}{2\sqrt{x}}\right),$

where: $H_n[x]$ is Hermite polynomial and if $M=1, A_{N,K}=\binom{N+\alpha}{N}.\frac{(\alpha+\beta+N+1)_k}{(\alpha+1)_k}, S_N^1[x]\to P_N^{(\alpha,\beta)}[1-2x],$

Where: $P_N^{(\alpha,\beta)}$ is Jacobi polynomial.

Hence many special cases can be investigated. The case of Jacobi polynomial is given by-

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(at+b)^{-v}.P_N^{(\alpha',\beta')}\left[1-2t^{\lambda}.(at+b)^{-\delta}\right].I\left[z_1t^{\sigma_1}(at+b)^{-w_1}\ldots.z_rt^{\sigma_r}(at+b)^{-w_r}\right]\right)\right\}(x)$$

$$=x^{\mu+\beta-1}b^{-v}\sum_{k=0}^{[N]}\frac{(-N)_k}{k!}\binom{N+\alpha'}{N}\frac{(\alpha'+\beta'+N+k)_k}{(\alpha'+1)_k}b^{-\delta k}.x^{\lambda k}.I^{0,n_2,\ldots.0,n_r+3\,:m^1,n^1\ldots\ldots\ldots m^{(r)},n^{(r)}\,:1\,0}_{p_2,q_2,\ldots.p_r+3,q_r+3\,:p^1,q^1\ldots.p^{(r)},q^{(r)}\,:0\,1}\left[\begin{matrix} z_1x^{\sigma_1}b^{-w_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{\sigma_r}b^{-w_r}\\ \frac{ax}{b}\end{matrix}\right|$$

$$\begin{matrix}[1-v-\delta k;w_1..w_r,1],[1-\mu-\lambda k;\sigma_1..\sigma_r,1],[1-\mu-\lambda k-\alpha-\beta-\eta;\sigma_1..\sigma_r,1]:(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}..(a_{rj};\alpha'_{rj},\alpha''_{rj})_{1,p_r}:\\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2}..(b_{rj};\beta'_{rj},\beta''_{rj})_{1,q_r}[1-v-\delta k;w_1..w_r,0],[1-\mu-\lambda k-\beta;\sigma_1..\sigma_r,1],[1-\mu-\lambda k-\eta;\sigma_1..\sigma_r,1]:\end{matrix}$$

$$\left.\begin{matrix}(a_j^1,\alpha_j^1)_{1,p^1}\ldots\,(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\;\_\_\_\_\_\_\\ \cdot\\ \cdot\\ \cdot\\ (b_j^1,\beta_j^1)_{1,q^1}\ldots\,(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}},\ldots(0,1)\end{matrix}\right].,$$

Provided the sufficient conditions of theorem (I) holds.

Conclusion

We have obtained the result namely theorem (I) and Theorem (II) which satisfied all the condition mention in the statement.

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